1. True: (a) (d) (f) False: (b) (c) (e)

2. (a) Suppose $x, y \in Z(G)$. Then $xg = gx$ and $yg = gy$ for all $g \in G$ by the definition of the center. Hence $gy^{-1} = y^{-1}g$ and so

$$(xy^{-1})g = xy^{-1}g = xgy^{-1} = gxy^{-1} = g(xy^{-1}).$$

By the definition of $Z(G)$, $xy^{-1} \in Z(G)$.

(b) Since $gz = zg$ for all $z \in Z(G)$ and all $g \in G$, it follows that $gZ(G) = Z(G)g$ for all $g \in G$.

3. Using the convention that 1-cycles are omitted, here are order: example.
1: e 2: (12)(34) 3: (123) 4: (12)(3456) 5: (12345) 6: (12)(34)(567)
7: (1234567)

4. (a) If $G$ is Abelian, $\phi(gh) = (gh)^2 = g^2h^2 = \phi(g)\phi(h)$.
If $\phi$ is a homomorphism, $\phi(g)\phi(h) = \phi(gh)$ and so $g^2h^2 = (gh)^2 = ghgh$. Multiplying by $g^{-1}$ on the left and $h^{-1}$ on the right, we get $gh = hg$.

(b) Let $g \in G$ have order $2n - 1$. We have $\phi(g^n) = g^{2n} = g^{2n-1}g = g$. (We did not actually use Abelian.)

5. Since $K$ is non-cyclic, its order cannot be a prime. Since $|N| = |G|/|K|$, the possible values of $|N|$ are $5^3 \times 7$ divided by numbers which are not prime. Thus we have
$5$ $5^2$ $7$ $5 \times 7$.

6. (a) The identity is 0 and the polynomials with zero derivative are the constants. Hence $\operatorname{Ker}\phi$ is the constants. Given any polynomial $p(x)$, let $q(x) = \int p(x)\,dx$ with any choice for the constant of integration. Then $\phi(q(x)) = p(x)$ and so $\phi$ is onto.

(b) Since the only polynomial mapped to 0 is 0, $\operatorname{Ker}\psi = \{0\}$. It is not onto because polynomials with odd degree terms are not in the image.

This example is interesting because, when $G$ is finite, a larger kernel means a smaller image. In this case, the reverse happened: $\phi$ has both a larger kernel and a larger image than $\psi$.

7. $\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{5^2}$ $\mathbb{Z}_{2^3} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$ $\mathbb{Z}_{2^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{5^2}$ $\mathbb{Z}_{2^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{5^2}$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$.

8. Suppose $a, b \in H$ and $g \in \mathrm{GL}(n, \mathbb{R})$. Since $\det(a)$ and $\det(b)$ are in $K$, so is $\det(ab^{-1}) = \det(a)/\det(b)$. Thus $H$ is a subgroup. Since $\det(gag^{-1}) = \det(a) \in K$, it follows that $gag^{-1} \in H$. Thus $gHg^{-1} \subseteq H$ and so $H$ is normal.

9. (a) $U = \{e^{i\theta}\}$, where $\theta$ is interpreted modulo $2\pi$; that is, $\theta$ is in the *set* $\mathbb{R}/(2\pi\mathbb{Z})$. Multiplying two elements in $U$ gives $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$. Hence the group operation in $U$ (multiplication) maps to the group operation in $\mathbb{R}/(2\pi\mathbb{Z})$ (addition).

(b) Every non-zero complex number has a unique polar coordinate representation $re^{i\theta}$ where $r$ is a positive real and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$. Multiplying two such numbers gives addition of the angles (which lie in $\mathbb{R}/(2\pi\mathbb{Z})$) and multiplication of the moduli (which lie in $\mathbb{R}^+$).