- Please put your name and ID number on your blue book.
- CLOSED BOOK, but BOTH SIDES of two pages of notes are allowed.
- Calculators are NOT allowed.
- *In a multipart problem, you can do later parts without doing earlier ones.*
- **You must show your work to receive credit.**

1. (12 pts.) Which are TRUE and which are FALSE? Do NOT give reasons.
   (a) If $\varphi : G \to H$ is a group homomorphism and $K \lhd H$, then $\varphi^{-1}(K) \lhd G$.
   (b) If $\varphi : G \to H$ is a group homomorphism and H is a cyclic group, then G is a cyclic group.
   (c) If $H \lhd G$ and $K \lhd H$, then $K \lhd G$.
   (d) Suppose p and q are primes. Then $\mathbb{Z}_{pq} \approx \mathbb{Z}_p \oplus \mathbb{Z}_q$ if and only if $p \neq q$.
   (e) If G is a group of order n and k divides n, then G has an element of order k.
   (f) If $g \in G$, then the order of g divides the order of G.

2. (12 pts.) Let G be a group and $Z(G)$ its center; that is, $Z(G) = \{g \in G \mid xg = gx \text{ for all } x \in G\}$.
   (a) Prove that $Z(G)$ is a subgroup of G.
   (b) Prove that $Z(G)$ is a normal subgroup of G. (Of course the "subgroup" part follows from (a).)

3. (12 pts.) What are the possible orders for the elements of the alternating group A_7? For each possible order, write down an element of that order.

4. (12 pts.) Let G be a group and let $\phi(g) = g^2$ for all $g \in G$.
   (a) Prove that the mapping $\phi : G \to G$ is a homomorphism if and only if G is Abelian.
   (b) If G is finite, Abelian and of odd order, prove that ϕ is a surjection (onto).

5. (12 pts.) Suppose G is a group of order $5^3 \times 7$ and $K = G/N$ is a *non-cyclic* factor group of G. What are the possible values for $|N|$?
   Note: We consider the one-element group to be cyclic.

6. (16 pts.) Let G be the group of all polynomials in x with real coefficients with the operation being addition of polynomials. Let $\phi(p(x)) = p'(x)$ (the derivative) and $\psi(p(x)) = p(x^2)$ for all $p \in G$.
   It is easily proved that ϕ and ψ are homomorphisms from G to G and you need not do so.
   (a) Compute Ker ϕ and prove that $\phi(G) = G$.
   (b) Compute Ker ψ and prove that $\psi(G) \neq G$.

7. (6 pts.) List up to isomorphism all Abelian groups of order $2^3 \times 5^2$. Each group in your list should be written as an external direct product and should have as many terms as possible. For example, $\mathbb{Z}_{12} \approx \mathbb{Z}_4 \oplus \mathbb{Z}_3$. Since $\mathbb{Z}_{12}$ has one term and $\mathbb{Z}_4 \oplus \mathbb{Z}_3$ has two, you would include $\mathbb{Z}_4 \oplus \mathbb{Z}_3$ in a list of Abelian groups of order 12 rather than $\mathbb{Z}_{12}$. You will lose points if your list contains duplicates (groups which are isomorphic).

8. (6 pts.) Let K be a subgroup of the multiplicative group $\mathbb{R}^*$. Let H be the set of matrices in $\mathrm{GL}(n, \mathbb{R})$ that have determinant in K.
   Prove that H is a normal subgroup of $\mathrm{GL}(n, \mathbb{R})$.
   Recall: $\mathrm{GL}(n, \mathbb{R})$ is the nonsingular $n \times n$ real matrices and $\det(AB) = (\det A)(\det B)$.

9. (12 pts.) Let U be the complex numbers of absolute value 1 viewed as a subgroup of the multiplicative group $\mathbb{C}^*$. Let $\mathbb{R}^+$ be the positive real numbers under multiplication.
   (a) Show that $U \approx \mathbb{R}/(2\pi\mathbb{Z})$. (Recall that $\mathbb{R}$ are the real numbers under addition.)
   (b) Show that $\mathbb{C}^* \approx \big(\mathbb{R}/(2\pi\mathbb{Z})\big) \oplus \mathbb{R}^+$ and interpret this in terms of polar coordinates.

END OF EXAM