1. (a) False (b) True (c) False (d) False (e) True
(f) False (g) True (h) True (i) False.

2. (a) If \( a, b \in \mathbb{Z}(G) \) and \( x \in G \), then \( xab = axb = xab \). Also \( ax = xa \) implies \( a^{-1}xa^{-1} = a^{-1}xa^{-1} \) and so \( xa^{-1} = a^{-1}x \).
(b) Suppose \( x \in G \). Then \( x\mathbb{Z}(G) = \{ xg \mid g \in \mathbb{Z}(G) \} = \{ gx \mid g \in \mathbb{Z}(G) \} = \mathbb{Z}(G)x \).

3. When multiplying permutations, odd times odd is even, odd times even is odd and even times even is even. Hence the product of the permutations \( \alpha_1, \ldots, \alpha_k \) will be even if and only if the number of \( \alpha_1, \ldots, \alpha_k \) which are odd is even. Since a cycle is odd if and only if it has even length, this tells us that a product of cycles is even if and only if the number of even length cycles in the product is even.

4. The simplest group is \( D_3 \). Let \( a \) and \( b \) be two different flips. All flips have order 2. Then \( ab \neq e \) and \( ab \) is a rotation. Hence it has order 3.

Another example is \( S_n \) with \( n \geq 3 \). Then \( a = (12), b = (13) \) and \( ab = (132) \). In a sense this is the same example since we can view \( S_3 \) as a subgroup of \( S_n \) and \( S_3 \cong D_3 \).

5. Since \( G \) is abelian \( (gh)^k = g^kh^k \) and so \( \varphi(gh) = (gh)^k = g^kh^k = \varphi(g)\varphi(h) \).

**Challenge:** One way is to use the structure theorem for finite abelian groups. Then \( \varphi(h_1, \ldots, h_n) = (h_1^k, \ldots, h_n^k) \). This will be an isomorphism if and only if \( h_i \mapsto h_i^k \) is a bijection for each \( i \). By our study of cyclic groups \( h^k \) is a generator of \( \langle h \rangle \) if and only if \( \gcd(k, |h|) = 1 \). In other words, \( h_i \mapsto h_i^k \) is a bijection if and only if \( \gcd(k, |h_i|) = 1 \). This holds for all \( i \) if and only if \( |G| \) has no factors in common with \( k \); that is \( \gcd(k, |G|) = 1 \).

6. The possible orders are \( 1, 2, 3, 4, 5, 6, 7, 10, 12 \). The identity has order 1. For \( k = 2, 3, 4, 5, 6, 7 \), a \( k \)-cycle has order \( k \). The product of a disjoint 2-cycle and 5-cycle has order 10. The product of a disjoint 3-cycle and 4-cycle has order 12.

7. (a) \( \mathbb{Z}_{16} \oplus \mathbb{Z}_{25} \approx \mathbb{Z}_{400} \)
(b) \( \mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{80} \oplus \mathbb{Z}_5 \)
(c) \( \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25} \approx \mathbb{Z}_{200} \oplus \mathbb{Z}_2 \approx \mathbb{Z}_8 \oplus \mathbb{Z}_{50} \)
(d) \( \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \cdots \)
(e) \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{25} \approx \cdots \)
(f) \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \cdots \)
(g) \( \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25} \approx \cdots \)
(h) \( \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \cdots \)
(i) \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25} \approx \cdots \)
(j) \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \approx \cdots \)

8. Let \( H \) be the set of matrices in \( G \) of determinant \( \pm 1 \). If \( A, B \in H \), then \( AB^{-1} \in H \) since \( \det(AB^{-1}) = \det A / \det B = \pm 1 \). Hence \( H \) is a subgroup of \( G \). Suppose \( C \in G \). Since \( \det CAC^{-1} = \det C \det A / \det C = \det A = \pm 1 \), \( CAC^{-1} \in H \). Thus \( H \) is normal.
9. This was discussed in class, but even if you were not in class, you should have enough knowledge to do it.

(a) Since \( \mathbb{R}^+ \) is contained in \( \mathbb{C}^* \) and is closed under multiplication and taking inverses, it is a subgroup. Every subgroup of an abelian group is normal.

(b) \( a\mathbb{R}^+ \) consists of all points on the half-line starting at the origin (but not including the origin) and passing through \( a \). In other words, it is all points in \( \mathbb{C}^* \) having the same arguments as \( a \).

(c) If a point in \( \mathbb{C}^* \) has polar coordinates \((r, \theta)\), then \( r \in \mathbb{R}^+ \) and \( \theta \in \mathbb{R}/2\pi\mathbb{Z} \). Furthermore, this correspondence between \( \mathbb{C}^* \) and pairs of elements \((r, \theta)\) is a bijection.

We need to show that the group operations behave correctly. The product of two complex numbers with polar coordinates \((r_1, \theta_1)\) and \((r_2, \theta_2)\) has polar coordinates \((r_1r_2, \theta_1 + \theta_2)\), where \( \theta_1 + \theta_2 \) is taken modulo \( 2\pi \). Since we combined \( r_1 \) and \( r_2 \) (resp. \( \theta_1 \) and \( \theta_2 \)) using the operation of \( \mathbb{R}^+ \) (resp. \( \mathbb{R}/2\pi\mathbb{Z} \)), we are done.