1. We need to verify that $Z(R)$ is a group under addition and closed under multiplication. Since $z_i \in Z(R)$ implies $(z_1 - z_2)r = z_1 r - z_2 r = r z_1 - r z_2 = r(z_1 - z_2)$ and $(z_1 z_2)r = z_1 r z_2 = r z_1 z_2 = r(z_1 z_2)$, this is true. Furthermore, $z_1 z_2 = z_2 z_1$ since $z_1 \in Z(R)$ and so $Z(R)$ is commutative.

2. Over the complex numbers, the polynomial splits as $(x+i)(x-i)(x-\omega)(x-\omega^2)$, where $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$, a third root of unity. None of these roots belong to $\mathbb{R}$.

   (a) Since none of these roots belong to $\mathbb{R}$, one of the roots is i, and all roots belong to $\mathbb{C} = \mathbb{R}(i)$, the splitting field is $\mathbb{C}$.

   (b) We adjoin i to $\mathbb{Q}$, but we still don't get ω, hence we must adjoin more. Among the possible answers are

$$\mathbb{Q}(i, \sqrt{3}) = \mathbb{Q}(i, \sqrt{-3}) = \mathbb{Q}(i, \omega) = \mathbb{Q}(i + \omega).$$

3. If a is a zero of $f(x)$, then $[F(a) : F] = \deg(f)$. Since $[F(a) : F]$ must divide $[E : F]$, $\deg(f)$ must divide n.

4. (a) The cyclic subgroups are formed by pairing one of $i = 0, 1 \in \mathbb{Z}_2$ with one of $j = 0, 1, 2, 3 \in \mathbb{Z}_6$. Then $\langle(0,0)\rangle$ is the trivial one-element group, $|\langle(0,2)\rangle| = 3$, $|\langle(0,1)\rangle| = |\langle(1,1)\rangle| = |\langle(1,2)\rangle| = 6$, and $|\langle(0,3)\rangle| = |\langle(1,0)\rangle| = |\langle(1,3)\rangle| = 2$. We were given the non-cyclic proper subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ of order 4.

   (b) The numbers correspond to subgroup orders. Hence we have

k	1	2	3	4	6	12
#	1	3	1	1	3	1

   and for all other k, none.

5. We use what we know about cyclic groups. Since $\mathrm{GF}(p^k)^*$ is a subgroup of $\mathrm{GF}(p^n)^*$ of order $p^k - 1$, it is generated by α^t where $t = \frac{p^n-1}{p^k-1}$.

6. (a) Write $n = 2^k$ and use induction on k: For $k = 0$, we are done by assumption (let $a = b$). For $k > 0$, $0 = b^b = b^{2^k} = \left(b^{2^{k-1}}\right)^2$ and so $b^{2^{k-1}} = 0$ by assumption.

   (b) Suppose $n \leq 2^k$. Since $b^n = 0$, it follows that $b^{2^k} = 0$. By (a), $b = 0$.

7. This can be done in various ways. Here's one. Since π is a zero of $x^5 - \pi^5$, $[\mathbb{Q}(\pi) : \mathbb{Q}(\pi^5)] < \infty$ and so everything in $\mathbb{Q}(\pi)$ is algebraic over $\mathbb{Q}(\pi^5)$. Since $\pi + \pi^{-1} \in \mathbb{Q}(\pi)$, we are done.

8. (a) Let $\star$ be a ring operation (plus, minus or times). We have

$$\begin{aligned}\phi(x \star y) &= (\phi_s(x \star y), \phi_t(x \star y)) = (\phi_s(x) \star \phi_s(y), \phi_t(x) \star \phi_t(y)) \\ &= (\phi_s(x), \phi_t(x)) \star (\phi_s(y), \phi_t(y)) = \phi(x) \star \phi(y).\end{aligned}$$

(b) Since the zero of $S \oplus T$ is $(0, 0)$. The kernel of ϕ is precisely those x such that $\phi_s(x) = 0$ and $\phi_t(x) = 0$. In other words, precisely those x which are in both $\mathrm{Ker}(\phi_s)$ and $\mathrm{Ker}(\phi_t)$.

(c) To be an isomorphism, the kernel must be $\{0\}$. Let $S = R/I$ and $\phi_s(x) = x+I$ and similarly for T. Now apply (b) and standard properties of ring homomorphisms to conclude that the kernel of the map in (c) is $I \cap J$.