1. $1-2x$ since $(1+2x)(1-2x)=1-4x^2=1$ in $\mathbb{Z}_4[x]$.

2. Five, since by looking at $\begin{matrix}0\,1\,0\,0\,0\,1\,0\,1\\1\,1\,1\,1\,0\,0\,1\,1\end{matrix}$ we see 5 differences. Calling the desired word x, we have $5=d(u,v)\le d(u,x)+d(x,v)\le 2+2=4$, a contradiction. Hence there is no such word.

3. The three zeros of x^3-2 are $a=2^{1/3}$, $b=a\omega$ and $c=a\omega^2$ where $\omega=e^{2\pi i/3}$. There are many possibilities. Here are three.
   - Adjoin any two of them to $\mathbb{Q}$.
   - Adjoin one of them and ω to $\mathbb{Q}$.
   - Adjoin $a+\omega$ to $\mathbb{Q}$.

   The first two are obviously splitting fields. The third is not so clear—but you weren't asked to prove the result.

4. (a) Since $\left(\sqrt{2}+\sqrt{5}\right)^2=7+2\sqrt{10}$, it follows that $\sqrt{10}\in E$ and so $F\subseteq E$.

   (b) Probably the simplest basis to find is 1, $\sqrt{2}+\sqrt{5}$; however, there are others such as 1, $\sqrt{2}$.

   (c) One possibility is 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{10}$.

5. (a) $|F|$ must be a power of p and all powers p^k with k a positive integer are possible.

   (b) If $[K:F]=n$, then K is a vector space over F of dimension n and so $|K|=|F|^n$.

   (c) Suppose $|F|=p^k$ and $|K|=|F|^n=p^{kn}$. By the uniqueness of finite fields (Theorem 22.1), we have $F=\mathrm{GF}(p^k)$ and $K=\mathrm{GF}(p^{kn})$. By the subfield theorem (Theorem 22.3), F is a subfield of K.