Q1. A family has 4 girls and 3 boys.

(a) How many ways can they sit in a row?
Ans: $7!$.

(b) How many ways can they sit in a row if boys and girls must alternate?
Ans: $4! \times 3!$ since the only possible seating pattern in GBGBGBG so we seat the boys ($4!$) and seat the girls ($3!$).

Q2. How many ways can t teams each of size s be made from st people? The teams have no names or other distinguishing features. Three versions were given depending on student ID number:

$$s = 2, \quad t = 4; \qquad\qquad s = 3, \quad t = 3; \qquad\qquad s = 4, \quad t = 2.$$

Ans: One could label the teams (count ordered teams). This will be $t!$ times the number of teams since there are $t!$ ways to assign t labels to t different sets of people. The number of labeled teams is $\binom{st}{s,s,\ldots,s}$ and so the answer is $\frac{(st)!}{t!(s!)^t}$. (One can do this with repeated binomial coefficients instead of multinomial coefficients: $\binom{st}{s}\binom{s(t-1)}{s}\cdots$.)

Q3. Let A, B and $C \subseteq B$ be sets. We make B^A into a probability space by selecting functions from A to B uniformly at random.

(a) What is the probability that a random f is an injection?
Ans: Since there are altogher b^a functions, each has probability $1/b^a$. An injection is an a-list without repetition from B, so there are $(b)_a$ of them. Thus the probability is $(b)_a/b^a$.

(b) What is the probability that $f(A) \subseteq C$ for a random f?
Ans: The condition simply says that f can be viewed as a function from A to C, of which there are c^a. Thus the probability is c^a/b^a. Someone with a bit more probability background could think as follows: The values of f can be chosen independently and the probability that $f(a) \in C$ is c/b. Thus the answer is $(c/b)^a$.

Q4. A permutation is given in cycle form. Write it in two line form and find its tenth power. Three versions were given depending on ID number.

$$(1,3,7)(2,9,4,8)(5,6) \qquad (1,2,5,4)(3,9)(6,8,7) \qquad (1,7)(2,4,9,6)(3,8,5).$$

Ans: Since they are similar, only the first is done here. $\binom{1\,2\,3\,4\,5\,6\,7\,8\,9}{3\,9\,7\,8\,6\,5\,1\,2\,4}$. To get the tenth power, one can count around 10 in each cycle. For a 3-cycle, this brings you one past your starting point, for a 2-cycle, back to your starting point and for a 4-cycle, two past. Thus the answer is $(1,3,7)(2,4)(8,9)(5)(6)$. Note that $(8,9) = (9,8)$ so either is okay. Also, the order in which the cycles is listed does not matter—only order within cycles matters.

Q5. Suppose X and Y are random variables with mean 0 and variance σ^2. Suppose that $\mathrm{Cov}(X,Y) = c$. Express the following in terms of σ and c.

$$\mathrm{E}(X^2) \qquad \mathrm{Var}(X+Y) \qquad \mathrm{Cov}(X+Y,\ X-Y)$$

Ans: We have $0 = \mathrm{E}(X) = \mathrm{E}(Y) = \mathrm{E}(X+Y) = \mathrm{E}(X-Y)$.
Since $\mathrm{Var}(X) = \mathrm{E}((X-\mathrm{E}(X))^2) = E(X^2)$, we have $\mathrm{E}(X^2) = \sigma^2$. [Also, $\mathrm{E}(Y^2) = 0$.]
We have

$$\mathrm{Var}(X+Y) = \mathrm{E}((X+Y-\mathrm{E}(X+Y))^2) = \mathrm{E}(X^2+2XY+Y^2) = \sigma^2+2c+\sigma^2,$$

where the last step uses linearity of expectation and the fact that X and Y have mean zero. Finally

$$\begin{aligned}\mathrm{Cov}(X+Y,X-Y) &= \mathrm{E}((X+Y-\mathrm{E}(X+Y))(X-Y-\mathrm{E}(X-Y)))\\ &= \mathrm{E}(X^2-Y^2) = \mathrm{E}(X^2)-\mathrm{E}(Y^2) = \sigma^2-\sigma^2 = 0.\end{aligned}$$

The last two could also be done by the linearity of covariance:

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Cov}(X+Y,X+Y)\\ &= \mathrm{Cov}(X,X)+\mathrm{Cov}(X,Y)+\mathrm{Cov}(Y,X)+\mathrm{Cov}(Y,Y)\\ &= 2\sigma^2+2c\\ \mathrm{Cov}(X+Y,X-Y) &= \mathrm{Cov}(X,X)-\mathrm{Cov}(X,Y)+\mathrm{Cov}(Y,X)-\mathrm{Cov}(Y,Y)\\ &= \sigma^2-c+c-\sigma^2 = 0.\end{aligned}$$

Q6. A tree was drawn on the blackboard and the following were requested:
breadth first vertex sequence (BFV),
depth first vertex sequence (DFV),
preorder sequence of vertices (PREV),
the ranks of the leaves.
The root of the tree was F. From F, edges led to H, C and G. From H, edges led to D and A. From C, an edge led to E. From B, an edge led to B.

Ans: BFV = F, H, C, G, D, A, E, B
DFV = F, H, D, H, A, H, F, C, E, C, F, G, B, G, F
PREV = F, H, D, A, C, E, G, B

Leaf	D	A	E	B
Rank	0	1	2	3

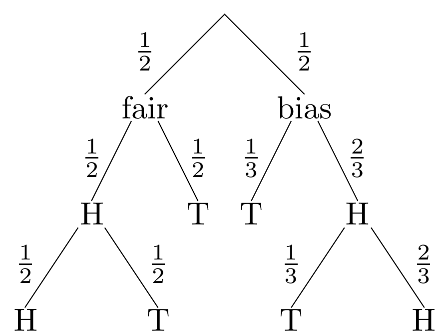
Q7. The permutations of $\{1,2,3,4,5,6\}$ are listed in lexicographic order. What is the rank of the following permutation? [Choice depends on ID number.]

$$2,1,5,3,6,4 \qquad 3,1,2,6,4,5 \qquad 4,2,1,3,6,5 \qquad 1,6,4,2,3,5.$$

Ans: All of these are done the same way: At each decision, count the number of leaves in decisions from the same vertex that are to the left. Here's an easy way to do it: If the permutation is $p_1,\ldots,p_n$, then the number of decisions to the left of p_k is the number of values of $p_{k+1},\ldots,p_n$ which are less than p_k, and each such decision leads to $(n-k)!$ leaves since it leads to all permutations of a set of size $n-k$. Here are the results

$2,1,5,3,6,4$	$1\times 5!+0\times 4!+2\times 3!+0\times 2!+1\times 1! = 133$
$3,1,2,6,4,5$	$2\times 5!+0\times 4!+0\times 3!+2\times 2!+0\times 1! = 244$
$4,2,1,3,6,5$	$3\times 5!+1\times 4!+0\times 3!+0\times 2!+1\times 1! = 385$
$1,6,4,2,3,5$	$0\times 5!+4\times 4!+2\times 3!+0\times 2!+0\times 1! = 108$

Q8. The tree is shown below. $P(H) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{2}{3}\times\frac{2}{3} = 25/72$.
$P(B|H) = P(B\cap H)/P(H) = \frac{1}{2}\times\frac{2}{3}\times\frac{2}{3}/(25/72) = 16/25$.



Q9. Give **SIMPLE** graphs satisfying the conditions for each problem **OR** explain why none exist.

(a) A graph with 3 vertices and 1 edge.
Ans: Except for labeling the vertices, the only solution is

(b) A graph with 3 vertices and 4 edges.
Ans: Impossible since $E \subseteq \mathcal{P}_2(V)$ and
$|E| \le |\mathcal{P}_2(V)| = \binom{|V|}{2} = \binom{3}{2} = 3 < 4 = |E|$.

(c) A graph with 3 vertices and 2 connected components.
Ans: Same answer as (a)

(d) A graph G with 4 vertices and one H with 5 vertices such that G is a subgraph of H.
Ans: Many answers are possible. The simplest is probably 5 vertices with labels 1–5 for H and 4 vertices with labels 1–4 for G. No edges are needed.

(e) A graph G with 4 vertices and one H with 5 vertices such that G is isomorphic to H.
Ans: Impossible since isomorphic graphs have the same number of vertices.

Q10. 1, 2, 1, 3, 1