- Please put your name and ID number on your blue book.
- CLOSED BOOK, but BOTH SIDES of two pages of notes are allowed.
- Calculators are NOT allowed.
- *In a multipart problem, you can do later parts without doing earlier ones.*
- **You must show your work to receive credit.**

1. (27 pts.) Which are TRUE and which are FALSE? Do NOT give reasons.
   (a) The nonzero reals under multiplication are a subgroup of the complex numbers under addition.
   (b) If $\varphi : G \to H$ is a group homomorphism and $K \triangleleft H$, then $\varphi^{-1}(K) \triangleleft G$.
   (c) If $\varphi : G \to H$ is a group homomorphism and H is a cyclic group, then G is a cyclic group.
   (d) If S_n is the symmetric group on $\{1, 2, \ldots, n\}$ and $\alpha \in S_n$, then $|\alpha| \leq n$.
   (e) Suppose p and q are primes. Then $\mathbb{Z}_{pq} \approx \mathbb{Z}_p \oplus \mathbb{Z}_q$ if and only if $p \neq q$.
   (f) If G is a group of order n and k divides n, then G has an element of order k.
   (g) If G is a cyclic group of order n and k divides n, then G has a normal subgroup of order k.
   (h) If $g \in G$, then the order of g divides the order of G.
   (i) If a permutation can be written as a product of 2-cycles, then either (1) it is the identity or (2) its order is even.

2. (12 pts.) Let G be a group and $Z(G)$ its center; that is, $Z(G) = \{g \in G \mid xg = gx \text{ for all } x \in G\}$.
   (a) Prove that $Z(G)$ is a subgroup of G.
   (b) Prove that $Z(G)$ is a normal subgroup of G. (Of course the "subgroup" part follows from (a).)

3. (6 pts.) We are looking at permutations. It is known (p.102) that a cycle is odd if and only if it has even length. Let α be a permutation written as a product of cycles. Prove that α is even if and only if the number of even length cycles in the product is even.

4. (6 pts.) Find a group G and elements a and b of G such that $|a| = |b| = 2$ and $|ab| = 3$.

THERE ARE MORE PROBLEMS

5. (7 pts.) Let G be an abelian group and let k be a positive integer. Define $\varphi : G \to G$ by $\varphi(g) = g^k$. Prove that φ is a homomorphism.

   Challenge: (**NOT part of the exam.**) Can you prove this: When $|G|$ is finite, φ is an isomorphism if and only if $\gcd(k, |G|) = 1$.

6. (10 pts.) What are the possible orders for the elements of the symmetric group S_7? For each possible order, write down an element of S_7 of that order.

7. (10 pts.) List up to isomorphism all abelian groups of order $400 = 2^4 5^2$. Do NOT include duplicates in your list; for example if you were asked for order 6 and listed the isomorphic groups $\mathbb{Z}_6$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_3$, you would have duplicates and would lose points.

8. (7 pts.) Prove that the set of matrices of determinant ± 1 in $G = \mathrm{GL}(n, \mathbb{R})$ is a normal subgroup of G.

   Recall: $\mathrm{GL}(n, \mathbb{R})$ is the nonsingular $n \times n$ real matrices and $\det(AB) = (\det A)(\det B)$.

9. (15 pts.) Let $\mathbb{C}^*$ be the nonzero complex numbers with multiplication the operation. Let $\mathbb{R}^+$ be the strictly positive reals with multiplication the operation. Let $\mathbb{R}$ be the real number with addition the operation.

   (a) Prove that $\mathbb{R}^+ \triangleleft \mathbb{C}^*$; that is, $\mathbb{R}^+$ is a normal subgroup of $\mathbb{C}^*$.

   (b) Suppose $a \in \mathbb{C}^*$. Think of $\mathbb{C}^*$ in the usual geometrical way as points in the plane ($x + yi$ corresponds to (x, y)), except for the origin, which corresponds to $0 \notin \mathbb{C}^*$. Describe $a\mathbb{R}^+$ geometrically.

   (c) Prove that $\mathbb{C}^* \approx \mathbb{R}^+ \oplus (\mathbb{R}/2\pi\mathbb{Z})$.
   Hint: You may want to think in terms of polar coordinates and/or use the fact that complex numbers can be written in the form $re^{i\theta}$.

Math 103B meets in Solis 110, not York 4080 A as originally scheduled.

END OF EXAM