- Please put your name and ID number on your blue book.
- The exam is CLOSED BOOK except for one page of notes.
- Calculators are NOT allowed.
- *In a multipart problem, you can do later parts without doing earlier ones.*
- **You must show your work to receive credit.**

1. (10 pts.) Consider the ring $R = \mathbb{Z}_2[x]/\langle x^2+1\rangle$.
   (a) How many elements does R contain?
   (b) Show that R is *not* an integral domain.

2. (5 pts.) Find all the zero divisors of $\mathbb{Z}_3 \oplus \mathbb{Z}_4$. (Recall that $R \oplus S$ is the set of pairs (r, s) with addition and multiplication done componentwise.)

3. (10 pts.) For $k \in \mathbb{Z}_n$, define $\varphi_k : \mathbb{Z}_n \to \mathbb{Z}_n$ by $\varphi_k(x) = kx$.
   (a) Prove that φ_k is a ring homomorphism if and only if $k^2 = k$ in $\mathbb{Z}_n$.
   Hint: $\varphi_k(1 \cdot 1) = \varphi_k(1)\varphi_k(1)$.
   (b) What is the kernel of $\varphi_4 : \mathbb{Z}_6 \to \mathbb{Z}_6$?

4. (15 pts.) Let R be a ring. An element of R is called *nilpotent* if some power of it is zero. (In particular, 0 is nilpotent.) A nonzero $a \in R$ is called a *zero divisor* if there are nonzero b and c such that $ab = 0$ and $ca = 0$. (The text only defined zero divisors for commutative rings.)
   (a) Prove that every nonzero nilpotent element of a ring is a zero divisor.

   Suppose R is a commutative ring and suppose $a, b \in R$ satisfy $a^n = 0$ and $b^k = 0$. It can be shown that $(a-b)^{n+k} = 0$. (You do NOT need to do this.)

   (b) Prove: If R is a commutative ring, then the nilpotent elements of R are an ideal.
   (c) It was shown in class that the only ideals in the ring $M_2(\mathbb{R})$ of 2×2 real matrices are the trivial ones $\{0\}$ and $M_2(\mathbb{R})$. Use this to show that "commutative" is necessary in (b).
   Hint: Look at $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

END OF EXAM