1. (a) Since $x^2 = -1$ in $R$, there are just four elements: $A$, $1 + A$, $x + A$ and $x + 1 + A$, where $A = \langle x^2 + 1 \rangle >$.

   (b) We can exhibit zero divisors: $(x + 1 + A)^2 = x^2 + 2x + 1 + A = x^2 + 1 + A = A$.

2. Since multiplication is componentwise $(r, s)$ is a zero divisor if and only if $r$ is a zero divisor of $\mathbb{Z}_3$ or $s$ is a zero divisor of $\mathbb{Z}_4$ and, in addition, $(r, s) \neq (0, 0)$ Hence the zero divisors are $(0, s)$, $(r, 0)$ and $(r', 2)$, where
   - $s$ is any nonzero element of $\mathbb{Z}_4$,
   - $r$ is any nonzero element of $\mathbb{Z}_3$ and
   - $r'$ is any element of $\mathbb{Z}_3$.

   That's enough for full credit. If you want to count them, there are $3 + 2 + 3 - 1 = 7$, where the $-1$ arises because $(0, 2)$ is counted twice, once with $s = 2$ and once with $r' = 0$.

3. (a) Regardless of the value of $k$, $\varphi_k(a + b) = k(a + b) = ka + kb = \varphi_k(a) + \varphi_k(b)$. Thus we only need to check multiplication. If $k^2 = k$, $\varphi_k(ab) = kab = kakb = \varphi_k(a)\varphi_k(b)$. Thus $k^2 = k$ is sufficient. To see that it is necessary, use the hint and the fact that $1 \cdot 1 = 1$: We must have $\varphi_k(1)^2 = \varphi_k(1)$. In other words, $k^2 = k$.

   (b) Just compute $\varphi_4(x)$ for $x \in \mathbb{Z}_6$ and list those $x$ that give 0: $\{0, 3\}$.

4. (a) Suppose $b$ is a nonzero nilpotent element. Let $n$ be the smallest power of $b$ that is zero. Since $b \neq 0$, $n > 1$. Thus $0 = b^n = b^{n-1}b = bb^{n-1}$. Since $n$ is as small as possible, $b^{n-1} \neq 0$ and so $b$ is a zero divisor.

   (b) Call the set of nilpotent elements $N$. Suppose $a, b \in N$ and $r \in R$. We must show that $a - b$, $ar$ and $ra$ all lie in $N$. The first follows from the remark preceding this part of the problem. If $a^n = 0$, commutativity gives us $(ra)^n = (ar)^n = a^n r^n = 0r^n = 0$ and so we are done.

   (c) Using the hint and letting $c = a + b$, note that $a^2 = b^2 = 0$ and $c^2 = I$, the $2 \times 2$ identity matrix. Thus $a$ and $b$ are nilpotent but $c$ is not since $c^{2n} = I$ and $c^{2n+1} = c$.