1. In each case, **give an example or explain why none exists**.
   (a) A permutation f of $\{1,2,3,4,5\}$ such that, for *some* $x \in \{1,2,3,4,5\}$, $f^{20}(x) \neq x$.
   A. In cycle form, choose f to have a 3-cycle and x to belong to the cycle. (There are 120 correct answers.)
   (b) A permutation f of $\{1,2,3,4,5\}$ such that, for *every* $x \in \{1,2,3,4,5\}$, $f^{20}(x) \neq x$.
   A. $f^{20}(x) = x$ if and only if x belongs to a cycle of length k for some divisor k of 20. Since the domain has 5 elements, $k = 1,\ 2,\ 4$, or 5. Thus, $f^{20}(x) \neq x$ implies that x belongs to a 3-cycle. Hence *every* $x \in \{1,2,3,4,5\}$ must belong to a 3-cycle, which is impossible.
   (c) A tree with exactly 10 vertices and exactly 10 edges.
   A. A tree with n vertices has $n-1$ edges, so this is impossible.

2. In each case, **give an example or explain why none exists**.
   (a) A function $f(n)$ such that $f(n)$ is in $O(n^2)$ but $f(n)$ is not $\Theta(n^2)$.
   A. Any function which grows slower than n^2 will work; for example, $f(n) = n$.
   (b) A function $f(n)$ such that $f(n)$ is $O(n \log n)$ but $f(n)$ is not $O(n^2)$.
   A. Impossible, since anything that doesn't grow faster than $n \log n$ does not grow faster than n^2. (Alternatively, you could note that $(n \log n)/n^2$ is bounded for large n.)
   (c) A probability space (U, P) and two subsets S and T of U such that $P(S) = P(T) = 2/3$ and $S \neq T$.
   A. There are many possible answers. For example, $U = \{a,b,c\}$, P is uniform, $S = \{a,b\}$ and $T = \{b,c\}$. Another: $S = U = \{a,b\}$, $T = \{a\}$ and $P(a)$ is chosen greater than 1/2.

3. A fair die is tossed. If n is the value that is seen, define the random variable X by $X = |n-2|$
   (a) **Compute** the probability that $X = k$ for $k = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6$.
   A. Each value of $n-3$ from -2 to 3 has probability 1/6. Thus

$$P(X\!=\!k) = \begin{cases} 1/6 & \text{for } k = 0,3; \\ 1/3 & \text{for } k = 1,2; \\ 0 & \text{otherwise.} \end{cases}$$

   (b) **Compute** the mean and variance of X. *Do the arithmetic.*
   A. The mean is $(0+3)/6 + (1+2)/3 = 3/2$. Since

$$\mathrm{E}(X^2) = (0+3^2)/6 + (1^2+2^2)/3 = 19/6,$$

   we have $\mathrm{var}(X) = 19/6 - (3/2)^2 = 11/12$.

4. The platoon commander knows:
   - If the air strike is successful, there is a 60% probability that the ground forces *will not* encounter enemy fire.
   - If the air strike is not successful, there is a 80% probability that the ground forces *will* encounter enemy fire.
   - There is a 70% probability that the air strike will be successful.

   Answer the following questions.

   (a) **What is the probability** that the ground forces *will not* encounter enemy fire?

   (b) The ground forces did not encounter enemy fire. **What is the probability** that the air strike was successful?

A. You can draw a decision tree. The first level branches according as the air strike is successful (A) or not (A'). The second level branches according as there is enemy fire (F) or not (F'). The leaves and their probabilities are

$$P(A \cap F) = 0.7 \times 0.4 = 0.28, \qquad P(A \cap F') = 0.7 \times 0.6 = 0.42,$$

$$P(A' \cap F) = 0.3 \times 0.8 = 0.24, \qquad P(A' \cap F') = 0.3 \times 0.2 = 0.06.$$

For (a), $P(F') = 0.42 + 0.06 = 0.48$ and for (b)

$$P(A \mid F') = \frac{P(A \cap F')}{P(F')} = \frac{0.42}{0.48} \approx 82\%.$$

5. After being dealt 4 cards, I have 3 of a kind and a 4th card that has a different face value.

   (a) **How many** such hands of 4 cards are there? (For counting, the order cards are dealt does not matter, only what is in the hand.)

   A. Choose the face value for the 3 of a kind, choose the suits for it, and choose the remaining card:

$$13 \times \binom{4}{3} \times 48 = 13 \times 4 \times 48 = 52 \times 48 = 2496.$$

   (b) I will be dealt a 5th card. **What is the probability** that, given the 4 cards I already have, I will end up with a hand that contains either 4 of a kind or a full house?

   A. Each of the remaining 48 cards is equally likely. To get the desired hand, I need the 4th card with the face value of the 3 of a kind or one of the 3 cards with a face value the same as my other card. Thus, I need one of 4 cards and so the probability is $4/48 = 1/12$.

6. **Prove**: If a graph has v vertices and n connected components, then it has at least $v-n$ edges.
Hint: A tree with t vertices has $t-1$ edges.

A. Here is an acceptable proof. Let v_i be the number of vertices in the ith component. Each component has a spanning tree. Thus the ith component has at least $v_i - 1$ edges. Adding up over all components, we get at least $v-n$ edges.

Here is a formal induction proof on n. For $n = 1$, a conncted graph with v vertices has a spanning tree and this has $v-1$ edges. Hence the original graph has at least $v-1$ edges. Now suppose $n > 1$. Let C be a connected component and let G be the remaining $n-1$ components. Let these two parts have v_C and v_G vertices, respectively. Note that $v = v_C + v_G$. By the result for $n = 1$, C has at least $v_C - 1$ edges. By the induction hypothesis, v_G, G has at least $v_G - (n-1)$ edges. Thus the number of edges in the graph is at least

$$(v_C - 1) + (v_G - (n-1)) = (v_C + v_G) - n = v - n.$$

7. Define a_n by $a_0 = 1$ and the recursion $a_n = (n/a_{n-1}) + a_{n-1}$ for $n > 0$.
Guess and prove a formula for a_n.
Suggestion: To help with your guessing, compute the first few values of a_n.

A. By computing a few values, it's easy to guess that $a_n = n+1$. You can prove this by induction or you can use the theorem in the text as follows: Note that $f(n) = n+1$ satisfies $f(0) = 1$ (the initial condition) and $f(n) = n/f(n-1) + f(n-1)$ (the recursion) and so $a_n = f(n)$.

8. The following algorithm computes x^n for n a nonnegative integer, where x is a complicated object and `MULT` is a procedure that multiplies such objects.

```
POW(x,n)
  If (n=0) Return 1
  Else
    Let q and r be the quotient and remainder when n is divided by 2.
    // Thus q = n/2 rounded down and r = n − 2q, which is 0 or 1.
    y = MULT(x,x)
    z = POW(y,q)                                  // Remark: A recursive call.
    If (r=0) Return z
    Else Return MULT(x,z)
    End if
  End if
End
```

Find a function $f(n)$ so that the number calls of `MULT` is $\Theta(f(n))$.
Hint: Use the Master Theorem for Recursions.

A. Let $T(n)$ be the number of calls of `MULT`. We have $T(n) = a_n + T(s(n))$ where a_n is 1 or 2 and $s(n) = q$ is $n/2$ rounded down.
The Master Theorem applies with $b = 0$, $c = 1/2$ and $w = 1$. Thus
$d = -\log(1)/\log(1/2) = 0$ and so $T(n)$ is $\Theta(\log n)$.