1. (a) False (b) True (c) False (d) False (e) True
   (f) False (g) True (h) True (i) False.

2. (a) If $a, b \in Z(G)$ and $x \in G$, then $xab = axb = xab$. Also $ax = xa$ implies $a^{-1}xa^{-1} = a^{-1}xa^{-1}$ and so $xa^{-1} = a^{-1}x$.
   (b) Suppose $x \in G$. Then $xZ(G) = \{xg \mid g \in Z(G)\} = \{gx \mid g \in Z(G)\} = Z(G)x$.

3. When multiplying permutations, odd times odd is even, odd times even is odd and even times even is even. Hence the product of the permutations $\alpha_1, \ldots, \alpha_k$ will be even if and only if the number of $\alpha_1, \ldots, \alpha_k$ which are odd is even. Since a cycle is odd if and only it has even length, this tells us that a product of cycles is even if and only if the number of even length cycles in the product is even.

4. The simplest group is D_3. Let a and b be two different flips. All flips have order 2. Then $ab \neq e$ and ab is a rotation. Hence it has order 3.

   Another example is S_n with $n \geq 3$. Then $a = (12)$, $b = (13)$ and $ab = (132)$. In a sense this is the same example since we can view S_3 as a subgroup of S_n and $S_3 \approx D_3$.

5. Since G is abelian $(gh)^k = g^k h^k$ and so $\varphi(gh) = (gh)^k = g^k h^k = \varphi(g)\varphi(h)$.

   Challenge: One way is to use the structure theorem for finite abelian groups. Then $\varphi(h_1, \ldots, h_n) = (h_1^k, \ldots, h_n^k)$. This will be an isomorphism if and only if $h_i \mapsto h_i^k$ is a bijection for each i. By our study of cyclic groups h^k is a generator of $\langle h \rangle$ if and only if $\gcd(k, |h|) = 1$. In other words, $h_i \mapsto h_i^k$ is a bijection if and only if $\gcd(k, |h_i|) = 1$. This holds for all i if and only if $|G|$ has no factors in common with k; that is $\gcd(k, |G|) = 1$.

6. The possible orders are 1,2,3,4,5,6,7,10,12. The identity has order 1. For $k = 2, 3, 4, 5, 6, 7$, a k-cycle has order k. The product of a disjoint 2-cycle and 5-cycle has order 10. The product of a disjoint 3-cycle and 4-cycle has order 12.

7. (a) $\mathbb{Z}_{16} \oplus \mathbb{Z}_{25} \approx \mathbb{Z}_{400}$
   (b) $\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{80} \oplus \mathbb{Z}_5$
   (c) $\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25} \approx \mathbb{Z}_{200} \oplus \mathbb{Z}_2 \approx \mathbb{Z}_8 \oplus \mathbb{Z}_{50}$
   (d) $\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \cdots$
   (e) $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{25} \approx \cdots$
   (f) $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \cdots$
   (g) $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25} \approx \cdots$
   (h) $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \cdots$
   (i) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25} \approx \cdots$
   (j) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \cdots$

8. Let H be the set of matrices in G of determinant ± 1. If $A, B \in H$, then $AB^{-1} \in H$ since $\det(AB^{-1} = \det A / \det B = \pm 1$. Hence H is a subgroup of G. Suppose $C \in G$. Since $\det CAC^{-1} = \det C \det A / \det C = \det A = \pm 1$, $CAC^{-1} \in H$. Thus H is normal.

9. This was discussed in class, but even if you were not in class, you should have enough knowledge to do it.

   (a) Since $\mathbb{R}^+$ is contained in $\mathbb{C}^*$ and is closed under multiplication and taking inverses, it is a subgroup. Every subgroup of an abelian group is normal.

   (b) $a\mathbb{R}^+$ consists of all points on the half-line starting at the origin (but not including the origin) and passing through a. In other words, it is all points in $\mathbb{C}^*$ having the same arguments as a.

   (c) If a point in $\mathbb{C}^*$ has polar coordinates (r, θ), then $r \in \mathbb{R}^+$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Furthermore, this correspondence between $\mathbb{C}^*$ and pairs of elements (r, θ) is a bijection.

   We need to show that the group operations behave correctly. The product two complex numbers with polar coordinates (r_1, θ_1) and (r_2, θ_2) has polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$, where $\theta_1 + \theta_2$ is taken modulo 2π. Since we combined r_1 and r_2 (resp. θ_1 and θ_2) using the operation of $\mathbb{R}^+$ (resp. $\mathbb{R}/2\pi\mathbb{Z}$), we are done.