1. (60 pts) In each case, give an example or explain why none exists.

   (a) A tree with exactly nine vertices and exactly nine edges.

   Ans: Impossible since a tree always has one less edge than it has vertices.

   (b) A permutation f on $\{1,2,3,4\}$ such that $f^{100} \neq f$.
   Notes: Remember that $f \neq g$ for functions with the same domain means there is at least one x such that $f(x) \neq g(x)$. Also remember that $f^{100}(x)$ means $f(f(\cdots f(x)\cdots))$, NOT $(f(x))^{100}$.

   Ans: The question is equivalent to asking that f^{99} not be the identity function. Such a permutation must have at least one cycle whose length does not divide 9. Since we are permuting a 4-set, the possible cycle lengths are 1, 2, 3 and 4. Thus f must be either a 4-cycle (there are six possibilities) or a product of two 2-cycles (there are three possibilities). Any of the nine possible answers is acceptable.

   (c) A simple graph with exactly five vertices that has a cycle containing six edges.

   Ans: Impossible, if we had six edges, the cycle would pass through at least one vertex twice.

   (d) A sample space U with a probability function P and two different elements s and t of U such that $P(s) > 1/2$ and also $P(t) > 1/2$.
   Note: $P(s)$ means the same thing as $P(\{s\})$.

   Ans: Impossible since $P(x) \geq 0$ and $1 = \sum_{x \in U} P(x) \geq P(s) + P(t)$.

   (e) A sample space U with a probability function P and two different subsets S and T of U such that $P(S) > 1/2$ and also $P(T) > 1/2$.

   Ans: There are many possibilities. For example, the uniform distribution on $U = \{1,2,3\}$ with $S = \{1,2\}$ and $T = \{1,2,3\}$.

   (f) Two functions $f(n)$ and $g(n)$ such that "$f(n)$ is $O(g(n))$" is TRUE and, *at the same time*, "$g(n)$ is $O(f(n))$" is FALSE.

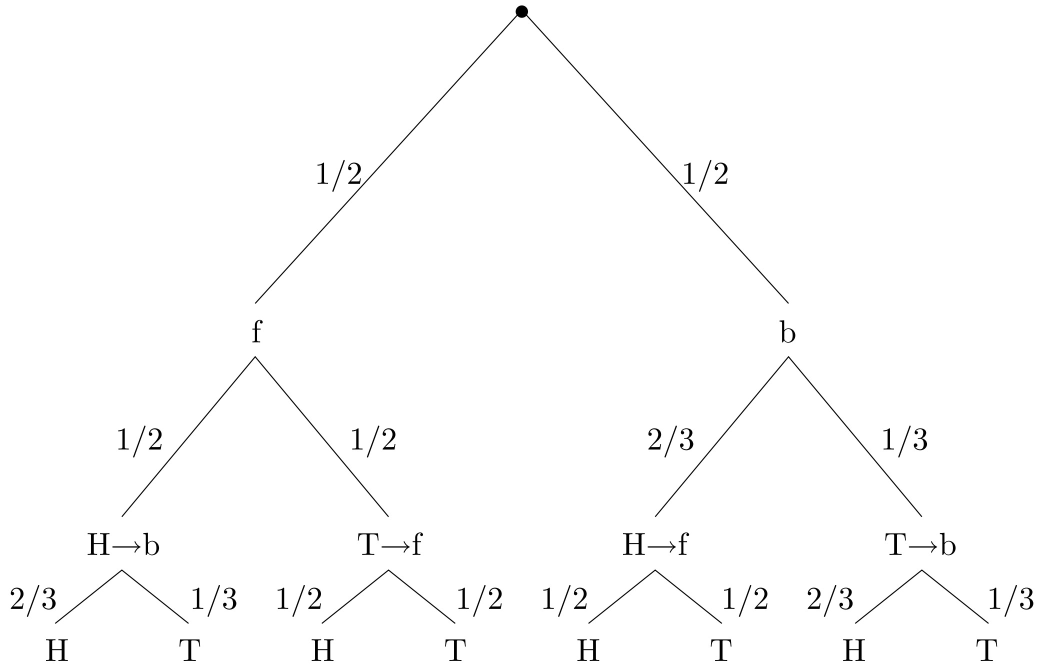
   Ans: There are many possibilities. A simple one is $f(n) = n$ and $g(n) = n^2$.

2. (20 pts) You have a fair coin and a coin that is biased 2/3 heads and 1/3 tails. You carry out the following procedure:

   - Choose a coin at random and toss it.
   - If the result of the toss is heads, switch coins so that you now hold the other coin.
   - Toss whichever coin you now hold.

   Draw the decision tree, label it to indicate probabilities and states, and use the tree to compute the probability that the final toss is heads.

Ans: I'll use f to denote fair coin and b to denote biased coin. Also, H and T are used for heads and tails and something like H→b means a head was tossed so we switched to the biased coin.



This gives a probability for heads of

$$\frac{1}{2} \times \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} \times \frac{2}{3}.$$

You can leave the answer this way, put it over a common denominator:

$$\frac{1}{6} + \frac{1}{8} + \frac{1}{6} + \frac{1}{9} = \frac{41}{72},$$

or write it as the decimal 0.5694.

3. (20 pts) Find a simple function $f(n)$ so that the running time of the following algorithm for multiplying the $n \times n$ matrices A and B is $\Theta(f(n))$.
Notes: You do NOT need to know what matrix multiplication is to do this problem. Remember to show your work!

```
MATRIXMULT(n,A,B,C)
  For i=1,...,n
    For j=1,...,n
      C(i,j)=0
      For k=1,...,n
        C(i,j) = C(i,j) + A(i,k)*B(k,j)
      End for
    End for
  End for
End
```

Ans: A single execution of "`C(i,j) = C(i,j) + A(i,k)*B(k,j)`" takes a constant amount of time. It is done $n \times n \times n = n^3$ times, so the answer is $\Theta(n^3)$.

4. (35 pts) The permutations of $\{1,2,3,4,5,6\}$ are listed in lex order.

   (a) What is the rank of the permutation 4, 3, 6, 1, 5, 2?

   Ans: $3 \times 5! + 2 \times 4! + 3 \times 3! + 0 \times 2! + 1 \times 1!$. You can leave it this way. Multiplied and summed, it becomes $360 + 48 + 18 + 1 = 427$.

   (b) What permutation is next after 4, 3, 6, 1, 5, 2?

   Ans: The answer is 4, 3, 6, 2, 1, 5. You could work out the permutation whose rank is $427 + 1$, however, it's easier to look at the tree for the last time we could take a more rightward edge and then go leftward thereafter.
   • Having chosen 5, there's only one choice remaining, so that's not it.
   • Having chosen 1, we could choose 2 or 5 and we chose the rightmost, so that's not it.
   • Having chosen 6, we could choose 1, 2, or 5 and chose 1. The choice 2 gives us the next choice to the right. Now we must stay left as much as possible in the tree by choosing first 1 and then 5.

5. (30 pts) Let $A = \{1, 2, \ldots, n\}$ and $B = \{1, \ldots, k\}$. We make B^A, the set of functions from A to B, into a probability space by selecting functions uniformly at random. Express the answers to the following in terms of n and k.

   (a) What is the probability that a random function is an injection?

   Ans: Since there are k^n functions and $k(k-1)\cdots(k-n+1)$ injections, the answer is $k(k-1)\cdots(k-n+1)/k^n$.

   (b) What is the probability that a random function is strictly decreasing?

   Ans: Since strictly decreasing functions correspond to subsets, there are $\binom{k}{n}$ of them. Thus the answer is $\binom{k}{n}/k^n$, which is the answer to (a) divided by $k!$.

   This can be done another way. Consider the function written in one-line form. Every injection is obtained exactly one by choosing a strictly decreasing function and then permuting the entries in the one-line form. If SD is the number of strictly decreasing functions, this shows that the answer to (a) is $SD \times k!$.

6. (20 pts) There are four Democrats and six Republicans. How many ways can a committee of five people be chosen if the committee must contain at least two Democrats and at least two Republicans?

Ans: The committee is either

2 Democrats AND 3 Republicans $\left(\text{doable in } \binom{4}{2} \times \binom{6}{3} \text{ ways}\right)$ OR

3 Democrats AND 2 Republicans $\left(\text{doable in } \binom{4}{3} \times \binom{6}{2} \text{ ways}\right)$

Thus we have

$$\binom{4}{2} \times \binom{6}{3} + \binom{4}{3} \times \binom{6}{2} = 6 \times 20 + 4 \times 15 = 180.$$

Any of these forms is okay.

7. (20 pts) Consider the recursion $a_n = a_{n-1} + (2n-1)$ for $n \geq 2$, with initial condition $a_1 = 1$. Guess and then prove by induction a formula for a_n.
Hint: Compute the first few values of a_n so you can make a guess at the general formula.

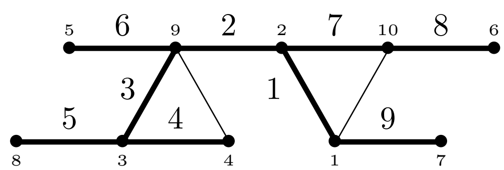
Ans: We compute $a_2 = 1 + 3 = 4$, $a_3 = 4 + 5 = 9$, $a_4 = 9 + 7 = 16$ and so conjecture that $a_n = n^2$. This is true for $n = 1$. Suppose $n > 1$. Using, in order, the recursion, the induction hypothesis, and algebra, we obtain

$$a_n = a_{n-1} + (2n-1) = (n-1)^2 + (2n-1) = n^2.$$

This completes the proof.

8. (20 pts) Copy the following graph in your exam. (You will need to make a large copy so the edge labels you are asked for are clear.)

Starting with vertex 1, construct a lineal (depth-first) spanning tree for the graph. Show the tree by numbering the edges 1, 2, and so on in the order they are added to the tree. (The numbers already on the vertices have nothing to do with the numbers you are to put on the edges.)
IMPORTANT: **Whenever you have a choice of edges, choose the edge that adds a vertex whose number is smallest.**



Ans: I've added the numbers to the edges of the graph. I've also marked the edges of the tree heavier so you can see the tree more easily.