

- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK, but calculators ARE allowed.
- **You must show your work to receive credit.**

1. (60 pts) In each case, give an example or explain why none exists.
 - (a) A tree with exactly nine vertices and exactly nine edges.
 - (b) A permutation f on $\{1, 2, 3, 4\}$ such that $f^{100} \neq f$.
Notes: Remember that $f \neq g$ for functions with the same domain means there is at least one x such that $f(x) \neq g(x)$. Also remember that $f^{100}(x)$ means $f(f(\dots f(x)\dots))$, NOT $(f(x))^{100}$.
 - (c) A simple graph with exactly five vertices that has a cycle containing six edges.
 - (d) A sample space U with a probability function P and two different elements s and t of U such that $P(s) > 1/2$ and also $P(t) > 1/2$.
Note: $P(s)$ means the same thing as $P(\{s\})$.
 - (e) A sample space U with a probability function P and two different subsets S and T of U such that $P(S) > 1/2$ and also $P(T) > 1/2$.
 - (f) Two functions $f(n)$ and $g(n)$ such that “ $f(n)$ is $O(g(n))$ ” is TRUE and, *at the same time*, “ $g(n)$ is $O(f(n))$ ” is FALSE.

2. (20 pts) You have a fair coin and a coin that is biased $2/3$ heads and $1/3$ tails. You carry out the following procedure:
 - Choose a coin at random and toss it.
 - If the result of the toss is heads, switch coins so that you now hold the other coin.
 - Toss whichever coin you now hold.

Draw the decision tree, label it to indicate probabilities and states, and use the tree to compute the probability that the final toss is heads.

3. (20 pts) Find a simple function $f(n)$ so that the running time of the following algorithm for multiplying the $n \times n$ matrices A and B is $\Theta(f(n))$.
Notes: You do NOT need to know what matrix multiplication is to do this problem. Remember to show your work!

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MATRIXMULT(n,A,B,C)
  For i=1,...,n
    For j=1,...,n
      C(i,j)=0
      For k=1,...,n
        C(i,j) = C(i,j) + A(i,k)*B(k,j)
      End for
    End for
  End for
End

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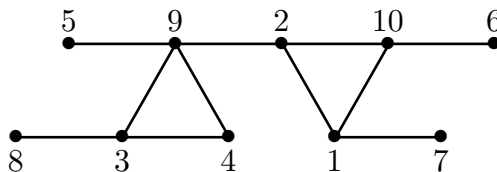
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4. (35 pts) The permutations of $\{1, 2, 3, 4, 5, 6\}$ are listed in lex order.
- What is the rank of the permutation 4, 3, 6, 1, 5, 2?
 - What permutation is next after 4, 3, 6, 1, 5, 2?
5. (30 pts) Let $A = \{1, 2, \dots, n\}$ and $B = \{1, \dots, k\}$. We make B^A , the set of functions from A to B , into a probability space by selecting functions uniformly at random. Express the answers to the following in terms of n and k .
- What is the probability that a random function is an injection?
 - What is the probability that a random function is strictly decreasing?
6. (20 pts) There are four Democrats and six Republicans. How many ways can a committee of five people be chosen if the committee must contain at least two Democrats and at least two Republicans?
7. (20 pts) Consider the recursion $a_n = a_{n-1} + (2n - 1)$ for $n \geq 2$, with initial condition $a_1 = 1$. Guess and then prove by induction a formula for a_n .
Hint: Compute the first few values of a_n so you can make a guess at the general formula.
8. (20 pts) Copy the following graph in your exam. (You will need to make a large copy so the edge labels you are asked for are clear.)

Starting with vertex 1, construct a lineal (depth-first) spanning tree for the graph. Show the tree by numbering the edges 1, 2, and so on in the order they are added to the tree. (The numbers already on the vertices have nothing to do with the numbers you are to put on the edges.)

IMPORTANT: Whenever you have a choice of edges, choose the edge that adds a vertex whose number is smallest.



END OF EXAM