1. Let A, B and C be finite sets. Suppose $f : A \to B$ and $g : B \to C$ are functions. Define $h : A \to C$ by $h(x) = g(f(x))$.
   (a) Prove or give a counterexample:
   If h is a surjection (an onto function), then f must be a surjection.
   A. False. Counterexample: $A = \{a\}$, $B = \{a, b\}$, $C = \{a\}$, $f(a) = a$, and $g(a) = b(b) = a$.
   (b) Prove or give a counterexample:
   If h is a surjection (an onto function), then g must be a surjection.
   A. True. Suppose $c \in C$. We must show that $g(b) = c$ for some $b \in B$. Since h is a surjection, there is some $a \in A$ such that $c = h(a) = g(f(a))$. Let $b = f(a)$.

2. A 5 person committee is to be chosen from a set of 6 men and 7 women.
   (a) How many possible committees are there?
   A. We must choose 5 people from 13, so the answer is $\binom{13}{5}$.
   (b) If the committee must contain at least 2 men and at least 2 women, how many possible committees are there?
   A. There are either (2 men AND 3 women) OR (3 men AND 2 women). By the Rules of Sum and Product, the answer is $\binom{6}{2} \times \binom{7}{3} + \binom{6}{3} \times \binom{7}{2}$.
   Choosing two of each sex and then a fifth committee member via $\binom{6}{2}\binom{7}{2}\binom{9}{1}$ overcounts. For example, if there are 3 women on the committee, the committee is counted 3 times depending on which woman is selected as the fifth committee member.

3. Prove that exactly half of the 2^{2n-1} compositions of $2n$ contain at most n parts.
   For example, when $n = 2$ the compositions of 4 with at most 2 parts are

$$4 \qquad 3+1 \qquad 2+2 \qquad 1+3.$$

   Warning: It was proved in a homework exercise that the average number of parts is $(2n + 1)/2$, but you cannot do the problem just by knowing the average number of parts. For example, the compositions in the set $\{2 + 2,\ 3 + 1,\ 1 + 3,\ 1 + 1 + 1 + 1\}$ have an average of $(2n + 1)/2$ parts but 3/4 of the compositions in the set have at most 2 parts.

A. The *result* about the average number of parts cannot be used, but the *method of proof* can be. Recall how we paired compostions. Write down $2n$ ones. Between every 2 ones write either a plus sign or a comma, for a total of $2n - 1$ symbols. Change plus signs to commas and vice versa. This is our pairing of compositions. If there are k plus signs in the original composition, then there are $k + 1$ parts. The second composition has $(2n - 1) - k$ plus signs and hence $2n - k$ parts. Since $(k + 1) + (2n - k) = 2n + 1$, exactly one of $k + 1$ and $2n - k$ is less than $(2n + 1)/2 = n + 1/2$. Thus exactly one compostion in the pair has at most n parts.

Note: Why does the problem involve compositions of an even number? If we had consider compositions of $2n - 1$, both compositions could have had n parts.

4. We want to count 4-bead necklaces that can be made using a supply of $k > 4$ different types of beads. (We allow rotations of a necklace, but not flipping over.)

   (a) How many necklaces can be made if each type of bead can be used as often as you wish?

   A. Consider an ordered list with the positions numbered 1,2,3,4. We apply Burnside's Lemma where G consists of the 4 permutations

$$g_0 = (1)(2)(3)(4) \qquad g_1 = (1,2,3,4) \qquad g_2 = (1,3)(2,4) \qquad g_3 = (1,4,3,2).$$

   Then $N(g_0) = k^4$, $N(g_1) = N(g_3) = k$, and $N(g_2) = k^2$. Thus the answer is $(k^4 + k^2 + 2k)/4$.

   (b) How many necklaces can be made if each type of bead can be used at most once in each necklace?

   A. You can use Burnside's Lemma as in (a). Now $N(g_0) = k(k-1)(k-2)(k-3)$ and $N(g_1) = N(g_2) = N(g_3) = 0$. (The zeroes arise because beads in the same cycle must be the same, but, since all beads are different, this is impossible unless each cycle has length 1.)

   Burnside's Lemma is not needed. Circular lists without repeats were discussed in Chapter 1 where a formula for any length was derived. Applying it to this case gives $k(k-1)(k-2)(k-3)/4$.