1. In each case, **give an example or explain why none exists**.

   (a) A permutation $f$ of $\{1, 2, 3, 4, 5\}$ such that $f^{20}$ has no fixed points.

   **A.** $f^{20}(x) = x$ if and only if $x$ belongs to a cycle of length $k$ for some divisor $k$ of 20. Since the domain has 5 elements, $k = 1,\ 2,\ 4$, or 5. Thus, $f^{20}(x) \neq x$ implies that $x$ belongs to a 3-cycle. Hence *every* $x \in \{1, 2, 3, 4, 5\}$ must belong to a 3-cycle, which is impossible.

   (b) A simple graph with 8 vertices, 2 connected components and 12 edges.

   **A.** There are many possibilities.

   (c) A connected simple graph with 8 vertices, 8 edges and no cycles.

   **A.** This is impossible. A connected simple graph with no cycles is a tree and a tree has one less edge than it has vertices.

2. Let $V = \{1, 2, \ldots, n\}$. An *oriented simple graph* with vertex set $V$ is a simple graph with vertex set $V$ where each edge is given a direction. In other words, given two vertices $v$ and $w$ in $V$, there are three choices: (a) no edge between them, (b) an edge $(v, w)$ or (c) an edge $(w, v)$. (Unlike a directed graph, you cannot have both $(v, w)$ and $(w, v)$ as edges and you cannot have loops.)
   Let $N = \binom{n}{2}$. We showed that there are $2^N$ simple graphs with vertex set $V$ and $\binom{N}{q}$ of them exactly $q$ edges.
   **Find** similar formulas for the number of oriented graphs with vertex set $V$ and for the number of them with exactly $q$ edges. *To receive credit you must justify your formulas; that is, explain how you got them.*

**A.** Between any pair of vertices there are 3 choices. Since there are $N = \binom{n}{2}$, the Rule of Product gives $3^N$ oriented graphs.
First choose the $q$ pairs of vertices to be connected by edges and then choose which direction to make each edge. By the Rule of Product, there are $\binom{N}{q}2^q$ oriented graphs with exactly $q$ edges.

3. Let $s_0 = 1$ and, for $n > 0$, let $s_n$ be the number of $n$-long sequences of As and Bs that never contain more than two Bs in a row. Thus AAABA and BBABB are counted in $s_5$, but ABBBA is not.

   **Obtain** a formula for $S(x) = \displaystyle\sum_{n=0}^{\infty} s_n x^n$, the generating function for $s_n$.

**A.** There are various ways this could be done. Here are two. By removing any terminal Bs and the last A, we have $s_n = s_{n-1} + s_{n-2} + s_{n-3}$ for $n \geq 4$. By counting, $s_1 = 2$, $s_2 = 4$, and $s_3 = 7$. We now need to construct the generating function for this recursion. Perhaps the easiest way is to let $s_0 = 1$ and note that, for $n \geq 0$,

$$s_n = a_n + s_{n-1} + s_{n-2} + s_{n-3},$$

where $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, and $a_n = 0$ for $n > 2$. Multiplying the recursion by $x^n$ and summing, we obtain $S(x) = A(x) + xS(x) + x^2S(x) + x^3S(x)$ and so

$$S(x) = \frac{A(x)}{1 - x - x^2 - x^3} = \frac{1 + x + x^2}{1 - x - x^2 - x^3}.$$

Another method is to use regular expressions to describe the sequences. A simple description is

$$\{\lambda, \text{B}, \text{BB}\}(\text{A}\{\lambda, \text{B}, \text{BB}\})^*,$$

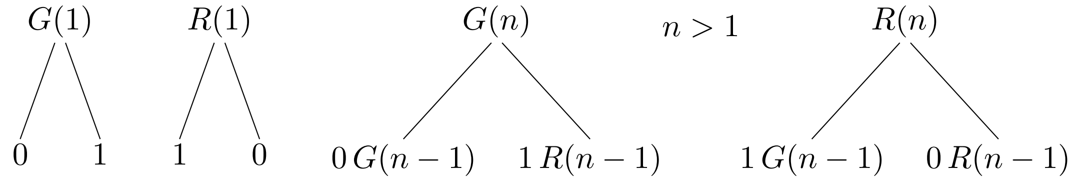
where $\lambda$ is the empty sequence. Here are generating functions

$$\begin{aligned}
\{\lambda, \text{B}, \text{BB}\} &: \quad 1 + x + x^2 \\
\text{A}\{\lambda, \text{B}, \text{BB}\} &: \quad x + x^2 + x^3 \\
(\text{A}\{\lambda, \text{B}, \text{BB}\})^* &: \quad \frac{1}{1 - (x + x^2 + x^3)} \\
\{\lambda, \text{B}, \text{BB}\}(\text{A}\{\lambda, \text{B}, \text{BB}\})^* &: \quad \frac{1 + x + x^2}{1 - x - x^2 - x^3}.
\end{aligned}$$

4. **Prove** the following theorem.
If $G$ is a (simple) graph, $P_G(x)$ is its chromatic polynomial, $n > k > 0$ are integers, and $P_G(n) = 0$, then $P_G(k) = 0$.
*Hint*: Remember the definition of the chromatic polynomial of a graph.

**A.** If $P_G(n) = 0$, then $G$ cannot be properly colored using $n$ colors. But then it cannot be properly colored with fewer than $n$ colors and so $P_G(k) = 0$ for $0 < k < n$.

5. Define $G(n)$ and $R(n)$ by the following local descriptions.




Here $0\,G(n-1)$ means that each of the sequences produced by $G(n-1)$ should have a zero placed in front of them.

(a) Using this definition,
**prove** that $G(n)$ and $R(n)$ each produce all $2^n$ $n$-long sequences of zeroes and ones.

**A.** We use induction on $n$. By looking at the two trees on the left, we see that the assertion is true for $n = 1$. For induction, we use the two trees on the right. Since $R(n)$ is done the same way as $G(n)$, we just do $G(n)$. Let the sequence be $a = a_1a_2\cdots a_n$. By the induction hypothesis, $G(n-1)$ and $R(n-1)$ both produce the sequence $a_2\cdots a_n$. If $a_1 = 0$, then $a$ is produced by $0\,G(n-1)$. If $a_1 = 1$, then $a$ is produced by $1\,R(n-1)$. This completes the induction step.

(b) **Find** the sequence of rank 67 in $G(7)$. You may find the following helpful $2^7 = 128$, $2^6 = 64$, $2^5 = 32$, et cetera.

**A.** Since $G(6)$ produces $2^6 = 64$ sequences, we want the sequence of rank $67-64 = 3$ in $1\,R(6)$. Since $G(5)$ produces $2^5 = 32$ sequences, we want the sequence of rank 3 in $11\,G(5)$. Similarly, we get eventually to the sequence of rank 3 in $11000\,G(2)$. Since $G(1)$ produces 2 sequences, we want the sequence of rank $3 - 2 = 1$ in $110001\,R(1)$, which is 1100010.

6. There are $2n$ clues to a game. One clue is distributed to each of $n$ husbands and wives. **How many ways** can the clues be redistributed so that it does not happen that any wife simply exchanges clues with her husband? In other words, we are looking for the number of permutations of the set

$$\{h_1, w_1, h_2, w_2, \ldots, h_n, w_n\}$$

that do *not* contain any of the cycles $(h_1, w_1)$, $(h_2, w_2)$, …, $(h_n, w_n)$.
*Hint*: This is an inclusion and exclusion problem.

**A.** Let $S_i$ be the number of permutations containing the cycle $(h_i, w_i)$. To compute $N_r$, we first choose $r$ husband-wife pairs that are to exchange clues and then permute the remaining $(2n-2r)$ clues. Thus $N_r = \binom{n}{r}(2n-2r)!$ and so the answer is
$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(2n-2r)!.$$

7. We are interested in RP-trees in which each non-leaf vertex may have either
(a) a left son and a right son, or (b) a left son only, or (c) a right son only.
Information is stored at every leaf and at every vertex with only one son. No information is stored at vertices that have two sons. Let $t_n$ be the number of such trees which have information stored at exactly $n$ vertices.
**Find** a formula for $T(x)$ similar to the formula $B(x) = x + B(x)^2$ we found for binary RP-trees.
*To receive credit you must justify your formula; that is, explain how you got it.*

**A.** A tree consists of exactly one of the following

- A single vertex, which is a leaf. Since information is stored at the vertex, this contributes $x$.
- A vertex joined to a left son. Since information is stored at the vertex and whatever vertices have in the son that have information, this contributes $xT(x)$ by the Rule of Product.
- Similarly for a right son.
- A vertex joined to two sons. Since no information is stored at the vertex, we get only the contribution of each son, giving $T(x)^2$.

Using the Rule of Sum, $T(x) = x + 2xT(x) + T(x)^2$.

8. On the second exam, we found the equation $T(x) = x+(x+1)T(x)^2$ for the generating function $T(x) = \sum_{n=1}^{\infty} t_n x^n$ for a certain kind of tree. It turns out that $t_n \sim an^b c^n$ for some $a$, $b$ and $c$. **Find** the values of $b$ and $c$ and **explain** how you got them.

**A.** One can solve the equation for $T(x)$ and apply Principle 11.6. It may be easier to use Principle 11.7 (Implicit functions). We do that. Hence $b = -3/2$ and $F(x, y)$ can be taken to be $(x + 1)y^2 + x - y$. Thus we have the equations

$$F(r, s) = (r + 1)s^2 + r - s = 0 \quad \text{and} \quad F_y(r, s) = 2(r + 1)s - 1 = 0.$$

From the second equation, $s = 1/2(r + 1)$. Putting this into the first equation:

$$\frac{r+1}{4(r+1)^2} + r - \frac{1}{2(r+1)} = 0 \quad \text{whence} \quad 1 + 4(r + 1)r - 2 = 0.$$

Thus $4r^2 + 4r - 1 = 0$ and so $r = \frac{-4+\sqrt{16+16}}{8} = \frac{\sqrt{2}-1}{2}$. Hence $c = \frac{2}{\sqrt{2}-1} = 2\left(\sqrt{2}+1\right)$.