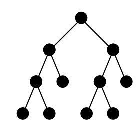
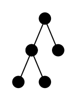
Math 184A Solutions to Second Exam 19 November 2001

1. Let $G = (V, E)$ where $V = \{0, 1, a, b, A, B\}$ and
   $E = \big\{\{0,1\},\ \{0,a\},\ \{0,b\},\ \{0,A\},\ \{0,B\},\ \ \{a,b\},\ \{A,B\}\big\}$.
   Sketch the simple graph G and
   compute its chromatic polynomial.

A. We omit the sketch. There are various ways to compute the chromatic polynomial. One can use formulas, but the easiest is to appeal directly to the definition. Given x colors, there are x ways to color vertex 0. There are then $x - 1$ ways to color each of 1, a, and A since they must differ from 0. There are then $x - 2$ ways to color each of b and B. (For example, b must differ from 0 and a, which leaves $x - 2$ colors for it.) By the Rule of Product from Chapter 1, $P_G(x) = x(x-1)^3(x-2)^2$.

2. **Compute** the rank of the binary RP-tree shown here. For your information, $b_1 = b_2 = 1$, $b_3 = 2$, $b_4 = 5$, $b_5 = 14$, $b_6 = 42$, and $b_7 = 132$.

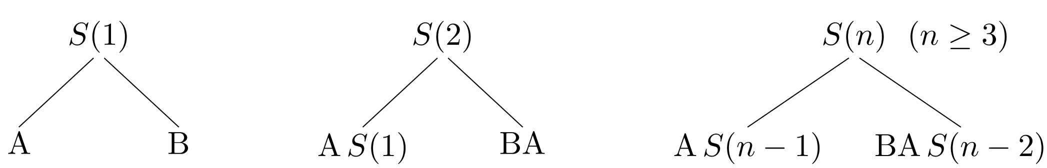


A. Let Let B be the given tree and let $T =$ . Then

$$\text{RANK}(B) = b_1 b_5 + b_2 b_4 + \text{RANK}(T) b_3 + \text{RANK}(T).$$

You can compute $\text{RANK}(T)$ or note that T is the second of the two trees counted by b_3 and so has rank 1. Thus the answer is $14 + 5 + 2 + 1 = 22$.

3. The local description of a decision tree for constructing sequences of A's and B's is given below. The notation BA $S(n-2)$ means place BA in front of each sequence produced by $S(n-2)$.



Let $S^*(n)$ denote the entire decision tree. Thus $S^*(1) = S(1)$ and $S^*(2)$ has the three leaves AA, AB, and BA.

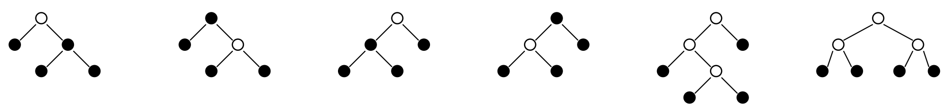
(a) **Find** a recursion for s_n, the number of leaves of $S^*(n)$.
Remember to include initial conditions.

A. From the pictures, we read off $s_1 = 2$, $s_2 = s_1 + 1 (= 3$, if you wish), and $s_n = s_{n-1} + s_{n-2}$ for $n \geq 3$.

(b) **Prove** that the leaves of $S^*(n)$ are sequences of length n and that their order from left to right is alphabetic.

A. We prove it by induction on n. For $n = 1$, it is clear from the picture. For $n = 2$, the entire tree has leaves AA, AB, and BA from left to right. Now we use induction for $n \geq 3$. Since A precedes B, the leaves of A $S^*(n-1)$ precede the leaves of BA $S^*(n-2)$ in alphabetic order. The leaves of $S^*(n-1)$ all have length $n-1$ and are in alphabetic order. Thus the leaves of A $S^*(n-1)$ all have length n and are in alphabetic order. The leaves of $S^*(n-2)$ all have length $n-2$ and are in alphabetic order. Thus the leaves of BA $S^*(n-2)$ all have length n and are in alphabetic order. This completes the proof.

4. A binary RP-tree has information stored at each leaf vertex. Each non-leaf vertex may or may not have information stored at it. Let t_n be the number of such trees with information stored at exactly n vertices and let $T(x) = \sum t_n x^n$ be the generating function. The following picture shows some of the nine trees that contribute to t_4. An empty circle indicates a vertex with no information.



Find a formula for $T(x)$ similar to the formula $B(x) = x + B(x)^2$ we found for binary RP-trees. *To receive credit you must justify your formula; that is, explain how you got it.*

A. A tree of the desired type is either
(a) a single vertex with information (since it is a leaf and so has information) OR
(b) a root with information joined to two trees of the same type OR
(c) a root without information joined to two trees of the same type.
The generating function for a vertex with information is $x^1 = x$ and that for a vertex without information is $x^0 = 1$. Using the Rule of Product in (b) and (c) and the Rule of Sum to combine the results we have

$$T(x) = x + xT(x)^2 + 1T(x)^2 = x + (x+1)T(x)^2.$$