Q1. A family has 3 girls and 3 boys.

(a) How many ways can they sit around a circular table?

Ans: The answer is 5!. As derived in class and in the text, the number of circular arrangements of n people is $(n-1)!$. We have $n = 6$.

(b) How many ways can they sit in a row if girls and boys must alternate?

Ans: The pattern of sexes is either BGBGBG or GBGBGB. We can

Choose a pattern (2 ways) AND

Seat the girls (3! ways) AND

Seat the boys (3! ways) for a total of $2 \times 3! \times 3!$

Another way to do this is to seat one person, and then someone of the opposite sex, and then someone of the same sex as the first person, and then someone of the same sex as the second. The seating of the two remaining people is forced. We get $6 \times 3 \times 2 \times 2$.

(If you misread the problem and thought it was a circular seating, you would have obtained 12 since there are 6 times as many row seatings as circular ones.)

Q2. See solutions to homework problems.

Q3. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. A function is chosen uniformly at random from B^A.

(a) What is the probability that it is an injection?

Ans: Since the probability is uniform, the probability is the number of injections divided by the number of functions in the probability space. There are $|B^A| = 5^3$ functions. The number of injections could be written in various ways:

$$5 \times 4 \times 3 = 5!/(5-3)! = 5!/2! = 60.$$

Thus the probability of an injection is $(5!/2!)/5^3$.

(b) What is the probability that it is a surjection?

Ans: Since $|A| < |B|$, there are no surjections in B^A. Hence the probability is zero.

To receive credit, be sure to show how you got your answers.

Q4. Here is a permutation f in cycle form: $(1, 6, 3, 2)\,(4, 7, 5)$.

(a) Write f in one-line form.

Ans: 6,1,2,7,4,3,5.

(b) Write f^{-1} in cycle form.

Ans: Read the cycles backwards: $(1, 2, 3, 6)\,(4, 5, 7)$.

Since a cycle need not start with the smallest element, $(1, 2, 3, 6)$ could be replaced by any of the other 3 ways of writing the cycle: $(2, 3, 6, 1)$, $(3, 6, 1, 2)$, $(6, 1, 2, 3)$. Similarly, $(4, 5, 7) = (5, 7, 4) = (7, 4, 5)$. Also, it doesn't matter which cycle is written first.

(c) Write f^{13} in cycle form.

Ans: Since $13 = 3 \times 4 + 1$, $f^{13} = f = (1, 6, 3, 2)\,(4, 7, 5)$.

(See answer to (b) about other forms for the two cycles.)

Q5. Given that

$$\mathrm{E}(X) = 5, \quad \mathrm{E}(Y) = 0, \quad \mathrm{var}(X) = 2, \quad \mathrm{var}(Y) = 1, \quad \mathrm{cov}(X, Y) = -1.$$

Compute $\mathrm{E}(X^2)$ and $\mathrm{var}(X + 3Y)$.
Ans: Since $\mathrm{var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$,

$$\mathrm{E}(X^2) = \mathrm{var}(X) + (\mathrm{E}(X))^2 = 2 + 5^2 = 27.$$

Using the theorem on properties of covariance:

$$\begin{aligned}\mathrm{var}(U + V) &= \mathrm{cov}(U + V, U + V) \\ &= \mathrm{cov}(U, U + V) + \mathrm{cov}(V, U + V) \\ &= \mathrm{cov}(U, U) + \mathrm{cov}(U, V) + \mathrm{cov}(V, U) + \mathrm{cov}(V, V) \\ &= \mathrm{var}(U) + 2\mathrm{cov}(U, V) + \mathrm{var}(V).\end{aligned}$$

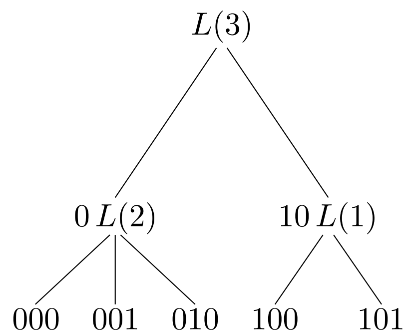
Thus we have

$$\mathrm{var}(X + 3Y) = \mathrm{var}(X) + 2 \times 3\mathrm{cov}(X, Y) + 3^2\mathrm{var}(Y) = 2 + 6(-1) + 9(1) = 5.$$

Q6. See the solutions to the homework problems.

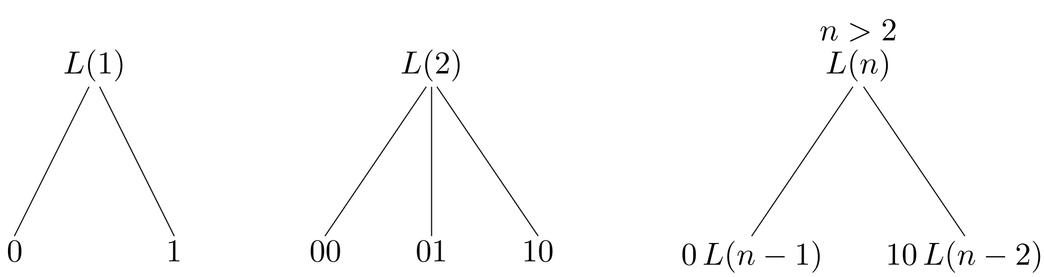
Q7. The local description of a decision tree for constructing sequences of 0's and 1's is given below. The notation $10\,L(n-2)$ means place 10 (1 and 0) in front of each sequence produced by $L(n-2)$.

(a) Draw the entire decision tree for $n = 3$. How many leaves does it have?
Ans: The tree, shown here, has 5 leaves.



(b) Let a_n be the number of leaves in the entire decision tree for $L(n)$. Obtain a recursion for a_n. *Remember to include initial conditions.*

Ans: The first two trees give the initial conditions $a_1 = 2$ and $a_2 = 3$. The rightmost tree gives the recursion $a_n = a_{n-1} + a_{n-2}$ for $n > 2$.



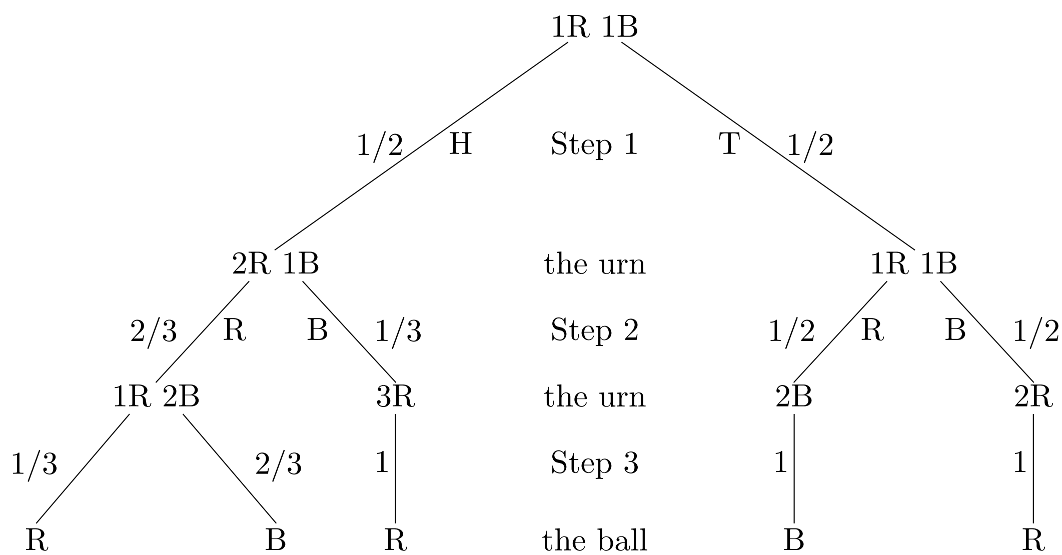
Q8. An urn contains one red ball and one blue ball. I do the following steps:

1. Flip a fair coin. If heads, add a red ball to the urn. If tails, add no balls to the urn.
2. Remove a ball at random from the urn and replace it with a ball of the opposite color.
3. Remove a ball at random from the urn.

Draw the decision tree and use it to answer the following:

(a) What is the probability that the ball removed in Step 3 is blue?

(b) If the ball in Step 3 is blue, what is the probability that the coin toss was heads?



(a) The probability of reaching B is $((1/2) \times (2/3) \times (2/3)) + ((1/2) \times (1/2) \times 1)$, which you can simplify to $(2/9) + (1/4) = 17/36$ if you wish.

(b) Since heads leads to the left of the two B leaves is my tree, the conditional probability of heads given B is $(2/9)/(17/36) = 8/17$. If you chose not to simplify any arithmetic, you would have

$$\frac{(1/2) \times (2/3) \times (2/3)}{((1/2) \times (2/3) \times (2/3)) + ((1/2) \times (1/2) \times 1)}.$$

Q9. Give a graph satisfying the conditions for each problem **OR** explain why none exists.

(a) A **simple** graph with 4 vertices and 7 edges.
Ans: Impossible. Since there can be no more than $\binom{4}{2} = 6$ edges. Here is a lengthier explanation: The edges are a subset of $\mathcal{P}_2(V)$ and $|\mathcal{P}_2(V)| = \binom{|V|}{2} = \binom{4}{2} = 6$, so there are at most 6 edges.

(b) A **simple** graph with 4 vertices, 2 edges, and 4 connected components.
Ans: Impossible. Since each component must have a vertex and there are 4 of each, each component must be a single vertex and so there can be no edges.

(c) A **simple** graph with 6 vertices, 6 edges, and 2 cycles.
Ans: Possible. Except for labeling the vertices, the only solution is