- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK except for both sides of two sheets of notes.
- Calculators are NOT allowed.
- *In a multipart problem, you can do later parts without doing earlier ones.*
- **You must show your work to receive credit.**

1. (10 pts.) If R is a ring, define its center $Z(R)$ to be those elements that commute with all elements of R; that is, $Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}$. Prove that $Z(R)$ is a commutative subring of R.

2. (12 pts.) In the following, we are thinking or $\mathbb{R}$ and $\mathbb{Q}$ as subfields of $\mathbb{C}$, the complex numbers. **Therefore, express your answers as subfields of $\mathbb{C}$.**

   (a) Find the splitting field of $(x^2 + 1)(x^2 + x + 1)$ over $\mathbb{R}$.

   (b) Find the splitting field of $(x^2 + 1)(x^2 + x + 1)$ over $\mathbb{Q}$.

   Remember to explain how you got your answers.

3. (10 pts.) Suppose $E \supset F$ are fields, $[E : F] = n$, $f(x) \in F[x]$ is irreducible over F and E contains a zero of $f(x)$. What are the possible values for the degree of $f(x)$? Why?

4. (15 pts.) Suppose $E \supset \mathbb{Q}$ is the splitting field of a polynomial over $\mathbb{Q}$. Further suppose that $\mathrm{Gal}(E/\mathbb{Q}) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_6$. This group has a total of 8 proper subgroups, all of which are cyclic except for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. ("Proper" means not the whole group and not the trivial one-element group.)

   (a) List all proper subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_6$ and give their orders.

   (b) For all $k > 0$, how many subfields F of E are there such that $[E : F] = k$? (Remember E and $\mathbb{Q}$!)

5. (10 pts.) Recall that $\mathrm{GF}(p^n)^*$ under multiplication is a cyclic group of order $p^n - 1$. Let α be a generator for the group. Suppose $\mathrm{GF}(p^k)$ is a subfield of $\mathrm{GF}(p^n)$. Prove that $\mathrm{GF}(p^k)^* = \langle \alpha^t \rangle$ for some t and determine the smallest such $t > 0$.

6. (15 pts.) Suppose R is a ring such that, if $a \in R$ and $a^2 = 0$, then $a = 0$. Suppose $b \in R$ and $b^n = 0$.

   (a) Prove that, if n is a power of two, then $b = 0$.

   (b) Prove that $b = 0$ regardless of whether or not n is a power of 2.

7. (10 pts.) Let $\pi \in \mathbb{R}$ have its usual meaning. Prove that $\pi + \pi^{-1}$ is algebraic over $\mathbb{Q}(\pi^5)$.

8. (18 pts.) Suppose R, S and T are rings and that $\phi_s : R \to S$ and $\phi_t : R \to T$ are ring homomorphisms. Define $\phi : R \to S \oplus T$ by $\phi(x) = (\phi_s(x), \phi_t(x))$.

   (a) Prove that ϕ is a homomorphism.

   (b) Recall that the kernel of a homomorphism is the set of elements that are mapped to zero. Prove that $\mathrm{Ker}(\phi) = \mathrm{Ker}(\phi_s) \cap \mathrm{Ker}(\phi_t)$.

   (c) Suppose that I and J are ideals of R and that $R \approx (R/I) \oplus (R/J)$ under that natural mapping $x \to (x + I, x + J)$. Prove that $I \cap J = \{0\}$.

END OF EXAM