- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK except for both sides of two sheets of notes.
- Calculators are NOT allowed.
- *In a multipart problem, you can do later parts without doing earlier ones.*
- **You must show your work to receive credit.**

1. (20 pts.) There is a polynomial $f(x) \in \mathbb{Q}[x]$ such that its splitting field E over $\mathbb{Q}$ has $\mathrm{Gal}(E/\mathbb{Q}) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Remember to show your work—don't just give numbers.

   (a) What is $[E:\mathbb{Q}]$ and why?

   (b) How many fields $K \subset E$ are there such that $[E:K]=2$?

   (c) How many fields $K \subset E$ are there such that $[K:\mathbb{Q}]=2$?

   (d) How many fields $K \subset E$ are there such that $[E:K]=3$?

2. (15 pts.) If A and B are ideals in a commutative ring R, then AB is the set of all sums $a_1b_1 + a_2b_2 + \cdots + a_nb_n$ where $a_i \in A$, $b_i \in B$ and $n > 0$. It can be shown that AB is an ideal of R.

   (a) Prove that $AB \subseteq A \cap B$.

   (b) Find an example of A and B with $R = \mathbb{Z}$ such that $AB \neq A \cap B$.

   (c) When $R = \mathbb{Z}$, find a simple necessary and sufficient condition to have $AB = A \cap B$. To receive full credit, prove that your condition is necessary and sufficient.

3. (12 pts.) If A and B are ideals in a ring R, then $A + B = \{a + b \mid a \in A,\ b \in B\}$.

   (a) Prove that the ideals are closed under this addition, that this addition is associative and that the ideal $\{0\}$ is an additive identity.

   (b) Do the ideals form an abelian group under addition? Justify your answer.

4. (5 pts.) Let D be an integral domain with unity 1. Prove that a subring R of D is an integral domain if and only if $1 \in R$.

5. (5 pts.) Show that the previous problem is false if "integral domain" is replaced by "field." That is, find a field F with unity 1 and a subring R of F such that $1 \in R$ but R is not a field.

THERE ARE MORE PROBLEMS

6. (5 pts.) Find the splitting field of $x^6 + x^2 + 1$ over $\mathbb{R}$. Justify your answer.

7. (12 pts.) Suppose G and H are subfields of a field E.

   (a) Prove that $G \cap H$ is a subfield of E.

   (b) If E is a finite field of characteristic p, show how to compute $|G \cap H|$ given $|G|$ and $|H|$. You need not justify your answer.

8. (6 pts.) Show that when $x^{128} - x$ is factored into irreducible factors over $\mathbb{Z}_2[x]$ there will be two linear factors and $\frac{128-2}{7} = 18$ factors of degree 7.
   Remark: $128 = 2^7$.

END OF EXAM