1. (4 pts.) Find a multiplicative inverse of $1 + 2x$ in $\mathbb{Z}_4[x]$. You must do the calculations that show your answer is a multiplicative inverse.

2. (4 pts.) Compute the Hamming distance between the two words $u = 01000101$ and $v = 11110011$. Also, either (a) find one word that is simultaneously within Hamming distance two of both $u$ and $v$ or (b) explain why there is no such word.

3. (6 pts.) Find the splitting field of $x^3 - 2$ over $\mathbb{Q}$. You should use real and/or complex numbers in your description of the field. For example, give the splitting field of $x^2 - 3$ over $\mathbb{Q}$ as “$\mathbb{Q}(\sqrt{3})$,” NOT as “$\mathbb{Q}[x]/\langle x^2 - 3 \rangle$” and NOT as “$\mathbb{Q}(a)$ where $a$ is a zero of $x^2 - 3$.”

4. (18 pts.) Let $E = \mathbb{Q}(\sqrt{2} + \sqrt{5})$ and $F = \mathbb{Q}(\sqrt{10})$.
   (a) Prove that $F$ is a subfield of $E$.
   (b) Find a basis for $E$ as a vector space over $F$. You need not prove that it is a basis.
   (c) Find a basis for $E$ as a vector space over $\mathbb{Q}$. You need not prove that it is a basis.

5. (18 pts.) Suppose $F$ and $K$ are fields and that $F$ is a finite field of characteristic $p$.
   (a) Describe explicitly all the values that $|F|$ can have. For example, DO NOT say “the size of any field with characteristic $p$. If it were correct (which it is NOT), you could say something like “$p$ and $p^2 - 1$.”
   (b) Prove: If $K$ is a finite extension of $F$, then $|K| = |F|^n$ for some integer $n$.
   (b) Prove: If $|K| = |F|^n$ for some integer $n$, then $K$ is a finite extension of $F$.

END OF EXAM