- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK, but you may have two pages of notes.
- Calculators are NOT allowed.
- **You must show your work to receive credit.**

1. In each case, **give an example or explain why none exists**.
   (a) A permutation f of $\{1,2,3,4,5\}$ such that f^{20} has no fixed points.
   (b) A simple graph with 8 vertices, 2 connected components and 12 edges.
   (c) A connected simple graph with 8 vertices, 8 edges and no cycles.

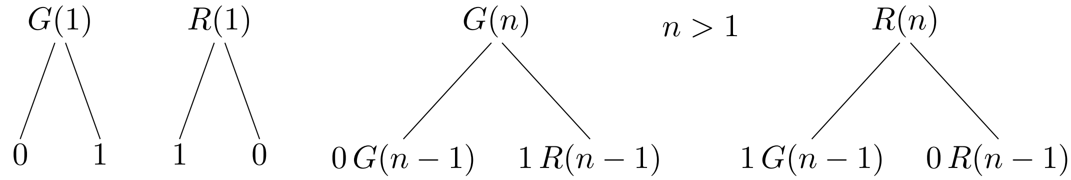
2. Let $V = \{1,2,\ldots,n\}$. An *oriented simple graph* with vertex set V is a simple graph with vertex set V where each edge is given a direction. In other words, given two vertices v and w in V, there are three choices: (a) no edge between them, (b) an edge (v,w) or (c) an edge (w,v). (Unlike a directed graph, you cannot have both (v,w) and (w,v) as edges and you cannot have loops.)
   Let $N = \binom{n}{2}$. We showed that there are 2^N simple graphs with vertex set V and $\binom{N}{q}$ of them exactly q edges.
   Find similar formulas for the number of oriented graphs with vertex set V and for the number of them with exactly q edges. *To receive credit you must justify your formulas; that is, explain how you got them.*

3. Let $s_0 = 1$ and, for $n > 0$, let s_n be the number of n-long sequences of As and Bs that never contain more than two Bs in a row. Thus AAABA and BBABB are counted in s_5, but ABBBA is not.
   Obtain a formula for $S(x) = \sum_{n=0}^{\infty} s_n x^n$, the generating function for s_n.

4. **Prove** the following theorem.
   If G is a (simple) graph, $P_G(x)$ is its chromatic polynomial, $n > k > 0$ are integers, and $P_G(n) = 0$, then $P_G(k) = 0$.
   Hint: Remember the definition of the chromatic polynomial of a graph.

THERE IS MORE

5. Define $G(n)$ and $R(n)$ by the following local descriptions.



Here $0\,G(n-1)$ means that each of the sequences produced by $G(n-1)$ should have a zero placed in front of them.

(a) Using this definition, **prove** that $G(n)$ and $R(n)$ each produce all 2^n n-long sequences of zeroes and ones.

(b) **Find** the sequence of rank 67 in $G(7)$. You may find the following helpful $2^7 = 128$, $2^6 = 64$, $2^5 = 32$, et cetera.

6. There are $2n$ clues to a game. One clue is distributed to each of n husbands and wives. **How many ways** can the clues be redistributed so that it does not happen that any wife simply exchanges clues with her husband? In other words, we are looking for the number of permutations of the set

$$\{h_1, w_1, h_2, w_2, \ldots, h_n, w_n\}$$

that do *not* contain any of the cycles (h_1, w_1), (h_2, w_2), ..., (h_n, w_n).
Hint: This is an inclusion and exclusion problem.

7. We are interested in RP-trees in which each non-leaf vertex may have either
(a) a left son and a right son, or (b) a left son only, or (c) a right son only.
Information is stored at every leaf and at every vertex with only one son. No information is stored at vertices that have two sons. Let t_n be the number of such trees which have information stored at exactly n vertices.
Find a formula for $T(x)$ similar to the formula $B(x) = x + B(x)^2$ we found for binary RP-trees.
To receive credit you must justify your formula; that is, explain how you got it.

8. On the second exam, we found the equation $T(x) = x + (x+1)T(x)^2$ for the generating function $T(x) = \sum_{n=1}^{\infty} t_n x^n$ for a certain kind of tree. It turns out that $t_n \sim a n^b c^n$ for some a, b and c. **Find** the values of b and c and **explain** how you got them.

END OF EXAM