- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK, but TWO PAGES OF NOTES ARE ALLOWED.
- Calculators are NOT ALLOWED. You need not evaluate binomial coefficients.
- **You must show your work to receive credit.**

1. In each case, **give an example or explain why none exists**.
   (a) A permutation f of $\{1,2,3,4,5\}$ such that, for *some* $x \in \{1,2,3,4,5\}$, $f^{20}(x) \neq x$.
   (b) A permutation f of $\{1,2,3,4,5\}$ such that, for *every* $x \in \{1,2,3,4,5\}$, $f^{20}(x) \neq x$.
   (c) A tree with exactly 10 vertices and exactly 10 edges.

2. In each case, **give an example or explain why none exists**.
   (a) A function $f(n)$ such that $f(n)$ is $O(n^2)$ but $f(n)$ is not $\Theta(n^2)$.
   (b) A function $f(n)$ such that $f(n)$ is $O(n \log n)$ but $f(n)$ is not $O(n^2)$.
   (c) A probability space (U,P) and two subsets S and T of U such that $P(S) = P(T) = 2/3$ and $S \neq T$.

3. A fair die is tossed. If n is the value that is seen, define the random variable X by $X = |n-3|$
   (a) **Compute** the probability that $X = k$ for $k =$ 0, 1, 2, 3, 4, 5, 6.
   (b) **Compute** the mean and variance of X. *Do the arithmetic.*

4. The platoon commander knows:
   - If the air strike is successful, there is a 60% probability that the ground forces *will not* encounter enemy fire.
   - If the air strike is not successful, there is a 80% probability that the ground forces *will* encounter enemy fire.
   - There is a 70% probability that the air strike will be successful.

   Answer the following questions.
   (a) **What is the probability** that the ground forces *will not* encounter enemy fire?
   (b) The ground forces did not encounter enemy fire. **What is the probability** that the air strike was successful?

5. After being dealt 4 cards, I have 3 of a kind and a 4th card that has a different face value.
   (a) **How many** such hands of 4 cards are there? (For counting, the order cards are dealt does not matter, only what is in the hand.)
   (b) I will be dealt a 5th card. **What is the probability** that, given the 4 cards I already have, I will end up with a hand that contains either 4 of a kind or a full house?
   (A full house is a pair and 3 of a kind.)

THERE IS MORE

6. **Prove**: If a graph has v vertices and n connected components, then it has at least $v - n$ edges.
   Hint: A tree with t vertices has $t - 1$ edges.

7. Define a_n by $a_0 = 1$ and the recursion $a_n = (n/a_{n-1}) + a_{n-1}$ for $n > 0$.
   Guess and prove a formula for a_n.
   Suggestion: To help with your guessing, compute the first few values of a_n.

8. The following algorithm computes x^n for n a nonnegative integer, where x is a complicated object and `MULT` is a procedure that multiplies such objects.

```
POW(x,n)
  If (n=0) Return 1
  Else
    Let q and r be the quotient and remainder when n is divided by 2.
    // Thus q = n/2 rounded down and r = n − 2q, which is 0 or 1.
    y = MULT(x,x)
    z = POW(y,q)                              // Remark: A recursive call.
    If (r=0) Return z
    Else Return MULT(x,z)
    End if
  End if
End
```

   Find a function $T(n)$ so that the number calls of `MULT` is $\Theta(T(n))$.
   Hint: Use the Master Theorem for Recursions.

Theorem (Master Theorem for Recursions) *Suppose that there are*

(i) numbers N and $0 < c < 1$,
(ii) a sequence $a_1, a_2, \ldots$,
(iii) functions $s_1, s_2, \ldots, s_w$, and T

such that

(a) $T(n) > 0$ for all $n > N$ and $a_n \geq 0$ for all $n > N$;
(b) $T(n) = a_n + T(s_1(n)) + T(s_2(n)) + \cdots + T(s_w(n))$ for all $n > N$;
(c) a_n is $\Theta(n^b)$ for some $b \geq 0$;
(d) $|s_i(n) - cn|$ is $O(1)$ for $i = 1, 2, \ldots, w$.

Let $d = -\log(w)/\log(c)$. Then

$$T(n) \text{ is } \begin{cases} \Theta(n^d) & \text{if } b < d, \\ \Theta(n^d \log n) & \text{if } b = d, \\ \Theta(n^b) & \text{if } b > d. \end{cases}$$

END OF EXAM