1. (36 pts) A *five digit number* is a sequence of five digits, the first of which is **NOT** zero. Thus, 12345 and 10101 are valid but 01234 and 1234 are **NOT** valid.

   (a) How many five digit numbers are there?

   Ans: One way is to note there are 9 choices for the first digit and 10 for each of the remaining giving 9×10^4. You could also note that it is simply all numbers less than 10^5 and at least 10^4 giving $10^5 - 10^4 = 9 \times 10^4$.

   (b) How many five digit numbers have all digits different? (as in 12345 but not 10101)

   Ans: There are 9 choices for the first digit, 9 for the second (anything but the first), 8 for the third (anything but the first two), and so on, giving $9 \times 9 \times 8 \times 7 \times 6$.

   (c) How many five digit numbers have no digit appearing just once? (So 11111 and 10010 are okay, but 10111 and 12312 are not.)

   Ans: Either one digit appears 5 times OR two digits appear, one twice and one three times. There are 9 possibilities for the first case. For the second case, choose the two positions for the pair of digits in $\binom{5}{2}$ ways, choose the first digit in the number in 9 ways (anything but zero), and choose the other digit in 9 ways. Thus we have $9 + \binom{5}{2} \times 9 \times 9$.

2. (16 pts.) Nine people, including Alice, are to be divided into two teams of four people each, plus a referee. If all divisions are equally likely, what is the probability that Alice is the referee? (No, it doesn't matter if the teams are distinguishable or not.) **Be sure to explain how you got your answer.**

Ans: There are nine choices for the referee, all of which are equally likely. Thus, the probability that Alice is the referee is 1/9.

3. (24 pts.) An integer k from 1 to 9 is picked uniformly at random.
   Let $X(k) = 1$ if k is odd and $X(k) = 0$ if k is even.
   Let $Y(k)$ be the remainder when k is divided by 3.

   (a) Draw a table like the one here and fill in the probabilities.

   Ans: First we construct a table of X and Y for each possible outcome. Since each outcome has probability 1/9, a simple count lets us fill the table.

	1	2	3	4	5	6	7	8	9
X	1	0	1	0	1	0	1	0	1
Y	1	2	0	1	2	0	1	2	0

$X \setminus Y$	0	1	2
0	1/9	1/9	2/9
1	2/9	2/9	1/9

(b) Compute $\mathrm{Cov}(X, Y)$.

Ans: We need $\mathrm{E}(X)$ and $\mathrm{E}(Y)$. From the table or otherwise, $P(X = 0) = 4/9$ and $P(X = 1) = 5/9$ and so $\mathrm{E}(X) = 5/9$. Also, $P(Y = 0) = P(Y = 1) = P(Y = 2) = 1/3$ and so $\mathrm{E}(Y) = 1/3 + 2 \times (1/3) = 1$. We can compute $\mathrm{Cov}(X, Y)$ from the formula $\mathrm{E}((X - \mathrm{E}(X))(Y - \mathrm{E}(y)))$ or from the formula $\mathrm{E}(XY) - \mathrm{E}(X) \times \mathrm{E}(Y)$. I prefer the latter since there are less fractions. From the table, adding the entries where $XY \neq 0$, since $XY = 0$ values contribute nothing to the expectation,

$$\mathrm{E}(X, Y) = 1 \times 1 \times (2/9) + 1 \times 2 \times (1/9) = 4/9.$$

Thus $\mathrm{Cov}(X, Y) = 4/9 - (5/9) \times 1 = -1/9$.

4. (24 pts.) Suppose a strictly decreasing function $f : \{1, 2\} \to \{1, 2, \ldots, n\}$ is chosen uniformly at random. The random variable X is defined by $X(f) = f(1)$.

(a) Describe choosing f in terms of choosing subsets of a set. (Specify the set, what subsets are chosen, and how they are chosen.) If S is a subset associated with f, what is X in terms of S?

Ans: f corresponds to choosing a 2-element subset of $\{1, 2, \ldots, n\}$ uniformly at random. X is the larger element of the subset S.

(b) Derive the formula

$$P(X = k) = \begin{cases} (k-1) \;/\; \binom{n}{2} & \text{for } 1 \leq k \leq n, \\ 0 & \text{for } k < 1 \text{ and } k > n. \end{cases}$$

Ans: The larger element must be at least 2 and at most n. Hence $P(X = k) = 0$ if $k < 2$ or $k > n$. This agrees with the given formula. The number of 2-element subsets whose larger element is k is $k - 1$ since the smaller element must be one of $1, \ldots, (k-1)$. Since there are $\binom{n}{2}$ possible 2-element subsets and all are equally likely, the formula follows.