

## MATH 15B REVIEW PROBLEMS

**Disclaimer:** These problems were written by me [Nick Loehr] and represent types of problems that can be solved based on the material Dr. Bender covered in 15B. The subject matter and difficulty level of these questions *do not necessarily correspond* to what you will see on the final exam. However, I've tried to design problems that cover lots of the topics mentioned in the course. The problems are divided into two groups. The reasonable problems are intended to be easy or medium-level in difficulty. Some of the interesting problems may be harder or trickier, but should be solvable based on what's been talked about in the course. If you want even more difficult problems, just consult the course reader. **Look on the course web page for a list of topics that Dr. Bender decided to exclude from the final exam after I wrote most of these review problems.**

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### I. Reasonable Problems.

1. How many four-digit numbers (with no leading zeroes) are multiples of 5 AND have all digits distinct?
2. Define a "valid word" to be any string of letters (from the 26 letters A through Z, repetitions allowed) with length less than five and containing at least one vowel (A, E, I, O, U). How many valid words are there?
3. How many functions  $f : \{1, 2, \dots, 100\} \rightarrow \{1, 2, \dots, 100\}$  are there with the property that  $f(1) = 3$  and  $f(47) = 1$  and  $f(19) = 100$  and  $f(2) = 2$  and  $f(50) < 4$ ? What's the answer if  $f$  is also required to be a permutation?
4. An urn contains three balls labelled A, B, and C. Ball A has radius 1, ball B has radius 2, and ball C has radius 3. One ball is drawn from the urn. Assume that the probability of drawing a ball is proportional to its *volume*.
  - (a) Find  $P(\text{ball A is drawn})$ ,  $P(\text{ball B is drawn})$ , and  $P(\text{ball C is drawn})$ .
  - (b) Let  $X$  be the radius of the drawn ball. Compute  $E[X]$  and  $\text{Var}(X)$ .
5. Solve these recurrences. Verify that your solutions are correct by induction.
  - (a)  $f_{n+2} = -6f_{n+1} - 9f_n$  for  $n \geq 0$ ;  $f_0 = 2$  and  $f_1 = 0$ .
  - (b)  $f_{n+2} + f_n = 0$  for  $n \geq 1$ ;  $f_1 = f_2 = 1$ .
6. Draw a decision tree to make an alphabetical list of all four-letter words containing only the letters A, T, and H, and such that:
  - (i) Two A's are never adjacent.
  - (ii) An H not at the end of the word is always immediately followed by an A.
  - (iii) There are never three consecutive T's.Based on this tree, what is the rank of the word HATH?
7. Let  $H$  be the (simple) graph with vertex set  $\{1, 2, 3, 4, 5\}$  and no edges. How many different subgraphs does  $H$  have? How many connected components does  $H$  have?
8. Let  $G$  be a simple graph, such that the vertex set of  $G$  is  $V = \{1, 2, 3, 4, 5, 6\}$  and the edge set of  $G$  is  $E = \{\{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{1, 6\}, \{1, 4\}\}$ . Let each edge of  $G$  have a weight equal to the sum of the labels of the two endpoints of the edge.
  - (a) Draw a picture of  $G$ , labelling the edges with their weights.
  - (b) Use Prim's algorithm (and (a)) to find a minimal spanning tree for  $G$ . Start at vertex 2, and indicate the order in which edges are added to the tree.

- (c) Use Kruskal's algorithm (and (a)) to find a minimal spanning tree for  $G$ , indicating the order in which the algorithm accepts new edges. [The order may not be unique.]  
 (d) Ignore edge weights for this part. Find all depth-first (lineal) spanning trees for  $G$  starting at vertex 4.

9. Consider the following functions of  $n$ :

$$\log_2 n, \log_2(n^2), n, n^2, n^2 + n, 5000n + 4.$$

For which functions  $f(n)$  and  $g(n)$  in the above list is it true that  $f(n)$  is  $\Theta(g(n))$ ? For which functions is it true that  $f(n)$  is  $O(g(n))$ ?

10. Give examples of the following situations (all of them *are* possible!).

- A function  $f : \{1, 2, 3, 4\} \rightarrow \{a, b\}$  that is not surjective.
- A simple graph with 4 vertices that has *no* spanning tree.
- A rooted tree with 7 leaves and 3 internal vertices.
- A simple graph with weighted edges that has more than one minimal spanning tree.
- A random experiment in which the sample space is a finite set but where the points of the set are *not* all equally likely.
- A permutation  $f$  of  $\{1, 2, 3, 4, 5, 6\}$  such that  $f$  is not the identity function, but  $f^2 = f \circ f$  is the identity function.
- A simple graph with 6 vertices, 6 edges, and exactly 2 cycles.
- *Independent* events  $A$  and  $B$ , defined on the sample space  $S = \{1, 2, \dots, 24\}$  where every point has probability  $1/24$ , such that  $P(A) = 1/3$  and  $P(B) = 1/4$ .
- *Disjoint* events  $C$  and  $D$ , defined on the sample space  $S = \{1, 2, \dots, 24\}$  where every point has probability  $1/24$ , such that  $P(C) = 1/3$  and  $P(D) = 1/4$ .
- A function  $g(n)$  such that  $g(n)$  is  $O(n^e)$  for *every* real  $e > 0$ .

**Good supplemental problems from Lipschutz:**

- Counting: 2.56, 2.68, 2.76;
  - Sample spaces and events: 3.41, 3.55, 3.60;
  - Conditional probability: 4.42, 4.46, 4.50, 4.60, 4.67, 4.68, 4.78;
  - Random variables: 5.53, 5.54, 5.71, 5.80, 5.81;
  - Binomial, Normal, and Poisson random variables: 6.57, 6.65, 6.74, 6.82.
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## II. Interesting Problems.

1. Prove that, for all  $n \geq 1$  and  $k$  with  $0 \leq k \leq n$ , we have:  $n! \leq k!(n-k)!2^n$ .  
(Induction on  $n$  works — maybe. But there is a much shorter proof not using induction.)
2. How many rearrangements of the ten numbers  $1, 2, \dots, 10$  have the property that the first six numbers in the rearrangement are in increasing order, the last four numbers in the rearrangement are in increasing order, but the sixth number is larger than the seventh number? (For example,  $[1,3,5,7,8,9,2,4,6,10]$  would be such a rearrangement since  $1 < 3 < 5 < 7 < 8 < 9$  and  $2 < 4 < 6 < 10$  but  $9 > 2$ .)
3. Consider the vertex set  $\{1, 2, \dots, n\}$  where  $n \geq 6$ .
  - (a) How many simple graphs with this vertex set have  $n$  connected components? How many have  $n - 1$  connected components? How many have  $n - 2$  connected components?
  - (b) What are the answers to the questions in (a) if we ignore labels on the vertices and view isomorphic graphs as being the same? [This may be easier than (a).]
4. Given a rooted tree  $T$  with  $v$  vertices, how long is the list  $DFV(T)$ ?<sup>1</sup>
5. Let  $H$  be the *complete* simple graph with vertex set  $\{1, 2, 3, 4, 5\}$ . How many different subgraphs does  $H$  have?<sup>2</sup>
6. Let  $A$  be a set with  $n$  elements. Recall that a binary relation  $R$  on  $A$  is any subset of  $A \times A$ ; also recall that every function  $f : A \rightarrow A$  may be viewed as a special type of binary relation.
  - (a) How many binary relations are there on  $A$ ?
  - (b) How many functions  $f : A \rightarrow A$  are there?
  - (c) Given a binary relation  $R$  on  $A$ , show how to construct a graph with vertex set  $A$  that represents  $R$ . Which edges are in the graph? Are there loops, multiple edges, and/or directed edges? Answer the same questions for functions from  $A$  to  $A$ .
  - (d) A binary relation  $R$  on  $A$  is selected at random. What's the probability that  $R$  is a function? What's the probability that  $R$  is reflexive? What's the probability that  $R$  is symmetric, given that  $R$  is reflexive?
7. Consider graphs with vertex set  $\{1, 2, \dots, n\}$  in which edges are directed, loops are allowed, but multiple directed edges from a given initial point to a given final point are *not* allowed. How many such graphs are there with the property that every vertex and every edge in the graph belongs to exactly one (directed) cycle? [*Answer:  $n!$* ]
8. This long question is an attempt to explicate the following assertion in the reader: **The number of  $k$ -element multisets with elements from  $\{1, 2, \dots, n\}$  is  $\binom{n+k-1}{k}$ .**  
Define  $S$  to be the set of all  $k$ -element multisets with elements from  $\{1, 2, \dots, n\}$ . Define  $T$  to be the set of all *weakly increasing* (i.e., nondecreasing) lists of length  $k$  with elements from  $\{1, 2, \dots, n\}$ . Define  $U$  to be the set of all *strictly increasing* lists of length  $k$  whose elements come from  $\{1, 2, \dots, n+k-1\}$ . Define  $V$  to be the set of all  $k$ -element subsets of the set  $\{1, 2, \dots, n+k-1\}$ .

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<sup>1</sup>See Unit 3, Section 1 for terminology.

<sup>2</sup>The terms “complete” and “subgraph” are defined in Unit 4. You don't need to know them for the final exam.

(a) Prove that there are bijections  $f : T \rightarrow S$  and  $g : U \rightarrow V$ . [This is routine if you understand all the definitions.]

(b) Define  $h$  to be a function with domain  $T$  as follows. Suppose  $L$  is in  $T$ ; say  $L$  is the list  $(x_1, x_2, \dots, x_k)$  where every  $x_i$  is in  $\{1, 2, \dots, n\}$  and  $x_1 \leq x_2 \leq \dots \leq x_k$ . Define  $h(L)$  to be the list  $(x_1 + 0, x_2 + 1, x_3 + 2, \dots, x_k + k - 1)$ . [Example: If  $L$  was  $(1, 1, 2, 2, 3)$ , then  $h(L) = (1 + 0, 1 + 1, 2 + 2, 2 + 3, 3 + 4) = (1, 2, 4, 5, 7)$ .] Prove that  $h(L)$  is a list *in the set*  $U$  for every  $L$ . Thus, we can view  $U$  as the codomain of  $h$ , so that  $h$  is a function  $h : T \rightarrow U$ .

(c) Define  $j$  to be a function with domain  $U$  as follows. Suppose  $L'$  is in  $U$ ; say  $L'$  is the list  $(y_1, y_2, \dots, y_k)$  where every  $y_i$  is in  $\{1, 2, \dots, n + k - 1\}$  and  $y_1 < y_2 < \dots < y_k$ . Define  $j(L')$  to be the list  $(y_1 - 0, y_2 - 1, y_3 - 2, \dots, y_k - (k - 1))$ . Show  $j(U) \subset T$ . Thus,  $j$  is a function  $j : U \rightarrow T$ .

(d) Finally, show  $h$  is a two-sided inverse to  $j$ . Conclude that  $h$  and  $j$  are bijections; that there is a bijection from  $S$  to  $V$  and vice versa; and that the boldface statement at the beginning of this problem is true. Explain **why** these conclusions follow.

9. A room contains a fair FOUR-SIDED die (shaped like a tetrahedron), a dart, a circular dartboard, a square-shaped dartboard, a fair coin, an urn, and a supply of balls of various colors. The sides of the four-sided die are numbered 1, 2, 3, and 4. The circular dartboard is a circle of radius 3, which contains two smaller concentric circles with radii 2 and 1. The innermost region of this dartboard (inside the circle of radius 1) is red; the middle region (a ring between the circles of radius 1 and 2) is yellow; and the outer region (a ring between the circles of radius 2 and 3) is blue. Similarly, the square dartboard is a square with side-length 5, containing smaller concentric squares of side-lengths 3 and 1. On the square dartboard, the innermost region (inside the smallest square) is blue; the middle region is yellow; and the outer region is red. The urn initially contains one red ball, zero yellow balls, and three blue balls. A person enters the room and performs the following actions:

- The 4-sided die is thrown, and the number on the bottommost face is examined. This number of yellow balls is added to the urn.
- If the number from the die roll was odd, the dart is thrown at the circular dartboard; if the number from the die roll was even, the dart is thrown at the square-shaped dartboard. Assume the dart always hits the relevant dartboard, and every point on the board is equally likely to be hit.
- The fair coin is tossed zero times if the dart landed in the outermost region; one time if the dart landed in the middle region; or two times if the dart landed in the innermost region. On each toss (if the coin is tossed at all), a red ball is added to the urn if the coin comes up heads; a blue ball is removed from the urn if the coin comes up tails.
- Finally, a ball is randomly drawn from the urn and its color is noted. Every ball currently in the urn is equally likely to be drawn.

(a) What is the probability that the color of the ball drawn at the end is the same as the color of the region hit by the dart?

(b) What is the conditional probability that the number from the die roll was 4, given that a yellow ball was drawn at the end?

(c) Let  $X$  be the total number of balls in the urn just before the ball is randomly drawn in the final step. Find  $E(X)$ .