Solutions for Basic Counting and Listing

CL-1.1 This is a simple application of the Rules of Sum and Product.

(a) Choose a discrete math text OR a data structures text, etc. This gives $5 + 2 + 6 + 3 = 16$.

(b) Choose a discrete math text AND a data structures text, etc. This gives $5 \times 2 \times 6 \times 3 = 180$.

CL-1.2 We can form n digit numbers by choosing the leftmost digit AND choosing the next digit AND $\cdots$ AND choosing the rightmost digit. The first choice can be made in 9 ways since a leading zero is not allowed. The remaining $n-1$ choices can each be made in 10 ways. By the Rule of Product we have $9 \times 10^{n-1}$. To count numbers with at most n digits, we could sum up $9 \times 10^{k-1}$ for $1 \le k \le n$. The sum can be evaluated since it is a geometric series. This does not include the number 0. Whether we add 1 to include it depends on our interpretation of the problem's requirement that there be no leading zeroes. There is an easier way. We can pad out a number with less than n digits by adding leading zeroes. The original number can be recovered from any such n digit number by stripping off the leading zeroes. Thus we see by the Rule of Product that there are 10^n numbers with at most n digits. If we wish to rule out 0 (which pads out to a string of n zeroes), we must subtract 1.

CL-1.3 For each element of S you must make one of two choices: "x is/isn't in the subset." To visualize the process, list the elements of the set in any order: $a_1, a_2, \ldots, a_{|S|}$. We can construct a subset by

including a_1 or not AND

including a_2 or not AND

. . .

including $a_{|S|}$ or not.

CL-1.4 (a) By the Rule of Product, we have $9 \times 10 \times \cdots \times 10 = 9 \times 10^{n-1}$.

(b) By the Rule of Product, we have 9^n.

(c) By the Rule of Sum, (answer)$+9^n = 9 \times 10^{n-1}$ and so the answer is $9(10^{n-1} - 9^{n-1})$

CL-1.5 (a) This is like the previous exercise. There are 26^4 4-letter strings and there are $(26-5)^4$ 4-letter strings that contain no vowels. Thus we have $26^4 - 21^4$.

(b) We can do this in two ways:
First way: Break the problem into 4 problems, depending on where the vowel is located. (This uses the Rule of Sum.) For each subproblem, choose each letter in the list and use the Rule of Product. We obtain one factor equal to 5 and three factors equal to 21. Thus we obtain 5×21^3 for each subproblem and $4 \times 5 \times 21^3$ for the final answer.
Second way: Choose one of the 4 positions for the vowel, choose the vowel and choose each of the 3 consonants. By the Rule of Product we have $4 \times 5 \times 21 \times 21 \times 21$.

CL-1.6 The only possible vowel and consonant pattern satisfying the two nonadjacent vowels and initial and terminal consonant conditions is CVCVC. By the Rule of Product, there are $3 \times 2 \times 3 \times 2 \times 3 = 108$ possibilities.

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CL-1.7 To form a composition of n, we can write n ones in a row and insert either "$\oplus$" or "," in the spaces between them. This is a series of 2 choices at each of $n-1$ spaces, so we obtain 2^{n-1} compositions of n. The compositions of 4 are

$$4 = 3\oplus 1 = 2\oplus 2 = 2\oplus 1\oplus 1 = 1\oplus 3 = 1\oplus 2\oplus 1 = 1\oplus 1\oplus 2 = 1\oplus 1\oplus 1\oplus 1.$$

The compositions of 5 with 3 parts are

$$3\oplus 1\oplus 1 = 2\oplus 2\oplus 1 = 2\oplus 1\oplus 2 = 1\oplus 3\oplus 1 = 1\oplus 2\oplus 2 = 1\oplus 1\oplus 3.$$

CL-1.8 The allowable letters in alphabetic order are A, I, L, S, and T. There are 216 words that begin with L, and the same number that begin with S, and with T. The word we are asked to find is the last one that begins with L. Thus the word is of the form $LVCVCC$, $LVCCVC$, or $LCVCVC$. Since all of the consonants in our allowable-letters list come after the vowels, we want a word of the form $LCVCVC$. We need to start off $LTVCVC$. The next letter, a vowel, needs to be I (bigger than A in the alphabet). Thus we have $LTICVC$. Continuing in this way we get $LTITIT$. The next name in dictionary order starts off with S and is of the form $SVCVCC$. We now choose the vowels and consonants as small as possible: $SALALL$. But, this word doesn't satisfy the condition that adjacent consonants must be different. Thus the next legal word is $SALALS$.

CL-1.9 The ordering on the C_i is as follows:

$$C_1 = ((2,4),(2,5),(3,5)) \quad C_2 = (\mathrm{AA},\mathrm{AI},\mathrm{IA},\mathrm{II})$$

$$C_3 = (\mathrm{LL},\mathrm{LS},\mathrm{LT},\mathrm{SL},\mathrm{SS},\mathrm{ST},\mathrm{TL},\mathrm{TS},\mathrm{TT}) \quad C_4 = (\mathrm{LS},\mathrm{LT},\mathrm{SL},\mathrm{ST},\mathrm{TL},\mathrm{TS}).$$

The first seven are

(2,4)(AA)(LL)(LS), (2,4)(AA)(LL)(LT), (2,4)(AA)(LL)(SL),

(2,4)(AA)(LL)(ST), (2,4)(AA)(LL)(TL),

(2,4)(AA)(LL)(TS), (2,4)(AA)(LS)(LS).

The last 7 are

(3,5)(II)(TS)(TS), (3,5)(II)(TT)(LS), (3,5)(II)(TT)(LT),

(3,5)(II)(TT)(SL), (3,5)(II)(TT)(ST),

(3,5)(II)(TT)(TL), (3,5)(II)(TT)(TS).

The actual names can be constructed by following the rules of construction from these strings of symbols (e.g, (3,5)(II)(TT)(LS) says place the vowels II in positions 3,5, the nonadjacent consonants are TT and the adjacent consonants are LS to get LSITIT).

CL-1.10 (a) One way to do this is to list all the possible multisets in some order. If you do this carefully, you will find that there are 15 of them. Unfortunately, it is easy to miss something if you do not choose the order carefully. One way to do this is to first write all the a's in the multiset, then all the b's and then all the c's. For example, we would write the multiset $\{a, b, c, a\}$ as $aabc$. We can now list these in lex order:

$$aaaa,\ aaab,\ aaac,\ aabb,\ aabc,\ aacc,\ abbb,\ abbc,$$
$$abcc,\ accc,\ bbbb,\ bbbc,\ bbcc,\ bccc,\ cccc$$

For (b), the answer is that there are an infinite number because an element can be repeated any number of times. In fact, an infinite number of multisets can be formed by using just a.

CL-2.1 (a) We can arrange n people in $n!$ ways. Use $n = 7$.

(b) Arrange b boys ($b!$ ways) AND arrange g girls ($g!$ ways) AND choose which list comes first (2 ways). Thus we have $2(b!\,g!)$. Here $b = 3$ and $g = 4$ and the answer is 288.

(c) As in (b), we arrange the girls and the boys separately, AND then we interleave the two lists as GBGBGBG. Thus we get $4!\,3! = 144$.

CL-2.2 This refers to the previous solution.

(a) Use $n = 6$.

(b) $b = g = 3$ and the answer is 72.

(c) We can interleave in two ways, as BGBGBG or as GBGBGB and so we get $2(3!\,3!) = 72$.

CL-2.3 For (a) we have the circular list discussed in the text and the answer is therefore $n!/n = (n-1)!$.
For (b), note that each circular list gives two ordinary lists — one starting with the girls and the other with the boys. Hence the answer is $2(b!\,g!)/2 = b!\,g!$. For the two problems we have $4!\,3! = 144$ and $3!\,3! = 36$.
For (c), it is impossible if $b < g$ since this forces two girls to sit together. If we have $b = g$, circular lists are possible. As in the unrestricted case, each circular list gives $n = b+g = 2g$ linear lists by cutting it arbitrarily. Thus we get $2(g!)^2/2g = g!\,(g-1)!$, which in this case is $3!\,2! = 12$.

CL-2.4 Each of the 7 letters ABMNRST appears once and each of the letters CIO appears twice. Thus we must form a list of length k from the 10 distinct letters. The solutions are

$$\begin{aligned} k = 2\text{:} &\quad 10 \times 9 = 90 \\ k = 3\text{:} &\quad 10 \times 9 \times 8 = 720 \\ k = 4\text{:} &\quad 10 \times 9 \times 8 \times 7 = 5040 \end{aligned}$$

CL-2.5 Each of the 7 letters ABMNRST appears once and each of the letters CIO appears twice.

- For $k = 2$, the letters are distinct OR equal. There are $(10)_2 = 90$ distinct choices. Since the only repeated letters are CIO, there are 3 ways to get equal letters. This gives 93.
- For $k = 3$, we have either all distinct ($(10)_3 = 720$) OR two equal. The two equal can be worked out as follows

choose the repeated letter (3 ways) AND

choose the positions for the two copies of the letter (3 ways) AND

choose the remaining letter ($10 - 1 = 9$ ways).

By the Rules of Sum and Product, we have $720 + 3 \times 9 \times 3 = 801$.

CL-2.6 (a) The letters are EILST. The number or 3-words is $(5)_3 = 60$.

(b) The answer is $5^3 = 125$.

(c) The letters are EILST, with T occurring 3-times, L occurring 2-times. Either the letters are distinct OR one letter appears twice OR one letter appears three times. We have seen that the first can be done in 60 ways. To do the second, choose one of L and T to repeat, choose one of the remaining 4 different letters and choose where that letter is to go, giving $2 \times 4 \times 3 = 24$. To do the third, use T. Thus, the answer is $60 + 24 + 1 = 85$.

CL-2.7 (a) Stripping off the initial R and terminal F, we are left with a list of at most 4 letters, at least one of which is an L. There is just 1 such list of length 1. There are $3^2 - 2^2 = 5$ lists of length 2, namely all those made from E, I and L minus those made from just E and I. Similarly, there are $3^3 - 2^3 = 19$ of length 3 and $3^4 - 2^4 = 65$. This gives us a total of 90.

(b) The letters used are E, F, I, L and R in alphabetical order. To get the word before RELIEF, note that we cannot change just the F and/or the E to produce an earlier word. Thus we must change the I to get the preceding word. The first candidate in alphabetical order is F, giving us RELF. Working backwards in this manner, we come to RELELF, RELEIF, RELEF and, finally, RELEEF.

CL-2.8 (a) If there are 4 letters besides R and F, then there is only one R and one F, for a total of 65 spellings by the previous problem. If there are 3 letters besides R and F, we may have R$\cdots$F, R$\cdots$FF or RR$\cdots$F, which gives us $3 \times 19 = 57$ words by the previous problem. We'll say there are 3 RF patterns, namely RF, RFF and RRF. If there 2 letters besides R and F, there are 6 RF patterns, namely the three just listed, RFFF, RRFF and RRRF. This gives us $6 \times 5 = 30$ words. Finally, the last case has the 6 RF patterns just listed as well as RFFFF, RRFFF, RRRFF and RRRRF for a total of 10 patterns. This give us 10 words since the one remaining letter must be L. Adding up all these cases gives us $65 + 57 + 30 + 10 = 162$ possible spellings. Incidentally, there is a simple formula for the number of n long RF patterns, namely $n - 1$. Thus there are

$$1 + 2 + \ldots + (n - 1) = n(n - 1)/2$$

of length at most n. This gives our previous counts of 1, 3, 6 and 10.

(b) Reading toward the front of the dictionary from RELIEF we have RELIEF, RELFFF, RELFF, RELF, RELELF, RELEIF, RELEFF,..., and so the spelling five before RELIEF is RELEIF.

CL-2.9 There are $n!/(n - k)!$ lists of length k. The total number of lists (not counting the empty list) is

$$\begin{aligned}\frac{n!}{(n-1)!} + \frac{n!}{(n-2)!} + \cdots + \frac{n!}{1!} + \frac{n!}{0!}\\ = n!\left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{(n-1)!}\right)\\ = n!\sum_{i=0}^{n-1}\frac{1^i}{i!}.\end{aligned}$$

Since $e = e^1 = \sum_{i=0}^{\infty} 1^i/i!$, it follows that the above sum is close to e.

CL-3.1 Choose values for pairs
AND choose suits for the lowest value pair
AND choose suits for the middle value pair
AND choose suits for the highest value pair.
This gives $\binom{13}{3}\binom{4}{2}^3 = 61,776$.

CL-3.2 Choose the lowest value in the straight (A to 10) AND choose a suit for each of the 5 values in the straight. This gives $10 \times 4^5 = 10240$.

Although the previous answer is acceptable, a poker player may object since a "straight flush" is better than a straight — and we included straight flushes in our count. Since a straight flush is a straight all in the same suit, we only have 4 choices of suits for the cards instead of 4^5. Thus, there are $10 \times 4 = 40$ straight flushes. Hence, the number of straights which are not straight flushes is $10240 - 40 = 10200$.

CL-3.3 If there are n 1's in the sequence, there are $n-1$ spaces between the 1's. Thus, there are 2^{n-1} compositions of n. A composition of n with k parts has $k-1$ commas The number of ways to insert $k-1$ commas into $n-1$ positions is $\binom{n-1}{k-1}$.

CL-3.4 Note that EXERCISES contains 3 E's, 2 S's and 1 each of C, I, R and X. We can use the multinomial coefficient

$$\binom{n}{m_1, m_2, \ldots, m_k} = \frac{n!}{m_1!\, m_2! \cdots m_k!}$$

where $n = m_1 + m_2 + \ldots + m_k$. Take $n = 9$, $m_1 = 3$, $m_2 = 2$ and $m_3 = m_4 = m_5 = m_6 = 1$. This gives $9!/3!\,2! = 30240$. This calculation can also be done without the use of a multinomial coefficient as follows. Choose 3 of the 9 possible positions to use for the three E's AND choose 2 of the 6 remaining positions to use for the two S's AND put a permutation of the remaining 4 letters in the remaining 4 places. This gives us $\binom{9}{3} \times \binom{6}{2} \times 4!$.

CL-3.5 An arrangement is a list formed from 13 things each used 4 times. Thus we have $n = 52$ and $m_i = 4$ for $1 \le i \le 13$ in the multinomial coefficient

$$\binom{n}{m_1, m_2, \ldots, m_k} = \frac{n!}{m_1!\, m_2! \cdots m_k!}.$$

CL-3.6 (a) The first 4 names in dictionary order are LALALAL, LALALAS, LALALAT, LALALIL.

(b) The last 4 names in dictionary order are TSITSAT, TSITSIL, TSITSIS, TSITSIT.

(c) To compute the names, we first find the possible consonant vowel patterns. They are CCVCCVC, CCVCVCC, CVCCVCC and CVCVCVC. The first three each contain two pairs of adjacent consonants, one isolated consonant and two vowels. Thus each corresponds to $(3 \times 2)^2 \times 3 \times 2^2$ names. The last has four isolated consonants and three vowels and so corresponds to $3^4 \times 2^3$ names. In total, there are 1,944 names.

CL-3.7 The first identity can be proved by writing the binomial coefficients in terms of factorials. It can also be proved from the definition of the binomial coefficient: Choosing a set of size k from a set of size n is equivalent to choosing a set of size $n-k$ to throw away, namely the things not chosen.

The total number of subsets of an n element set is 2^n. On the other hand, we can divide the subsets into collections T_j, where T_i contains all the i element subsets. The number of subsets in T_i is $\binom{n}{i}$. Apply the Rule of Sum.

CL-3.8 $S(n,n) = 1$: The only way to partition an n element set into n blocks is to put each element in a block by itself, so $S(n,n) = 1$.

$S(n,n-1) = \binom{n}{2}$: The only way to partition an n element set into $n-1$ blocks is to choose two elements to be in a block together and put the remaining $n-2$ elements in $n-2$ blocks by themselves. Thus it suffices to choose the 2 elements that appear in a block together and so $S(n,n-1) = \binom{n}{2}$.

$S(n,1) = 1$: The only way to partition a set into one block is to put the entire set into the block.

$S(n,2) = (2^n-2)/2$: We give two solutions. Note that $S(n,k)$ is the number of k-sets $\mathcal{S}$ where the entries in $\mathcal{S}$ are nonempty subsets of a given n-set T and each element of T appears in exactly one entry of $\mathcal{S}$. We will count k-lists, which is $k!$ times the number of k-sets. We choose a subset for the first block (first list entry) and use the remaining set elements for the second block. Since an n-set has 2^n, this would seem to give $2^n/2$; however, we must avoid empty blocks. In the ordered case, there are two ways this could happen since either the first or second list entry could be the empty set. Thus, we must have $2^n - 2$ instead of 2^n. The answer is $(2^n-2)/2$.

Here is another way to compute $S(n,2)$. Look at the block containing n. Once it is determined, the entire two block partition is determined. The block containing n can be gotten by starting with n and adjoining one of the $2^{n-1}-1$ proper subsets of $\{1,2,\ldots,n-1\}$.

CL-3.9 We use the hint. Choose i elements of $\{1,2,\cdots,n\}$ to be in the block with $n+1$ AND either do nothing else if $i = n$ OR partition the remaining elements. This gives $\binom{n}{n}$ if $i = n$ and $\binom{n}{i}B_{n-i}$ otherwise. If we set $B_0 = 1$, the second formula applies for $i = n$, too. Since $i = 0$ OR $i = 1$ OR $\cdots$ OR $i = n$, the result follows.

(b) To calculate B_n for $n \le 5$: We have $B_0 = 1$ from (a). Using the formula in (a) for $n = 0,1,2,3,4$ in order, we obtain $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$ and $B_5 = 52$.

CL-3.10 (a) There is exactly one arrangement — 1,2,3,4,5,6,7,8,9.

(b) We do this by counting those arrangements that have $a_i \le a_{i+1}$ except, perhaps, for $i = 5$. Then we subtract off those that also have $a_5 < a_6$. In set terms:

- S is the set of rearrangements for which $a_1 < a_2 < a_3 < a_4 < a_5$ and $a_6 < a_7 < a_8 < a_9$,

- T is the set of rearrangements for which $a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7 < a_8 < a_9$, and
- we want $|S \setminus T| = |S| - |T|$.

An arrangement in S is completely determined by specifying the set $\{a_1, \ldots, a_5\}$, of which there are $\binom{9}{5} = 126$. In (a), we saw that $|T| = 1$. Thus the answer is $126 - 1 = 125$.

CL-4.1 Let the probability space consist of all $\binom{6}{2} = 15$ pairs of horses and use the uniform probability. Thus each pair has probability 1/15. Since each horse is in exactly 5 pairs, the probability of your choosing the winner is $5/15 = 1/3$, regardless of which horse wins.

Here is another way. You could choose your first horse and your second horse, so the space consists of 6×5 choices. The probability that your first choice was the winner is 1/6. The probability that your second choice was the winner is also 1/6. Since these events are disjoint, the probability of picking the winner is $1/6 + 1/6 = 1/3$.

Usually the probability of winning a bet on a horse race depends on picking the fastest horse after much study. The answer to this problem, 1/3, doesn't seem to have anything to do with studying the horses? Why?

CL-4.2 The sample space is $\{0, 1, \ldots, 36, 00\}$. We have $P(0) = P(1) = \cdots = P(36)$ and $P(00) = 1.05P(0)$. Thus

$$1 = P(0) + \cdots + P(36) + P(00) = 38.05P(0).$$

Hence $P(0) = 1/38.05$ and so $P(00) = 1.05/38.05 = 0.0276$.

CL-4.3 Let the event space be $\{A, B\}$, depending on who finds the key. Since Alice searches 20% faster than Bob, it is reasonable to assume that $P(A) = 1.2\,P(B)$. The odds that Alice finds the key are $P(A)/P(B) = 1.2$, that is, 1.2:1, which can also be written as 6:5. Combining $P(A) = 1.2\,P(B)$ with $P(A) + P(B) = 1$, we find that $P(A) = 1.2/2.2 = 0.545$.

CL-4.4 Let A be the event that you pick the winner and B the probability that you pick the horse that places. From a previous exercise, $P(A) = 1/3$ Similarly, $P(B) = 1/3$. We want $P(A \cup B)$. By the principle of inclusion and exclusion, this is $P(A) + P(B) - P(A \cap B)$. Of all $\binom{6}{2} = 15$ choices, only one is in $A \cap B$. Thus $P(A \cap B) = 1/15$ and the answer is $1/3 + 1/3 - 1/15 = 3/5$.

CL-4.5 Since probabilities are uniform, we simply count the number of events that satisfy the conditions and divide by the total number of events, which is m^n for n balls and m boxes. First we will do the problems in an ad hoc manner, then we'll discuss a systematic solution. We use (a′)–(c′) to denote the answers for (d).

(a) We place one ball in the first box AND one in the second AND so on. Since this can be done in 4! ways, the answer is $4!/4^4 = 3/32$.

(a′) We must have one box with two balls and one ball in each of the other three boxes. We choose one box to contain two balls AND two balls for the box AND distribute the three remaining balls into three boxes as in (a). This gives us $4 \times \binom{5}{2} \times 3! = 240$. Thus the answer is $240/4^5 = 15/64$.

(b) This is somewhat like (a′). Choose a box to be empty AND choose a box to contain two balls AND choose two balls for the box AND distribute the other two balls into the other two boxes. This gives $4 \times 3 \times \binom{4}{2} \times 2! = 144$. Thus the answer is $144/4^4 = 9/16$.

(b′) This is more complicated since the ball counts can be either 3,1,1,0 or 2,2,1,0. As in (b), there are $4 \times 3 \times \binom{5}{3} \times 2! = 240$ to do the first. In the second, there are $\binom{4}{2} \times 2 = 12$ ways to designate the boxes and $\binom{5}{2} \times \binom{3}{2} = 30$ ways to choose the balls for the boxes that contain two each. Thus there are 360 ways and the answer is $(240 + 360)/4^5 = 75/128$.

(c) Simply subtract the answer for (a) from 1 since we are asking for the complementary event. This gives $29/32$. For (c′) we have $39/64$.

We now consider a systematic approach. Suppose we want to assign n balls to m boxes so that exactly $k \le m$ of the boxes contain balls. Call the balls $1, 2, \ldots, n$ First partition the set of n balls into k blocks. This can be done in $S(n, k)$ ways, where $S(n, k)$ is the Stirling number discussed in Section 3. List the blocks in some order (pick your favorite; e.g., numerical order based on the smallest element in the block). Assign the first block to a box AND assign the second block to a box AND, etc. This can be done in $m(m-1)\cdots(m-k+1) = m!/(m-k)!$ ways. Hence the number of ways to distribute the balls is $S(n, k)m!/(m-k)!$ and so the probability is $S(n, k)m!/(m-k)!\,m^n$. For our particular problems, the answers are

(a) $S(4,4)4!/0!\,4^4 = 3/32$	(a′) $S(5,4)4!/0!\,4^5 = 15/64$
(b) $S(4,3)4!/1!\,4^4 = 9/16$	(b′) $S(5,3)4!/1!\,4^5 = 75/128$.

The moral here is that if you can think of a systematic approach to a class of problems, it is likely to be easier than solving each problem separately.

CL-4.6 (a) Since the die is thrown k times, the sample space is S^k, where $S = \{1, 2, 3, 4, 5, 6\}$. Since the die is fair, all 6^k sequences in S^k are equally likely. We claim that exactly half have an even sum and so $P(E) = 1/2$. Why do half have an even sum? Here are two proofs.

- Let $N_o(n)$ be the number of odd sums in the first n throws and let $N_e(n)$ be the number of even sums. We have

  $$N_e(k) = 3N_e(k-1) + 3N_o(k-1) \qquad \text{and} \qquad N_o(k) = 3N_o(k-1) + 3N_e(k-1)$$

  because an even sum is obtained from an even by throwing 2, 4, or 6 and from an odd by throwing 1, 3, or 5; and similarly for an odd sum. Thus $N_e(k) = N_o(k)$. Since the probability on S^k is uniform, the probability of an even sum is $1/2$.

- Let S_o be all the k-lists in S^k with odd sum and let S_e be those with even sum. Define the function $f : S^k \to S^k$ as follows

  $$f(x_1, x_2 \ldots, x_k) = \begin{cases} (x_1 + 1, x_2, \ldots, x_k), & \text{if } x_1 \text{ is odd;} \\ (x_1 - 1, x_2, \ldots, x_k), & \text{if } x_1 \text{ is even.} \end{cases}$$

  We leave it to you to convince yourself that this function is a bijection between S_o and S_e. (A bijection is a one-to-one correspondence between elements of S_o and S_e.)

(b) The sample space for drawing cards n times is S^n where S is the Cartesian product

$$\{\mathrm{A}, 2, 3, \ldots, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}\} \times \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

The probability of any point in S^n is $(1/52)^n$. The number of draws with no king is $(52-4)^n$ and so the probability of none is $(48/52)^n = (12/13)^n$. The probability of at least one king is $1-(12/13)^n$.

(c) The equiprobable sample space is gotten by distinguishing the marbles $M = \{w_1, w_2, w_3, r_1, \ldots\}$ and defining the sample space by

$$S = \{(m, m') : m \text{ and } m' \text{ are distinct elements of } M\}.$$

If E_r is the event that both m and m' are red, then $P(E_r) = 4{*}3/|S|$ where $|S| = 12{*}11$.

RELATED PROBLEMS TO THINK ABOUT: What is the probability of two white and two blue marbles being drawn if four marbles are drawn without replacement? Of two white and two blue marbles being drawn if four marbles are drawn with replacement?

CL-4.7 This is nearly identical to the example on hypergeometric probabilities. The answer is $C(5,3)C(10,3)/C(15,6)$.

CL-4.8 Let $B = \{1, 2, \ldots, 10\}$.

(a) The sample space S is the set of all subsets of B of size 2. Thus $|S| = \binom{10}{2} = 45$. Since each draw is equally likely, we just need to know how many pairs have an odd sum. One of the balls must have an odd label and the other an even label. The number of pairs with this property is 5×5 since there are 5 odd labels and 5 even labels. Thus the probability is $25/45 = 5/9$.

(b) The sample space S is the set of ordered pairs (b_1, b_2) with $b_1 \neq b_2$ both from B. Thus $|S| = 10 \times 9 = 90$. To get an odd sum, one of b_1 and b_2 must be even and the other odd. Thus there are 10 choices for b_1 AND then 5 choices for b_2. The probability is $50/90 = 5/9$.

(c) The sample space is $S = B \times B$ and $|S| = 100$. The number of pairs (b_1, b_2) is 50 as in (b). Thus the probability is $50/100 = 1/2$.

CL-4.9 This is an inclusion and exclusion type of problem. There are three ways to approach such problems:

- Have a variety of formulas handy that you can plug into. This, by itself, is not a good idea because you may encounter a problem that doesn't fit any of the formulas you know.
- Draw a Venn diagram and use the information you have to compute the probability of as many regions as you can. If there are more than 3 sets, the Venn diagram is too confusing to be very useful. With 2 or 3 sets, it is a good approach.
- Carry out the preceding idea without the picture. We do this here.

Suppose we are dealing with k sets, $A_1, \ldots, A_k$. We need to know what the regions in the Venn diagram are. Each region corresponds to $T_1 \cap \cdots \cap T_k$ where T_i is either A_i or A_i^c. In our case, $k = 2$ and so the probabilities of the regions are

$$P(A \cap B) \qquad P(A \cap B^c) \qquad P(A^c \cap B) \qquad P(A^c \cap B^c).$$

We get A by combining $A\cap B$ and $A\cap B^c$. We get B by combining $A\cap B$ and $A^c\cap B$. By properties of sets, $(A\cup B)^c = A^c\cap B^c$. Thus our data corresponds to the three equations

$$P(A\cap B)+P(A\cap B^c)=3/8 \qquad P(A\cap B)+P(A^c\cap B)=1/2 \qquad P(A^c\cap B^c)=3/8.$$

We have one other equation: The probabilities of all four regions sum to 1. This gives us four equations in four unknowns whose solution is

$$P(A\cap B)=1/4 \qquad P(A\cap B^c)=1/8 \qquad P(A^c\cap B)=1/4 \qquad P(A^c\cap B^c)=3/8.$$

Thus the answer to the problem is $1/4$.

When we are not asked for the probability of all regions, it is often possible to take shortcuts. That is the case here. From $P((A\cup B)^c)=3/8$ we have $P(A\cup B) = 1-3/8=5/8$. Since $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ and three of the four terms in this equation are known, we can easily solve for $P(A\cap B)$.

CL-4.10 This is another Venn diagram problem. This time we'll work with number of people instead of probabilities. Let C correspond to the set of computer science majors, W the set of women and S to the entire student body. We are given

$$\begin{aligned}|C| &= 20\%\times 5,000 = 1,000\\ |W| &= 58\%\times 5,000 = 2,900\\ |C\cap W| &= 430.\end{aligned}$$

(a) We want $|W\cap C^c|$, which equals $|W|-|W\cap C| = 2,470$. You should be able to see why this is so by the Venn diagram or by the method used in the previous problem.

(b) The number of men who are computer science majors is the number of computer science majors who are not women. This is $|C|-|C\cap W| = 1,000-430 = 570$. The number of men in the student body is $42\%\times 5,000 = 2,100$. Thus $2,100-570 = 1,530$ men are not computer science majors.

(c) The probability is $\frac{430}{5,000} = 0.086$.

(c) Since there are $58\%\times 5,000 = 2,900$ women, the probability is $\frac{430}{2,900}$.

CL-4.11 Since the coin is fair $P(H)=1/2$, what about $P(W)$, the probability that Beatlebomb wins? Recall the meaning of the English phrase "the odds that it will occur." This is trivial but important, as the phrase is used often in everyday applications of probability. If you don't recall the meaning, see the discussion of odds in the text. From the definition of odds, you should be able to show that $P(W)=1/101$. If we had studied "independent" events, you could immediately see that the answer to the questions is $(1/2)\times(1/101) = 1/202$, but we need a different approach which lets independent events sneak in through the back door.

Let the sample space be $\{H,T\}\times\{W,L\}$, corresponding to the outcome of the coin toss and the outcome of the race. From the previous paragraph $P(\{(H,W),(T,W)\}) = 1/101$. Since the coin is fair and the coin toss doesn't influence the race, we should have $P((H,W)) = P((T,W))$. Since

$$P(\{(H,W),(T,W)\}) = P((H,W))+P((T,W)),$$

It follows after a little algebra that $P(H, W)) = 1/202$.

CL-4.12 This is another example of the hypergeometric probability. Do you see why? The answer is $C(37, 11)C(2, 2)/C(39, 13)$.

CL-4.13 It may seem at first that you need to break up the problem according to what the other players have been dealt. Not so! You should be able to see that the results would have been the same if you had been dealt your fifth card *before* the other players had been dealt their cards. Now it's not hard to work things out. After you've been dealt 4 cards, there are 48 cards left. Of those, the fourth card in the 3 of a kind ($4\diamondsuit$ in the example) and any of the 3 cards with the same value as your odd card ($10\heartsuit$ $10\diamondsuit$ $10\clubsuit$ in the example) improve your hand. That's 4 cards out of 48, so the probability is $4/48 = 1/12$.

CL-4.14 (a) Let words of length 6 formed from three G's and three B's stand for the arrangements in terms of Boys and Girls; for example, BBGGBG or BBBGGG. There are $\binom{6}{3} = 6!/(3!\,3!) = 20$ such words. Four such words correspond to the three girls together: GGGBBB, BGGGBB, BBGGGB, BBBGGG. The probability of three girls being together is $4/20 = 1/5$.

(b) If they are then seated around a circular table, there are two additional arrangements that will result in all three girls sitting together: GGBBBG and GBBBGG. The probability is $6/20 = 3/10$.

CL-4.15 You can draw the Venn diagram for three sets and, for each of the eight regions, count how much a point in the region contributes to the addition and subtraction. This does not extend to the general case. We give another proof that does.

Let S be the sample space and let T be a subset of S Define the function χ_T with domain S by

$$\chi_T(s) = \begin{cases} 1 & \text{if } s \in T, \\ 0 & \text{if } s \notin T. \end{cases}$$

This is called the *characteristic function* of T.[1] We leave it to you to check that

$$\chi_{T^c}(s) = 1 - \chi_T(s), \quad \chi_{T\cap U}(s) = \chi_T(s)\,\chi_U(s), \quad \text{and} \quad P(S) = \sum_{s\in S} P(s)\chi_T(s).$$

[1] χ is a lower case Greek letter and is pronounced like the "ki" in "kind."

Using these equations and a little algebra, we have

$$\begin{aligned}
P(A^c \cap B^c \cap C^c) &= \sum_{s\in S} P(s)\chi_{A^c\cap B^c\cap C^c}(s) \\
&= \sum_{s\in S} P(s)\big(1-\chi_A(s)\big)\big(1-\chi_B(s)\big)\big(1-\chi_C(s)\big) \\
&= \sum_{s\in S} P(s) - \sum_{s\in S} P(s)\chi_A(s) - \sum_{s\in S} P(s)\chi_B(s) - \sum_{s\in S} P(s)\chi_C(s) \\
&\quad + \sum_{s\in S} P(s)\chi_A(s)\chi_B(s) + \sum_{s\in S} P(s)\chi_A(s)\chi_C(s) \\
&\quad + \sum_{s\in S} P(s)\chi_B(s)\chi_C(s) - \sum_{s\in S} P(s)\chi_A(s)\chi_B(s)\chi_C(s) \\
&= 1 - P(A) - P(B) - P(C) \\
&\quad + P(A\cap B) + P(A\cap C) \\
&\quad + P(B\cap C) - P(A\cap B\cap C).
\end{aligned}$$

CL-4.16 Let the stick have unit length and let x be the distance from the end of the stick where the break is made. Thus $0 \le x \le 1$. The longer piece will be at least twice the length of the shorter if $x \le 1/3$ or if $x \ge 2/3$. The probability of this is $1/3 + 1/3 = 2/3$. You should be able to fill in the details.

CL-4.17 Let x and y be the places where the stick is broken. Thus, (x, y) is chosen uniformly at random in the square $S = (0,1) \times (0,1)$. Three pieces form a triangle if the sum of the lengths of any two is always greater than the length of the third. We must determine which regions in S satisfy this condition.

Suppose $x < y$. The lengths are then x, $y - x$, and $1 - y$. The conditions are

$$x + (y - x) > 1 - y, \quad x + (1 - y) > y - x, \quad \text{and} \quad (y - x) + (1 - y) > x.$$

With a little algebra, these become

$$y > 1/2, \quad y < x + 1/2, \quad \text{and} \quad x < 1/2,$$

respectively. If you draw a picture, you will see that this is a triangle of area $1/8$.

If $x > y$, we obtain the same results with the roles of x and y reversed. Thus the total area is $1/8 + 1/8 = 1/4$. Since S has area 1, the probability is $1/4$.

CL-4.18 Look where the center of the coin lands. If it is within $d/2$ of a lattice point, it covers the lattice point. Thus, there is a circle of diameter d about each lattice point and the coin covers a lattice point if and only if it lands in one of the circles. We need to compute the fraction of the plane covered by these circles. Since the pattern repeats in a regular fashion, all we need to do is calculate the fraction of the square $\{(x, y) | 0 \le x \le 1, 0 \le y \le 1\}$ that contains parts of circles. There is a quarter circle about each of the points (0,0), (0,1), (1,0) and (1,1) inside the square. Since the circle has diameter at most 1, the quarter circles have no area in common and so their total area equals the area of the coin, $\pi d^2/4$. Since the area of the square is 1, the probability that the coin covers a lattice point is $\pi d^2/4$.

CL-4.19 Select the three points uniformly at random from the circumference of the circle and label them 1, 2, 3 going clockwise around the circle from the top of the circle. Let E_1 denote the event consisting of all such configurations where points 2 and 3 lie in the half circle starting at 1 and going clockwise (180 degrees). Let E_2 denote the event that points 2 and 1 lie in the half circle starting at 2 and going clockwise 180 degrees. Let E_3 be defined similarly. Note that the events E_1, E_2, and E_3 are mutually exclusive. (Draw a picture and think about this.) By our basic probability axioms, the probability of the union is the sum of the probabilities $P(E_1) + P(E_2) + P(E_3)$. To compute $P(E_1)$, imagine point 1 on the circle, consider its associated half circle and, before looking at the other two points, ask "What is the probability that they lie in that half circle?" Let x be the number of degrees clockwise from point 1 to point 2 and y the number from 1 to 3. Thus (x, y) is a point chosen uniformly at random in the square $[0, 360) \times [0, 360)$. For event E_1 to occur, (x, y) must lie in $[0, 180) \times [0, 180)$, which is 1/4 of the original square. Thus $P(E_1) = 1/4$. (This can also be done using independent events: the locations of points 2 and 3 are chosen independently so one gets $(1/2) \times (1/2)$.) The probabilities of E_2 and E_3 are the same for the same reason. Thus $P(E_1) + P(E_2) + P(E_3) = 3/4$.

What is the probability that k points selected uniformly at random on the circumference of a circle lie the same semicircle? Use the same method. The answer is $k/(2^{k-1})$.

Solutions for Functions

Fn-1.1 (a) We know the domain and range of f. f is not an injection. Since no order is given for the domain, the attempt to specify f in one-line notation is meaningless (the ASCII order $+, <, >, ?$, is a possibility, but is unusual enough in this context that explicitly specifying it would be essential). If the attempt at specification makes any sense, it tells us that f is a surjection. We cannot give it in two-line form since we don't know the function.

(b) We know the domain and range of f and the domain has an implicit order. Thus the one-line notation specifies f. It is an injection but not a surjection. In two-line form it is $\begin{pmatrix} 1 & 2 & 3 \\ ? & < & + \end{pmatrix}$.

(c) This function is specified and is an injection. In one-line notation it would be (4,3,2), and, in two-line notation, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$.

Fn-1.2 (a) If f is an injection, then $|A| \le |B|$. **Solution**: Since f is an injection, every element of A maps to a different element of B. Thus B must have at least as many elements as A.

(b) If f is a surjection, then $|A| \ge |B|$. **Solution**: Since f is a surjection, every element of B is the image of at least one element of A. Thus A must have at least as many elements as B.

(c) If f is a bijection, then $|A| = |B|$. **Solution**: Combine the two previous results.

(d) If $|A| = |B|$, then f is an injection if and only if it is a surjection. **Solution**: Suppose that f is an injection and not a surjection. Then there is some $b \in B$ which is not the image of any element of A under f. Hence f is an injection from A to $B - \{b\}$. By (a), $|A| \le |B - \{b\}| < |B|$, contradicting $|A| = |B|$.
Now suppose that f is a surjection and not an injection. Then there are $a, a' \in A$ such that $f(a) = f(a')$. Consider the function f with domain restricted to $A - \{a'\}$. It is still a surjection to B and so by (b) $|B| \le |A - \{a'\}| < |A|$, contradicting $|A| = |B|$.

(e) If $|A| = |B|$, then f is a bijection if and only if it is an injection or it is a surjection. **Solution**: By the previous part, if f is either an injection or a surjection, then it is both, which is the definition of a bijection.

Fn-1.3 (a) Since ID numbers are unique and every student has one, this is a bijection.

(b) This is a function since each student is born exactly once. It is not a surjection since D includes dates that could not possibly be the birthday of any student; e.g., it includes yesterday's date. It is not an injection. Why? You may very well know of two people with the same birthday. If you don't, consider this. Most entering freshman are between 18 and 19 years of age. Consider the set F of those freshman and their possible birth dates. The maximum number of possible birth dates is $366+365$, which is smaller than the size of the set F. Thus, when we look a the function on F it is not injective.

(c) This is not a function. It is not defined for some dates because no student was born on that date. For example, D includes yesterday's date

(d) This is not a function because there are students whose GPAs are outside the range 2.0 to 3.5. (We cannot *prove* this without student record information, but we can be sure it is true.)

(e) We cannot *prove* that it is a function without gaining access to student records; however, we can be sure that it is a function since we can be sure that each of the 16 GPAs between 2.0 and 3.5 will have been obtained by many students. It is not a surjection since the codomain is larger than the domain. It is an injection since a student has only one GPA.

Fn-1.4 $\{(1,a),\ (2,b),\ (3,c)\}$ is not a relation because $c \notin B$. The others are relations. Among the relations, $\{(1,a),\ (2,b),\ (1,d)\}$ is not a functional relation because the value of the function at 3 is not defined and $\{(1,a),\ (2,b)\ (3,d),\ (1,b)\}$ is not a function because the value of the function at 1 is not uniquely defined. Thus only $\{(3,a),\ (2,b),\ (1,a)\}$ is a functional relation.
Only the inverse of $\{(1,a),\ (2,b),\ (1,d)\}$ is a functional relation. We omit the explanation.

Fn-2.1 (a) For (1,5,7,8) (2,3) (4) (6): $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 3 & 2 & 4 & 7 & 6 & 8 & 1 \end{pmatrix}$ is the two-line form and (5,3,2,4,7,6,8,1) is the one-line form. (We'll omit the two-line form in the future since it is simply the one-line form with $1, 2, \ldots$ placed above it.) The inverse is (1,8,7,5) (2,3) (4) (6) in cycle form and (8,3,2,4,1,6,5,7) in one-line form.

(b) For $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 3 & 7 & 2 & 6 & 4 & 5 & 1 \end{pmatrix}$: The cycle form is (1,8) (2,3,7,5,6,4). Inverse: cycle form is (1,8) (2,4,6,5,7,3); one-line form is (8,4,2,6,7,5,3,1).

(c) For (5,4,3,2,1), which is in one-line form: The cycle form is (1,5) (2,4) (3). The permutation is its own inverse.

(d) (5,4,3,2,1), which is in cycle form: This is not the standard form for cycle form. Standard form is (1,5,4,3,2). The one-line form is (5,1,2,3,4). The inverse is (1,2,3,4,5) in cycle form and (2,3,4,5,1) in one-line form.

Fn-2.2 Write one entire set of interchanges as a permutation in cycle form. The interchanges can be written as (1,3), (1,4) and (2,3). Thus the entire set gives $1 \to 3 \to 2$, $2 \to 3$, $3 \to 1 \to 4$ and $4 \to 1$. In cycle form this is (1,2,3,4). Thus five applications takes 1 to 2.

Fn-2.3 (a) Imagine writing the permutation in cycle form. Look at the cycle containing 1, starting with 1. There are $n-1$ choices for the second element of the cycle AND then $n-2$ choices for the third element AND $\cdots$ AND $(n-k+1)$ choices for the kth element. Prove that the number of permutations in which the cycle generated by 1 has length n is $(n-1)!$: The answer is given by the Rule of Product and the above result with $k=n$.

(b) For how many permutations does the cycle generated by 1 have length k? We write the cycle containing 1 in cycle form as above AND then permute the remaining $n-k$ elements of $\underline{n}$ in any fashion. For the k long cycle containing 1, the above result gives $\frac{(n-1)!}{(n-k)!}$ choices. There are $(n-k)!$ permutations on a set of size $n-k$. Putting this all together using the Rule of Product, we get $(n-1)!$, a result which does not depend on k.

(c) Since 1 must belong to some cycle and the possible cycle lengths are $1, 2, \ldots, n$, summing the answer to (b) over $1 \le k \le n$ will count all permutations of $\underline{n}$ exactly once. In our case, the sum is $(n-1)! + \cdots + (n-1)! = n \times (n-1)! = n!$.

This problem has shown that if you pick a random element in a permutation of an n-set, then the length of the cycle it belongs to is equally likely to be any of the values from 1 to n.

Fn-2.4 Let e be the identity permutation of A. Since $e \circ f = f$ for any permutation of A, we have $e \circ e = e$. Applying this many times $e^k = e \circ e \circ \cdots \circ e = e$ for any $k > 0$. We will use this in discussing the solution.

(a) We can step around the cycle as in Example 8 and see that after 3 steps we are back where we started from. Three hundred steps simply does this one hundred times. Instead of phrasing it this way, we could say $(1,2,3)^3 = e$ and so $(1,2,3)^{300} = ((1,2,3)^3)^{100} = e^{100}e$.

(b) Since we step around each cycle separately,

$$\big((1,3)(2,5,4)\big)^{300} = (1,3)^{300}(2,5,4)^{300} = e^{300/2}e^{300/3} = e.$$

(c) A permutation of a k-set cannot have a cycle longer than k. Thus the possible cycle lengths for permutations of $\underline{5}$ are 1, 2, 3, 4 and 5. A cycle of any of these lengths raised to the 60th power is the identity. For example $(a,b,c,d)^{60} = ((a,b,c,d)^4)^{15} = e^{15} = e$. Thus $f^{60} = e$. Finally $f^{61} = f^{60}f = ef = f$.

Fn-3.1 (a) The domain and range of f are specified and f takes on exactly two distinct values. f is not an injection. Since we don't know the values f takes, f is not completely specified; however, it cannot be a surjection because it would have to take on all four values in its range.

(b) Since each block in the coimage has just one element, f is an injection. Since $|\text{Coimage(f)}| = 5 = |\text{range of } f|$, f is a surjection. Thus f is a bijection and, since the range and domain are the same, f is a permutation. In spite of all this, we don't know the function; for example, we don't know $f(1)$, but only that it differs from all other values of f.

(c) We know the domain and range of f. From $f^{-1}(2)$ and $f^{-1}(4)$, we can determine the values f takes on the union $f^{-1}(2) \cup f^{-1}(4) = \underline{5}$. Thus we know f completely. It is neither a surjection nor an injection.

(d) This function is a surjection, cannot be an injection and has no values specified.

(e) This specification is nonsense. Since the image is a subset of the range, it cannot have more than four elements.

(f) This specification is nonsense. The number of blocks in the coimage of f equals the number of elements in the image of f, which cannot exceed four.

Fn-3.2 (a) The coimage of a function is a partition of the domain with one block for each element of $\text{Image}(f)$.

(b) You can argue this directly or apply the previous result. In the latter case, note that since $\text{Coimage}(f)$ is a partition of A, $|\text{Coimage}(f)| = |A|$ if and only if each block of Coimage(f) contains just one element. On the other hand, f is an injection if and only if no two elements of A belong to the same block of Coimage(f).

(c) By the first part, this says that $|\text{Image}| = |B|$. Since Image(f) is a subset of B, it must equal B.

Fn-3.3 (a) The list is $321, 421, 431, 432, 521, 531, 532, 541, 542, 543$.

(b) The first number is $\binom{x_1-1}{3} + \binom{x_2-1}{2} + \binom{x_3-1}{1} + 1 = \binom{2}{3} + \binom{1}{2} + \binom{0}{1} + 1 = 1$. The last number is $\binom{4}{3} + \binom{3}{2} + \binom{2}{1} + 1 = 10$. The numbers $\binom{x_1-1}{3} + \binom{x_2-1}{2} + \binom{x_3-1}{1} + 1$ are, consecutively, $1, 2, \ldots 10$ and represent the positions of the corresponding strings $x_1x_2x_3$ in the list.

(c) The list is $123, 124, 125, 134, 135, 145, 234, 245, 345$.

(d) If, starting with the list of (c), you form the list $(6-x_1)(6-x_2)(6-x_3)$, you get $543, 542, 541, 532, 531, 521, 432, 431, 421, 321$ which is the list of (a) in reverse order. Thus the formula of (b) gives the positions $\rho(x_x, x_2, x_3)$ in reverse order of the list (c). Subtract $11 - \rho(x_x, x_2, x_3)$ to get the position in forward order.

(e) Successor: 98421. Predecessor: 97654.

(f) Let $x_1 = 9$, $x_2 = 8$, $x_3 = 3$, $x_2 = 2$ and $x_1 = 1$. Using the idea in part (b) of this exercise, the answer is

$$\begin{aligned}
&\binom{x_1-1}{5} + \binom{x_2-1}{4} + \binom{x_3-1}{3} + \binom{x_4-1}{2} + \binom{x_5-1}{1} \\
&= \binom{8}{5} + \binom{7}{4} + \binom{2}{3} + \binom{1}{2} + \binom{0}{1} \\
&= 56 + 35 + 0 + 0 + 0 \;=\; 91.
\end{aligned}$$

Fn-3.4 (a) The first distribution of balls to boxes corresponds to the strictly decreasing string 863. The next such string in lex order on all strictly decreasing strings of length 3 from $\underline{8}$ is 864. To get the corresponding distribution, place the three moveable box boundaries under positions 8, 6, and 4 and put balls under all other positions in $\underline{8}$. The predecessor to 863 is 862. The second distribution corresponds to 542. Its successor is 543, its predecessor is 541.

(b) The formula $p(x_1, x_2, x_3) = \binom{x_1-1}{3} + \binom{x_2-1}{2} + \binom{x_3-1}{1} + 1$ gives the position of the string $x_1x_2x_3$ in the list of decreasing strings of length three from $\underline{8}$. We solve the equation $p(x_1, x_2, x_3) = \binom{8}{3}/2 = 28$ for the variables x_1, x_2, x_3. Equivalently, find x_1, x_2, x_3 such that $\binom{x_1-1}{3} + \binom{x_2-1}{2} + \binom{x_3-1}{1} = 27$. First try to choose $x_1 - 1$ as large as possible so that $\binom{x_1-1}{3} \le 27$. A little checking gives $x_1 - 1 = 6$, with $\binom{x_1-1}{3} = \binom{6}{3} = 20$. Subtracting, $27 - 20 = 7$. Now choose $x_2 - 1$ as large as possible so that $\binom{x_1-1}{2} \le 7$. This gives $x_2 - 1 = 4$ with $\binom{x_2-1}{2} = \binom{4}{2} = 6$. Now subtract $7 - 6 = 1$ and choose $x_3 - 1 = 1$. Thus, $(x_1, x_2, x_3) = (7, 5, 2)$. The first element in the second half of the list is the next one in lex order after 752 which is 753. The corresponding distributions of ball into boxes can be obtained in the usual way.

Fn-3.5 (a) $2, 2, 3, 3$ is not a restricted growth (RG) function because it doesn't start with 1. $1, 2, 3, 3, 2, 1$ is a restricted growth function. It starts with 1 and the first occurrence of each integer is exactly one greater than the maximum of all previous integers.

$1,1,1,3,3$ is not an RG function. The first occurrence of 3 is *two* greater than the max of all previous integers.
$1,2,3,1$ is an RG function.

(b) We list the blocks $f^{-1}(i)$ in order of i. Observe that all partitions of 4 occur exactly once as coimages of the RG functions.

$$\begin{array}{lll} 1111 \to \{1,2,3,4\} & 1112 \to \{1,2,3\},\{4\} & 1121 \to \{1,2,4\},\{3\} \\ 1122 \to \{1,2\},\{3,4\} & 1123 \to \{1,2\},\{3\},\{4\} & 1211 \to \{1,3,4\},\{2\} \\ 1212 \to \{1,3\},\{2,4\} & 1213 \to \{1,3\},\{2\},\{4\} & 1221 \to \{1,4\},\{2,3\} \\ 1222 \to \{1\},\{2,3,4\} & 1223 \to \{1\},\{2,3\},\{4\} & 1231 \to \{1,4\},\{2\},\{3\} \\ 1232 \to \{1\},\{2,4\},\{3\} & 1233 \to \{1\},\{2\},\{3,4\} & 1234 \to \{1\},\{2\},\{3\},\{4\} \end{array}$$

(c) 11111, 11112, 11121, 11122, 11123 $\to \{\{1,2,3\},\{4\},\{5\}\}$
11211, 11212, 11213, 11221, 11222 $\to \{\{1,2\},\{3,4,5\}\}$
11223, 11231, 11232, 11233, 11234 $\to \{\{1,2\},\{3\},\{4\},\{5\}\}$

Fn-3.6 $S(6,3)(5)_3 = 90 \times 5 \times 4 \times 3 = 5400$.

Fn-3.7 The set B of balls is the domain and the set C of cartons is the range. Every function in C^B describes a different one of the ways to put balls into cartons. Since 2 cartons are to remain empty, we are interested in functions f with $|\text{Image}(f)| = 3$. Thus the answer to this exercise is exactly the same as for the previous exercise.

Fn-3.8 By the theorem in the text and Example 14, these are all the same. By the method in Example 14, the answer is $\binom{4+6-1}{6} = \binom{9}{6} = \binom{9}{3} = 84$.

Fn-4.1

$h_{X,Y}$	0	1	2	3	4	f_X
0	1/16	0	0	0	0	1/16
1	0	4/16	0	0	0	4/16
2	0	3/16	3/16	0	0	6/16
3	0	0	2/16	2/16	0	4/16
4	0	0	0	0	1/16	1/16
f_Y	1/16	7/16	5/16	2/16	1/16	

The row index is X and the column index is Y.

$E(X) = 2$, $\text{Var}(X) = \sigma_X = 1$ $\quad E(Y) = 1.69$, $\text{Var}(Y) = 0.96$, $\sigma_Y = 0.98$

(c) $\text{Cov}(X,Y) = 0.87$

(d) $\rho(X,Y) = 0.87/(1)(0.98) = +0.89$ Since the correlation is close to 1, X and Y move up and down together. In fact, you can see from the table for the joint distribution that X and Y are often equal.

Fn-4.2 (a) You should be able to supply reasons for each of the following steps

$$\begin{aligned} \text{Cov}(aX+bY, aX-bY) &= E[(aX+bY)(aX-bY)] - E[(aX+bY)]E[(aX-bY)] \\ &= E[a^2X^2 - b^2Y^2] - [aE(X) - bE(Y)][aE(X) + bE(Y)] \\ &= E[a^2X^2 - b^2Y^2] - [a^2(E(X))^2 - b^2(E(Y))^2] \\ &= a^2[E(X^2) - (E(X))^2] - b^2[E(Y^2) - (E(Y))^2] \\ &= a^2\text{Var}(X) - b^2\text{Var}(Y) \end{aligned}$$

Alternatively, using the bilinear and symmetric properties of Cov:

$$\begin{aligned} \text{Cov}(aX+bY, aX-bY) &= a^2\text{Cov}(X,X) - ab\text{Cov}(X,Y) + ba\text{Cov}(Y,X) + b^2\text{Cov}(Y,Y) \\ &= a^2\text{Var}(X) - b^2\text{Var}(Y) \end{aligned}$$

(b) Here is the calculation:

$$\begin{aligned}\mathrm{Var}[(aX+bY)(aX-bY)] &= \mathrm{Var}[a^2X^2-b^2Y^2)] \\ &= a^4\mathrm{Var}(X^2) - 2a^2b^2\mathrm{Cov}(X^2,Y^2) + b^4\mathrm{Var}(Y^2)\end{aligned}$$

Fn-4.3 We begin our calculations with no assumptions about the distribution for X. Expand the argument of the expectation and then use linearity of expectation to obtain.

$$E((aX+b)^2) = E(a^2X^2 + 2abX + b^2)) = a^2E(X^2) + 2abE(X) + b^2\,.$$

(The last term comes from the fact that $E(b^2) = b^2$ since b^2 is a constant.) By definition, $\mathrm{Var}(X) + (E(X))^2 = E(X^2)$. Thus

$$E((aX+b)^2) = a^2\Big(\mathrm{Var}(X) + (E(X))^2\Big) + 2abE(X) + b^2\,.$$

With a little algebra this becomes,

$$E((aX+b)^2) = a^2\mathrm{Var}(X) + (aE(X)+b)^2\,.$$

Specializing to the particular distributions for parts (a) and (b), we have the following.

(a) $E((aX+b)^2) = a^2np(1-p) + (anp+b)^2$.

(b) $E((aX+b)^2) = a^2\lambda + (a\lambda+b)^2$.

Fn-4.4 We make the dubious assumption that the misprints are independent of one another. (This would not be the case if the person preparing the book was more careless at some times than at others.)

Focus your attention on page 8. Go one by one through the misprints m_1, m_2, …, m_{200} asking the question, “Is misprint m_i on page 8?”

By the assumptions of the problem, the probability that the answer is “yes” for each m_i is 1/100. Thus, we are dealing with the binomial distribution $b(k; 200, 1/100)$. The probability of there being less than four misprints on page 8 is

$$\sum_{k=0}^{3} b(k; 200, 1/100) = \sum_{k=0}^{3} \binom{200}{k} (1/100)^k (99/100)^{200-k}.$$

Using a calculator, we find the sum to be 0.858034.

Using the Poisson approximation, we set $\lambda = np = 2$ and compute the easier sum

$$e^{-2}2^0/0! + e^{-2}2^1/1! + e^{-2}2^2/2! + e^{-2}2^3/3!,$$

which is 0.857123 according to our calculator.

Fn-4.5 From the definition of Z and the independence of X and Y, Tchebycheff’s inequality states that

$$P(|Z - aE(X) - bE(y)| \ge \epsilon) \le \frac{\mathrm{Var}(X) + \mathrm{Var}(Y)}{\epsilon^2}\,.$$

Applying this to the two parts (a) and (b), we get

(a) $P(|Z - a\gamma - b\delta| \geq \epsilon) \ \leq \dfrac{\gamma + \delta}{\epsilon^2}$.

(b) $P(|Z - anr - bns \geq \epsilon) \leq \dfrac{nr(1-r) + ns(1-s)}{\epsilon^2}$.

Fn-4.6 We are dealing with $b(k; 1000, 1/10)$. The mean is $np = 100$ and the variance is $npq = 90$. The standard deviation is thus, 9.49. The exact solution is

$$\sum_{k=85}^{115} b(k; 1000, 1/10) = \sum_{k=85}^{115} \binom{1000}{k} (1/10)^k (9/10)^{1000-k}.$$

Using a computer with multi-precision arithmetic, the exact answer is 0.898. To apply the normal distribution, we would compute the probability of the event $[100, 115]$ using the normal distribution with mean 100 and standard deviation 9.49. In terms of the standard normal distribution, we compute the probability of the event $[0, (115 - 100)/9.49] = [0, 1.6]$ (rounded off). If you have access to values for areas under the standard normal distribution, you can find that the probability is 0.445. We double this to get the approximate answer: 0.89.

Fn-4.7 We have

$$\begin{aligned} E(X) &= E((1/n)(X_1 + \cdots + X_n)) = (1/n)E(X_1 + \cdots + X_n) \\ &= (1/n)(E(X_1) + \cdots + E(X_n)) = (1/n)(\mu + \cdots + \mu) = \mu \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{Var}((1/n)(X + 1 + \cdots + X_n)) = (1/n)^2 \,\mathrm{Var}(X + 1 + \cdots + X_n) \\ &= (1/n)^2(\mathrm{Var}(X + 1) + \cdots + \mathrm{Var}(X_n)) = (1/n)^2(n\sigma^2) = \sigma^2/n. \end{aligned}$$

Since X has mean μ, it is a reasonable approximation to μ. Of course, it's important to know something about the accuracy.

(c) Since Var(X)$= \sigma^2/n$, we have $\sigma_X = \sigma/\sqrt{n}$. If we change from n to N, σ_X changes to $\sigma/\sqrt{N}$. Since we want to improve accuracy by a factor of 10, we want to have $\sigma/\sqrt{N} = (1/10)(\sigma/\sqrt{n})$. After some algebra, this gives us $N = 100n$. In other words we need to do 100 times as many measurements!

Solutions for Decision Trees and Recursion

DT-1.1 PREV: C, CC, CCV, CCVC, CCVCC, CCCVCV, CV, CVC, CVCC, CVCCV, CVCV, CVCVC, V, VC, VCC, VCCV, VCCVC, VCV, VCVC, VCVCC, VCVCV.
POSV: CCVCC, CCVCV, CCVC, CCV, CC, CVCCV, CVCC, CVCVC, CVCV, CVC, CV, C, VCCVC, VCCV, VCC, VCVCC, VCVCV, VCVC, VCV, VC, V.
BFV: C, V, CC, CV, VC, CCV, CVC, VCC, VCV, CCVC, CVCC, CVCV, VCCV, VCVC, CCVCC, CCVCV, CVCCV, CVCVC, VCCVC, VCVCC, VCVCV.

DT-1.2 You will need the decision trees for lex and insertion order for permutations of $\underline{3}$ and $\underline{4}$. The text gives the tree for insertion order for $\underline{4}$, from which the tree for $\underline{3}$ can be found — just stop one level above the leaves of $\underline{4}$. You should construct the tree for lex order.

(a) To answer this, compare the leaves. For $n = 3$, permutations $\sigma = 123$, 132, and 321 have $\mathrm{RANK}_L(\sigma) = \mathrm{RANK}_I(\sigma)$. For $n = 4$ the permutations $\sigma = 1234$, 1243, and 4321 have $\mathrm{RANK}_L(\sigma) = \mathrm{RANK}_I(\sigma)$.

(b) From the tree for (a), $\mathrm{RANK}_L(2314) = 8$.
Rather than draw the large tree for $\underline{5}$, we use a smarter approach to compute $\mathrm{RANK}_L(45321) = 95$. To see the latter, Note that all permutations on $\underline{5}$ that start with 1, 2, or 3 come before 45321. There are $3 \times 24 = 72$ of those. This leads us to the subtree for permutations of $\{1, 2, 3, 5\}$ in lex order. It looks just like the decision tree for $\underline{4}$ with 4 replaced by 5. (Why is this?) Since $\mathrm{RANK}_L(4321) = 23$, this makes a total of $72 + 23 = 95$ permutations that come before 45321 and so $\mathrm{RANK}_L(45321) = 95$. If you find this unclear, you should try to draw a picture to help you understand it.

(c) $\mathrm{RANK}_I(2314) = 16$. What about $\mathrm{RANK}_I(45321)$? First does 1, then 2, and so on. After have done all but 5, we are at the rightmost leaf of the tree for $\underline{4}$. It has 23 leaves to the left of it. When we insert 5, each of these leaves is replaced by 5 new leaves because there are 5 places to insert 5. This gives us $5 \times 23 = 115$ leaves. Finally, of the 5 places we could insert 5 into 4321, we chose the 4th so there are 3 additional leaves to the left of it. Thus the rank is $115 = 3 = 118$.

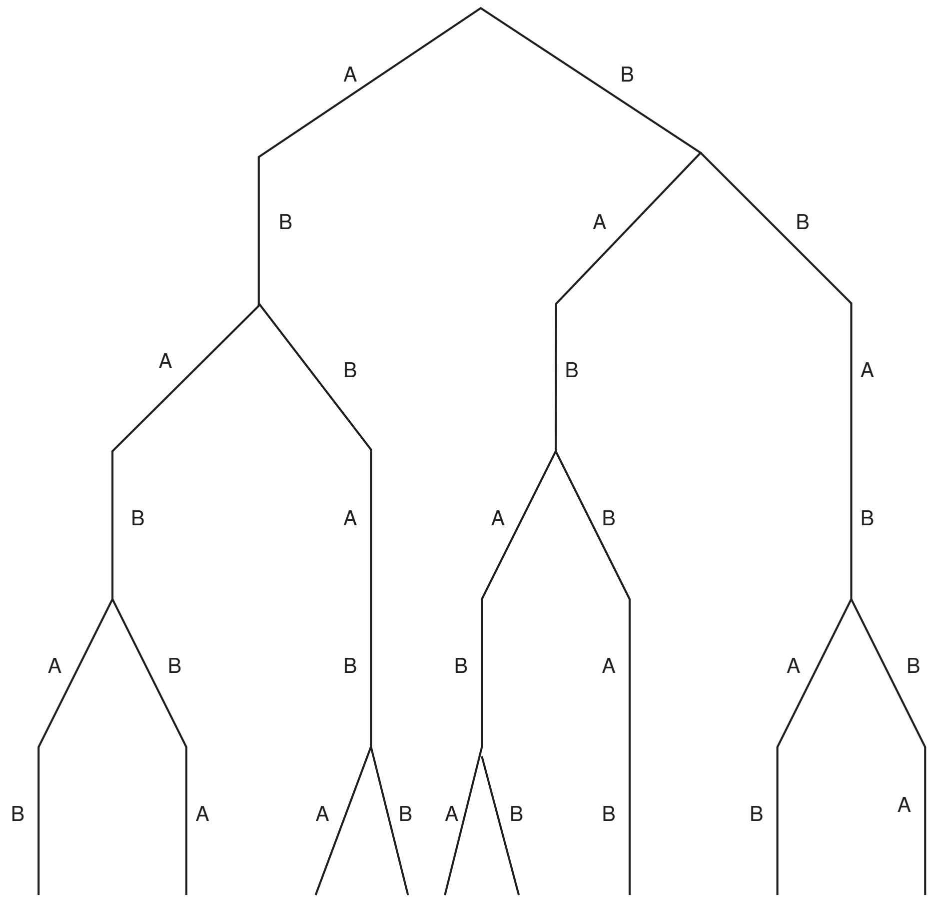
(d) $\mathrm{RANK}_L(3241) = 15$.

(e) $\mathrm{RANK}_I(4213) = 15$.

(f) The first 24 permutations on $\underline{5}$ consist of 1 followed by a permutation on $\{2, 3, 4, 5\}$. Since our goal is the permutation of rank 15, it is in this set. By (d), RANK_L of 3241 is 15 for $n = 4$. Thus $\mathrm{RANK}_L(4352) = 15$ in the lex list of permutations on $\{2, 3, 4, 5\}$.

Solutions for Decision Trees and Recursion

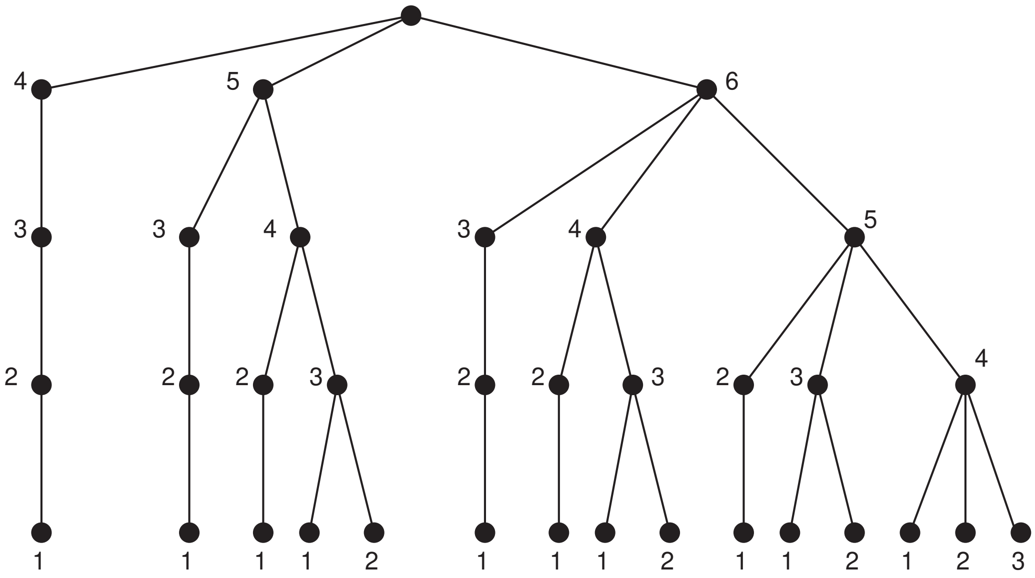
DT-1.3 Here is the tree



The list in lex order:

ABABAB ABABBA, ABBABA, ABBABB BABABA BABABB BABBAB BBABAB BBABBA

DT-1.4 Here is a decision tree for $D(\underline{6}^4)$. The leaves correspond to the elements of $D(\underline{6}^4)$ in lex order, obtained by reading the sequence of vertex labels from the root to the leaf.



(a) The rank of 5431 is 3. The rank of 6531 is 10.

(b) 4321 has rank 0 and 6431 has rank 7.

(c) The first 5 leaves correspond to $D(\underline{5}^{\underline{4}})$.

(d) $D(\underline{6}^{\underline{4}})$ is bijectively equivalent to the set, $\mathbf{P}(\underline{6},4)$, of all subsets of $\underline{6}$ of size 4. Under this bijection, an element such as $5431 \in D(\underline{6}^{\underline{4}})$ corresponds to the set $\{1,3,4,5\}$.

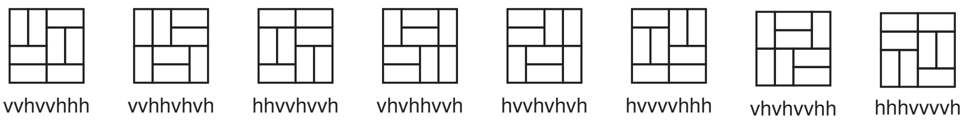
DT-1.5 For PREV and POSV, omit Step 2. For PREV, begin Step 3 with the sentence

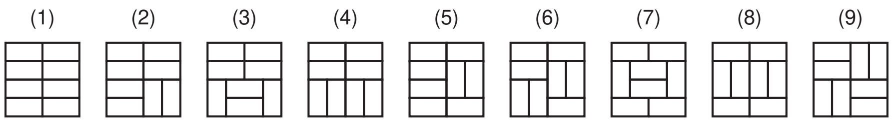
> "If you have not used any edges leading out from the vertex, list the vertex."

For POSV, change Step 3 to

> "If there are no unused edges leading out from the vertex, list the vertex and go to Step 4; otherwise, go to Step 5."

DT-1.6 The problem is that the eight hibachi grills, though different as domino coverings, are all equivalent or "isomorphic" once they are made into grills. All eight in the first row below can be gotten by rotating and/or turning over the first grill.





There are nine different grills as shown in the picture. These nine might be called a "representative system" for the domino coverings up to "grill equivalence." Note that these nine representatives are listed in lex order according to their codes (starting with hhhhhhhh and ending with hvvhvvhh). They each have another interesting property: each one is lexicographically minimal among all patterns equivalent to it. The one we selected from the list of "screwup" grills (number (6)) has code hhhvvvvh and that is minimal among all codes on the first row of coverings.

This problem is representative of an important class of problems called "isomorph rejection problems." The technique we have illustrated, selecting a lex minimal system of representatives up to some sort of equivalence relation, is an important technique in this subject.

DT-2.1 We refer to the decision tree in Example 10. The permutation 87612345 specifies, by edge labels, a path from the root L($\underline{8}$) to a leaf in the decision tree. To compute the rank, we must compute the number of leaves "abandoned" by each edge just as was done in Example 14. There are eight edges in the path with the number of abandoned leaves equal to $7\times7!+6\times6!+5\times5!+0+0+0+0+0 = 35,280+4,320+600 = 40,200$. This is the RANK of 87612345 in the lex list of permutations on $\underline{8}$. Note that $8! = 40,320$, so the RANK 20,160 permutation is the first one of the second half of the list: 51234678.

DT-2.2 (a) The corresponding path in the decision tree is H(8, S, E, G), H(7, E, S, G), H(6, S, E, G), H(5, S, G, E), H(4, S, E, G), H(3, E, S, G), E $\xrightarrow{3}$ G.

(b) The move that produced the configuration of (a) was E $\xrightarrow{3}$ G. The configuration prior to that was Pole S: 6, 5, 2, 1; Pole E: 3; Pole G: 8, 7, 4.

(c) The move just prior to E $\xrightarrow{3}$ G was G $\xrightarrow{1}$ S. This is seen from the decision tree structure or from the fact that the smallest washer, number 1, moves every other time in the pattern S, E, G, S, E, G, etc. The configuration just prior to the move G $\xrightarrow{1}$ S was Pole S: 6, 5, 2; Pole E: 3; Pole G: 8, 7, 4, 1.

(d) The next move after E $\xrightarrow{3}$ G will be another move by washer 1 in its tiresome cycle S, E, G, S, E, G, etc. That will be S $\xrightarrow{1}$ E.

(e) The RANK of the move that produced (a) can be computed by summing the abandoned leaves associated with each edge of the path (a) in the decision tree. (See Example 14.) There are six edges in the path of part (a) with associated abandoned leaves being $2^7 = 128$, $2^6 = 64$, 0, 0, $2^3 = 8$, $2^2 - 1 = 3$. The total is 203.

DT-2.3 (a) 110010000 is preceded by 110010001 and is followed by 110110000. You can find this by first drawing the path from the root to 110110000. You will find that the last edge of the path goes to the right. Therefore, we can get the preceding element by going to the left instead. This changes the last element from 0 to 1 and all other elements remain fixed. To get the element that follows it, we want to branch to the right instead of the left. The last five edges to 110110000 all go to the right and the edge just before them, say e goes to the left. Instead of taking e, we take the edge that goes to the right. Now what? We must take edges to the left after this so that we end up as close to the original leaf 110010000 as possible. A trick: Since we are dealing with a Gray code, we know that there is only one change so that when we've found it we can just copy everything else. In this case we changed the underlined symbol in 110$\underline{0}$10000 (from 0 to 1) and so the others are the same.

(b) The first element of the second half of the list corresponds to a path in the decision tree that starts with a right-sloping edge and has all of the remaining eight edges left-sloping. That element is 110000000.

(c) Each right-sloping edge abandons 2^{n-k} leaves, if the edge is the k^{th} one in the path. For the path 111111111 the right-sloping edges are numbers 1, 3, 5, 7, and 9 (remember, after the first edge, a label 1 causes the direction of the path to change). Thus, the rank of 111111111 is $2^8 + 2^6 + 2^4 + 2^2 + 2^0 = 341$.

(d) To compute the element of RANK 372, we first compute the path in the decision tree that corresponds to the element. The first edge must be **(1) right sloping** (abandoning 256 leaves), since the largest rank of any leaf at the end of a path that starts left sloping is $2^8 - 1 = 255$. We apply this same reasoning recursively. The right sloping edge leads to 256 leaves. We wish to find the leaf of RANK $372 - 256 = 116$ in that list of 256 leaves. That means the second edge must be **(1) left sloping** (abandoning 0 leaves), so our path starts off **(1) right sloping**, **(1) left sloping**. This path can access 128 leaves. We want the leaf of RANK $116 - 0$ in this list. Thus we must access a leaf in the second half of the list of 128, so the third edge must be **(1) right sloping** (abandoning 64 leaves). In that second half we must find the leaf of RANK $116 - 64 = 52$.

Our path is now **(1) right sloping**, **(1) left sloping**, **(1) right sloping**. Following that path leads to 64 leaves of which we want the leaf of RANK 52. Thus, the fourth edge must be **(0) right sloping** (abandoning 32 leaves). This path of four edges leads to 32 leaves of which we must find the one of RANK $52 - 32 = 20$. Thus the fifth edge must also be **(0) right sloping** (abandoning 16 leaves). Thus we must find the leaf of RANK $20 - 16 = 4$. This means that the sixth edge must be **(1) left sloping** (abandoning 0 leaves), the seventh edge must be **(1) right sloping** (abandoning 4 leaves), and the last two edges must be left sloping: **(1) left sloping** (abandoning 0 leaves), **(0) left sloping** (abandoning 0 leaves). Thus the final path is 111001110.

DT-2.4 (a) Let $\mathcal{A}(n)$ be the assertion "H(n,S,E,G) takes the least number of moves." Clearly $\mathcal{A}(1)$ is true since only one move is required. We now prove $\mathcal{A}(n)$. Note that to do $\mathrm{S} \xrightarrow{\mathrm{n}} \mathrm{G}$ we must first move all the other washers to pole E. They can be stacked only one way on pole E, so moving the washers from S to E requires using a solution to the Towers of Hanoi problem for $n-1$ washers. By $\mathcal{A}(n-1)$, this is done in the least number of moves by H($n-1$,S,G,E). Similarly, H($n-1$,E,S,G) moves these washers to G in the least number of moves.

(b) For $n = 1,\ f_1 = 1$: $\mathrm{S} \xrightarrow{1} \mathrm{G}$
For $n = 2,\ f_2 = 3$: $\mathrm{S} \xrightarrow{1} \mathrm{E},\ \mathrm{S} \xrightarrow{2} \mathrm{G},\ \mathrm{E} \xrightarrow{1} \mathrm{G}$
For $n = 3,\ f_3 = 5$: $\mathrm{S} \xrightarrow{1} \mathrm{E},\ \mathrm{S} \xrightarrow{2} \mathrm{F},\ \mathrm{S} \xrightarrow{1} \mathrm{G},\ \mathrm{F} \xrightarrow{2} \mathrm{G},\ \mathrm{E} \xrightarrow{1} \mathrm{G}$

(c) Let $s(p,q)$ be the number of moves for G(p, q, S, E, F, G). The recursive step in the problem is described for $p > 0$, so the simplest case is $p = 0$ and $s(0,q) = h(q) = 2^q - 1$. In that case, (i) tells us what to do.

Otherwise, the number of moves in (ii) is $s(p,q) = 2s(i,j) + h_q$. To find the minimum, we look at all allowed values of i and j, choose those for which $s(i,j)$ is a minimum. This choice of i and j, when used in (ii) tells us which moves to make. In the following table, numbers on the rows refer to p and those on the columns refer to q.

Except for the s_p column, then entries are $s(p,q)$. The $p=0$ row is h_q by (i). To find $s(p,q)$ for $p>0$, we use (ii). To do this, we look along the diagonal whose indices sum to p, choose the minimum (It's location is (i,j).), double it and add h_q. For example, $s(5,2)$ is found by taking the minimum of the diagonal entries at (0,5), (1,4), (2,3), (3,2), and (4,1). Since these entries are 31, 17, 13, 13, and 19, the minimum is 13. Since this occurs at (2,3) and (3,2), we have a choice for (i,j). Either one gives us $2\times 13+h_2=29$ moves. To compute s_n we simply look along the $p+q=n$ diagonal and choose the minimum.

	s_p	1	2	3	4	5	6	(values of q)
0		1	3	7	15	31	63	$(s(0,q)=h_q)$
1	1	3	5	9	17	33	65	
2	3	7	9	13	21	27		
3	5	11	13	17	25			
4	9	19	21	25				
5	13	27	29					
6	17	35						Column labels are p.

(d) From the description of the algorithm,

- $s(p,q)=2\min s(i,j)+h_q$, where the minimum is over $i+j=p$ and
- $s_n=\min s(p,q)$, where the minimum is over $p+q=n$.

Putting these together gives us $s(p,q)=2s_p+h_q$ and so $s_n=\min(2s_p+h_q)$. The initial condition is $s_0=0$. In summary

$$s_n=\begin{cases} 0 & \text{if } n=0,\\ \displaystyle\min_{\substack{p+q=n\\ q>0}}(2s_p+h_q) & \text{if } n>0.\end{cases}$$

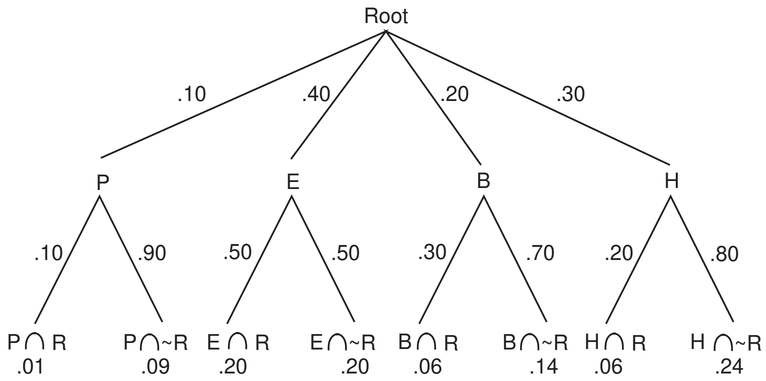
(e) Change the recursive procedure in the algorithm to use the moves for f_p instead of using those for $s(p,q)$. It follows that we can solve the puzzle in $2f_{n-j}+h_j$ moves.

DT-3.1 When there is replacement, the result of the first choice does not matter since the ball is placed back in the box. Hence the answer to both parts of (a) is 3/7.

(b) If the first ball is green, we are drawing a ball from three white and three green and so the probability is $3/6=1/2$. If the first ball is white, we are drawing a ball from two white and four green and so the probability is $2/6=1/3$.

DT-3.2 There are five ways to get a total of six: $1+5$, $2+4$, $3+3$, $4+2$, and $5+1$. All five are equally likely and so each outcome has probability 1/5. We get the answers by counting the number that satisfy the given conditions and multiplying by 1/5:
(a) 1/5, (b) 2/5, (c) 3/5.

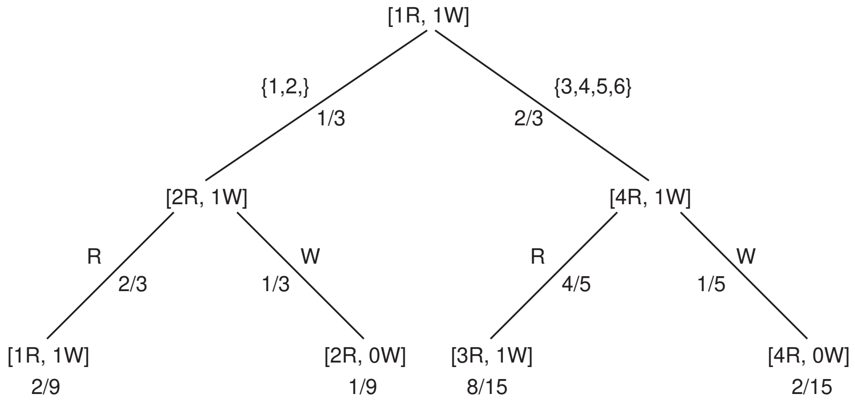
DT-3.3 Here is the decision tree for this problem



(a) We want to compute the conditional probability that a student is a humanities major, given that that student has read Hamlet. In the decision tree, if we follow the path from the Root to H to $H \cap R$, we get a probability of .06 at the leaf. We must divide this by the sum over all probabilities of such paths that end at $X \cap R$ (as opposed to $X \cap \sim R$). That sum is $0.01 + 0.20 + 0.06 + 0.06 = 0.33$. The answer is $0.06/0.33 = 0.182$.

(b) We compute the probabilities that a student has not read Hamlet and is a P (Physical Science) or E (Engineering) major: $0.09 + 0.20 = 0.29$. We must divide this by the sum over all probabilities of such paths that end at $X \cap \sim R$ (as opposed to $X \cap R$). The answer is $0.29/0.67 = 0.433$.

DT-3.4 Here is a decision tree where the vertices are urn compositions. The edges incident on the root are labeled with the outcome sets of the die and the probabilities that these sets occur. The edges incident on the leaves are labeled with the color of the ball drawn and the probability that such a ball is drawn. The leaves are labeled with the product of the probabilities on the edges leading from the root to that leaf.

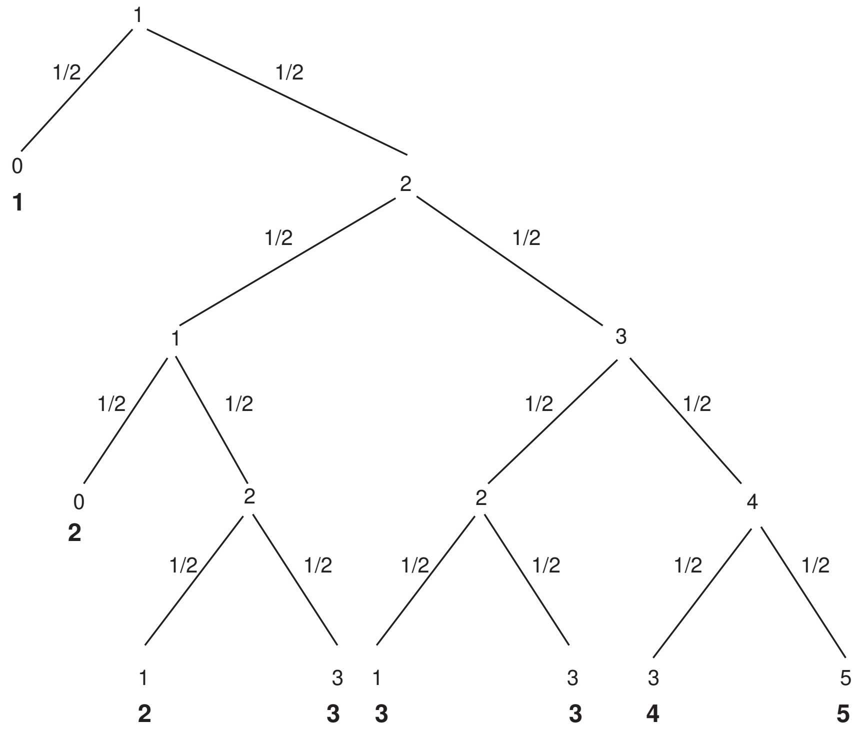


(a) To compute the conditional probability that a 1 or 2 appeared, given that a red ball was drawn, we take the probability 2/9 that a 1 or 2 appeared and a red ball was drawn and divide by the total probability that a red ball was drawn: $2/9 + 8/15 = 34/45$. The answer is $5/17 = 0.294$.

(b) We divide the probability that a 1 or 2 appeared and the final composition had more than one red ball (1/9) by the sum of the probabilities where the final composition had more than one red ball : $1/9 + 8/15 + 2/15 = 7/9 = 0.78$.

DT-3.5 A decision tree is shown below. The values of the random variable X are shown just below the amount remaining in the pot associated with each leaf. To compute $E(X)$ we sum the values of X times the product of the probabilities along the path from the root to that value of X. Thus, we get

$$E(X) = 1 \times (1/2) + 2 \times (1/8) + (2+3+3+3+4+5) \times (1/16) = 2\,.$$



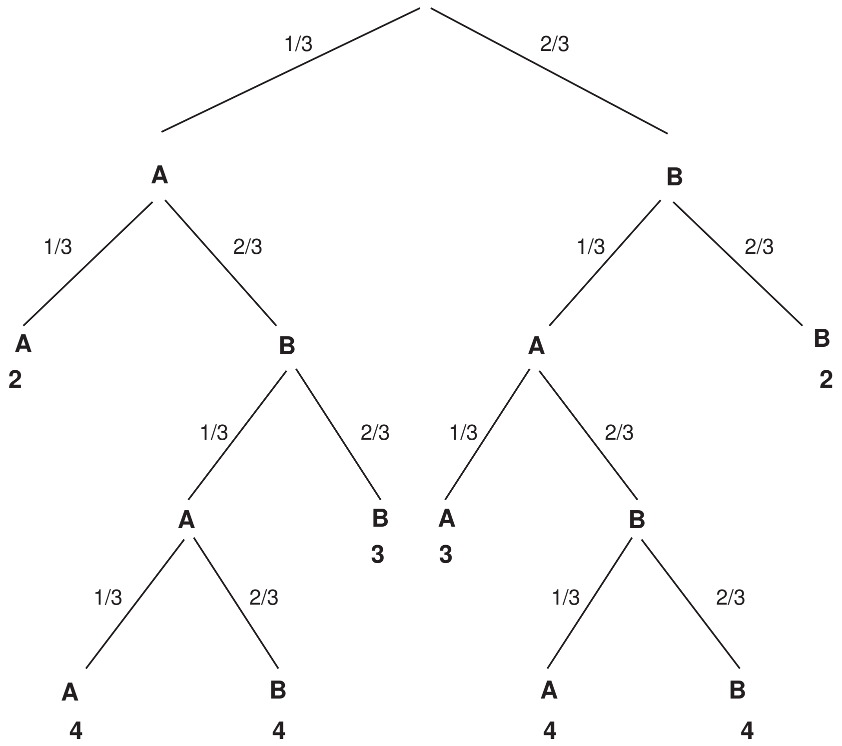
DT-3.6 A decision tree is shown below. Under the leaves is the length of the game (the height of the leaf). The expected length of the game is the sum of the products of the probabilities on the edges of each path to a leaf times the height of that leaf:

$$2((1/3)^2 + (2/3)^2)+$$

$$4((1/3)^3(2/3) + (1/3)^2(2/3)^2 + (1/3)^2(2/3)^2 + (1/3)(2/3)^3)+$$

$$3((1/3)(2/3)^2 + (1/3)^2(2/3)\,.$$

The expected number of games is about 2.69.



DT-3.7 We are given

$$P(F' \mid A) = 0.6, \quad P(F \mid A') = 0.8 \quad \text{and} \quad P(A) = 0.7\,.$$

You can draw a decision tree. The first level branches according as the air strike is successful (A) or not (A'). The probabilities, left to right, are 0.7 and $1 - 0.7 = 0.3$. The second level branches according as there is enemy fire (F) or not (F'). To compute the conditional probabilities on the edges, note that

$$P(F \mid A) = P(F' \mid A) = 1 - 0.6 = 0.4 \quad \text{and} \quad P(F' \mid A') = 1 - 0.8 = 0.2\,.$$

The leaves and their probabilities are

$$P(A \cap F) = 0.7 \times 0.4 = 0.28, \qquad P(A \cap F') = 0.7 \times 0.6 = 0.42,$$

$$P(A' \cap F) = 0.3 \times 0.8 = 0.24, \qquad P(A' \cap F') = 0.3 \times 0.2 = 0.06.$$

For (a), $P(F') = 0.42 + 0.06 = 0.48$ and for (b)

$$P(A \mid F') = \frac{P(A \cap F')}{P(F')} = \frac{0.42}{0.48} \approx 82\%.$$

DT-4.1 (a) $a_n = 1$ for all n.

(b) $a_0 = 0$, $a_1 = 0 + a_0 = 0$, $a_2 = 1 + a_1 = 1$, $a_3 = 1 + a_2 = 2$, $a_4 = 2 + a_3 = 4$.

(c) $a_0 = 1$, $a_1 = 1 + a_0 = 2$, $a_2 = 2 + a_1 = 4$, $a_3 = 3 + a_1 = 5$, $a_4 = 4 + a_2 = 8$.

(d) $a_0 = 0$, $a_1 = 1$, $a_2 = 1 + a_1a_1 = 2$, $a_3 = 1 + \min(a_1a_2, a_2a_1) = 1 + a_1a_2 = 3$, $a_4 = 1 + \min(a_1a_3, a_2a_2, a_3, a_1) = 1 + \min(3, 4) = 4$.

DT-4.2 $a_n = \lfloor n/2 \rfloor$, $b_n = (-1)^n \lfloor 1 + (n/2) \rfloor = (-1)^n(1 + \lfloor n/2 \rfloor)$, $c_n = n^2 + 1$, $d_n = n!$.

DT-4.3 $x^2 - 6x + 5 = 0$ has roots $r_1 = 1$ and $r_2 = 5$
$x^2 - x - 2 = 0$ has roots $r_1 = -1$ and $r_2 = 2$
$x^2 - 5x - 5 = 0$ has roots $\frac{5 \pm \sqrt{45}}{2}$.

DT-4.4 The characteristic equation is $x^2 - 6x + 9 = 0$, which factors as $(x - 3)^2 = 0$. Thus $r_1 = r_2 = 3$. We have $K_1 = a_0 = 0$ and $3K_2 = a_1 = 3$. Thus $a_n = n3^n$.

DT-4.5 Let $A_n = a_{n+2}$ so that $A_0 = 1$, $A_1 = 3$ and $A_n = 3A_{n-1} + 2A_{n-2}$ for $n > 2$. The characteristic equation is $x^2 - 3x - 2 = 0$ and has roots $r_1 = 1$, $r_2 = 2$. Thus $K_1 + K_2 = 1$ and $K_1 + 2K_2 = 3$ and so $K_1 = -1$ and $K_2 = 2$. We have $A_n = -1 + 2 \times 2^n = 2^{n+1} - 1$ and so $a_n = A_{n-2} = 2^{n-1} - 1$.

DT-4.6 The characteristic equation is $x^2 - 2x + 1 = (x - 1)^2 = 0$. Thus $r_1 = r_2 = 1$ and so $K_1 = a_0 = 2$ and $K_1 + K_2 = a_1 = 1$. We have $K_2 = 1 - K_1 = -1$ and so $a_n = 2 - n$.

DT-4.7 (a) Let $\mathcal{A}(n)$ be the assertion that $G(n) = (1 - A^n)/(1 - A)$. When $n = 1$, $G(1) = 1$ and $(1 - A^n)/(1 - A) = 1$, so the base case is proved. For $n > 1$, we have

$$\begin{aligned} G(n) &= 1 + A + A^2 + \ldots + A^{n-1} && \text{by definition,} \\ &= (1 + A + A^2 + \ldots + A^{n-2}) + A^{n-1} \\ &= \frac{1 - A^{n-1}}{1 - A} + A^{n-1} && \text{by } \mathcal{A}(n-1), \\ &= \frac{1 - A^n}{1 - A} && \text{by algebra.} \end{aligned}$$

(b) The recursion can be found by looking at the definition or by examining the proof in (a). It is $G(1) = 1$ and, for $n > 1$, $G(n) = G(n - 1) + A^{n-1}$.

(c) Applying the theorem is straightforward. The formula equals 1 when $n = 1$, which agrees with $G(1)$. By some simple algebra

$$\frac{1 - A^{n-1}}{1 - A} + A^{n-1} = \frac{(1 - A^{n-1}) + (A^{n-1} - A^n)}{1 - A} = \frac{1 - A^n}{1 - A},$$

and so the formula satisfies the recursion.

(d) Letting $A = y/x$ and cleaning up some fractions

$$\frac{1 - (y/x)^n}{1 - y/x} = \frac{y^n - x^n}{x - y}\, x^{n-1}.$$

Let $n = k + 1$, multiply by x^k and use the geometric series to obtain

$$\begin{aligned} \frac{x^{k+1}D - y^{k+1}}{x - y} &= x^k \Big(1 + (y/x) + (y/x)^2 + \cdots + (y/x)^k\Big) \\ &= x^k y^0 + x^{k-1} y^1 + \cdots + x^0 y^k. \end{aligned}$$

DT-4.8 We will Theorem 7 to prove our conjectures are correct.

(a) Writing out the first few terms gives A, $A/(1+A)$, $A/(1+2A)$, $A/(1+3A)$, etc. It appears that $a_k = A/(1+kA)$. Since $A > 0$, the denominators are never zero. When $k = 0$, $A/(1+kA) = A$, which satisfies the initial condition. We check the recursion:

$$\frac{A/(1+(k-1)A)}{1+A/(1+(k-1)A)} = \frac{A}{(1+(k-1)A)+A} = A/(1+kA),$$

which is the conjectured value for a_k.

(b) Writing out the first few terms gives C, $AC+B$, $A^2C+AB+B$, $A^3C+A^2B+AB+B$, $A^4C+A^3B+A^2B+AB+B$, etc. Here is one possible formula:

$$a_k = A^kC + B(1+A+A^2+\ldots+A^{k-1}).$$

Here is a second possibility:

$$a_k = A^kC + B\left(\frac{1-A^k}{1-A}\right).$$

Using the previous exercise, you can see that they are equal. We leave it to you to give a proof of correctness for both formulas, without using the previous exercise.

DT-4.9 We use Theorem 7. The formula gives the correct value for $k = 0$. The recursion checks because

$$\begin{aligned} A + B(k-1)\Big(((k-1)^2-1)/3\Big) + Bk(k-1) &= A + B(k-1)\Big((k^2-2k+1-1)-3k\Big) \\ &= A + B(k-1)k(k+1)/3 \\ &= A + Bk(k^2-1)/3. \end{aligned}$$

This completes the proof.

DT-4.10 (a) We apply Theorem 7, but there is a little complication: The formula starts at $k = 1$, so we cannot check the recursion for $k = 1$. Thus we need a_1 to be the initial condition. From the recursion, $a_1 = 2A - C$, which we take as our initial condition and use the recursion for $k > 1$. You should verify that the formula gives a_1 correctly and that the formula satisfies the recursion when $k > 1$.

(b) From the last part of Exercise 4.7 with $x = 2$ and $y = -1$, we obtain

$$a_k = A\left(\frac{2^{k+1}-(-1)^{k+1}}{3}\right) + (-1)^k(C-A).$$

Make sure you can do the calculations to derive this.

DT-4.11 Let p_k denote the probability that the gambler is ruined if he starts with $0 \le k \le Q$ dollars. Note that $p_0 = 1$ and $p_Q = 0$. Assume $1 < k \le Q$. Then the recurrence relation $p_{k-1} = (1/2)p_k + (1/2)p_{k-2}$ holds. Solving for p_k gives $p_k = 2p_{k-1} - p_{k-2}$. This looks familiar. It is a two term linear recurrence relation. But the setup was a little strange! We would expect to know p_0 and p_1 and would expect the values of p_k

to make sense for all $k \geq 0$. But here we have an interpretation of the p_k only for $0 \leq k \leq Q$ and we know p_0 and p_Q instead of p_0 and p_1. Such a situation is not for faint-hearted students.

We are going to keep going as if we knew what we were doing. The characteristic equation is $r^2 - 2r + 1 = 0$. There is one root, $r = 1$. That means that the sequence $a_k = 1$, for all $k = 0, 1, 2, \ldots$, is a solution and so is $b_k = k$, for $k = 0, 1, 2, \ldots$. We need to find A and B such that $Aa_0 + Bb_0 = 1$ and $Aa_Q + Bb_Q = 0$. We find that $A = 1$ and $B = -1/Q$. Thus we have the general solution

$$p_k = 1 - \frac{k}{Q} = \frac{Q-k}{Q} \qquad q_k = \frac{k}{Q}\,.$$

Note that p_k is defined for all $k \geq 0$ like it would be for any such linear two term recurrence. The fact that we are only interested in it for $0 \leq k \leq Q$ is no problem to the theory.

Suppose a rich student, Brently Q. Snodgrass the III, has $8,000$ dollars and he wants to play the coin toss game to make $10,000$ dollars so he has $2,000$ his parents don't know about. His probability of being ruined is $(10,000 - 8000)/10000 = 1/5$. His probability of getting his extra 2000 dollars is $4/5$. A poor student who only had 100 dollars and wanted to make 2000 dollars would have a probability of $(2,100 - 100)/2,100 = 0.95$ of being ruined. Life isn't fair.

There is one consolation. The expected number of times Brently will have to toss the coin to earn his 2,000 dollars is $16,000,000$. It will take him 69.4 weeks tossing 40 hours per week, one toss every 10 seconds. If he does get his 2000 dollars, he will have been working as a "coin tosser" for over a year at a salary of 72 cents per hour. He should get a minimum wage job instead!

Solutions for Basic Concepts in Graph Theory

GT-1.1 To specify a graph we must choose $E \in \mathcal{P}_2(V)$. Let $N = |\mathcal{P}_2(V)|$. (Note that $N = \binom{n}{2}$.) There are 2^N subsets E of $\mathcal{P}_2(V)$ and $\binom{N}{q}$ of them have cardinality q. This proves (a) and answers (b).

GT-1.2 The sum is the number of ends of edges since, if x and y are the ends of an edge, the edge contributes 1 to the value of $d(x)$ and 1 to the value of $d(y)$. Since each edge has two ends, the sum is twice the number of edges.

Since $\sum_v d(v)$ is even if and only if the number of odd summands is even, it follows that there are an even number of v for which $d(v)$ is odd.

GT-1.3 (a) The graph is isomorphic to Q. The correspondence between vertices is given by

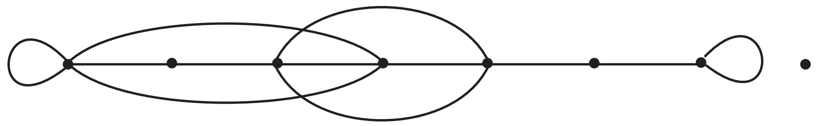
$$\phi = \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & A & C & E & F & D & G & B \end{pmatrix}$$

where the top row corresponds to the vertices of Q.

(b) The graph Q' is not isomorphic to Q. It can be made isomorphic by deleting one edge and adding another. You should try to figure out which edges these are.

GT-1.4 (a) $(0,2,2,3,4,4,4,5)$ is the degree sequence of Q. (b) If a pictorial representation of R can be created by labeling $P'(Q)$ with the edges and vertices of R, then R has degree sequence $(0,2,2,3,4,4,4,5)$ because the degree sequence is determined by ϕ.

(c) This is the converse of (b). It is false. The following graph has degree sequence $(0,2,2,3,4,4,4,5)$ but cannot be morphed into the form $P'(Q)$.



GT-1.5 (a) There is no graph Q with degree sequence $(1,1,2,3,3,5)$ since the sum of the degrees is odd. The sum of the degrees of a graph is $2|E|$ and must, therefore, be even.

(d) (answers (b) and (c) as well) There is a graph with degree sequence $(1,2,2,3,3,5)$, no loops or parallel edges allowed. Take

$$\phi = \begin{pmatrix} a & b & c & d & e & f & g & h \\ A & B & C & A & B & C & E & F \\ B & C & E & D & D & D & D & D \end{pmatrix}.$$

(e) (answers (f) as well) A graph with degree sequence $(3,3,3,3)$ has $(3+3+3+3)/2 = 6$ edges and, of course 4 vertices. That is the maximum $\binom{4}{2}$ of edges that a simple graph with 4 vertices can have. It is easy to construct such a graph. Draw the four vertices and make all possible connections. This graph is called the *complete* graph on 4 vertices.

(g) There is no simple graph (or graph without loops or parallel edges) with degree sequence $(3,3,3,5)$. See (f).

(h) Similar arguments to (f) apply to the complete graph with degree sequence $(4,4,4,4,4)$. Such a graph would have $20/2 = 10$ edges. But $\binom{5}{2} = 10$. To construct such a graph, use 5 vertices and make all possible connections.

(i) There is no such graph. See (h).

GT-1.6 Each of (a) and (c) has just one pair of parallel edges (edges with the same endpoints), while (b) and (d) each have two pairs of parallel edges. Thus neither (b) nor (d) is equivalent to (a) or (c). Vertex 1 of (b) has degree 4, but (d) has no vertices of degree 4. Thus (b) and (d) are not equivalent. It turns out that (a) and (c) are equivalent. Can you see how to make the forms correspond?

GT-1.7 (a) We know that the expected number of triangles behaves like $(np)^3/6$. This equals 1 when $p = 6^{1/3}/n$.

(b) By Example 6, the expected number of edges is $\binom{n}{2}p$, which behaves like $(n^2/2)p$ for large n. Thus we expect about $(6^{1/3}/2)n$

GT-1.8 Introduce random variables X_S, one for each $S \in \mathcal{P}_k(V)$. Reasoning as in the example, $E(X_S) = p^K$ where $K = \binom{k}{2}$, the number of edges that must be present. Thus the expected number of sets of k vertices with all edges present is $\binom{n}{k}p^K$.

For large n, this behaves like $n^k p^K/k!$, which will be 1 when $p = (k!/n^k)^{1/K}$. For large n, the expected number of edges behaves like $(n^2/2)(k!/n^k)^{1/K}$. This last number has the form Cn^α where $C = (k!)^{1/K}/2$ and $\alpha = 2 - k/K = 2 - 2/(k-1) = \frac{2(k-2)}{k-1}$.

GT-1.9 The first part comes from factoring out $\binom{n}{3}p^3$ from the last equation in Example 7. To obtain the inequality, replace $(1-p^3)$ with $(1-p^2)$, factor it out, and use $1+3(n-3) < 3n$.

GT-2.1 Since $E \subseteq \mathcal{P}_2(V)$, we have a simple graph. Regardless of whether you are in set C or S, following an edge takes you into the other set. Thus, following a path with an odd number of edges takes you to the opposite set from where you started while a path with an even number of edges takes you back to your starting set. Since a cycle returns to its starting vertex, it obviously returns to its starting set.

GT-2.2 (a) The graph is not Eulerian. The longest trail has 5 edges, the longest circuit has 4 edges.

(b) The longest trail has 9 edges, the longest circuit has 8 edges.

(c) The longest trail has 13 edges (an Eulerian trail starting at C and ending at D). The longest circuit has 12 edges (remove edge f).

(d) This graph has an Eulerian circuit (12 edges).

GT-2.3 (a) The graph is Hamiltonian.

(b) The graph is Hamiltonian.

(c) The graph is not Hamiltonian. There is a cycle that includes all vertices except K.

(d) The graph is Hamiltonian.

GT-2.4 (a) There are $|V \times V|$ potential edges to choose from. Since there are two choices for each edge (either in the digraph or not), we get 2^{n^2} simple digraphs.

(b) With loops forbidden, our possible edges include all elements of $V \times V$ except those of the form (v,v) with $v \in V$. Thus there are $2^{n(n-1)}$ loopless simple digraphs. An alternative derivation is to note that a simple graph has $\binom{n}{2}$ edges and we have 4 possible choices in constructing a digraph: (i) omit the edge, (ii) include the edge directed one way, (iii) include the edge directed the other way, and (iv) include two edges, one directed each way. This gives $4^{\binom{n}{2}} = 2^{n(n-1)}$. The latter approach is not useful in doing part (c).

(c) Given the set S of possible edges, we want to choose q of them. This can be done in $\binom{|S|}{q}$ ways. In the general case, the number is $\binom{n^2}{q}$ and in the loopless case it is $\binom{n(n-1)}{q}$.

GT-2.5 (a) Let $V = \{u,v\}$ and $E = \{(u,v),(v,u)\}$.

(b) For each $\{u,v\} \in \mathcal{P}_2(V)$ we have three choices: (i) select the edge (u,v), (ii) select the edge (v,u) or (iii) have no edge between u and v. Let $N = |\mathcal{P}_2(V)| = \binom{n}{2}$. There are 3^N oriented simple graphs.

(c) We can choose q elements of $\mathcal{P}_2(V)$ and then orient each of them in one of two ways. This gives us $\binom{N}{q}2^q$.

GT-2.6 (a) For all $x \in S$, $x|x$. For all $x,y \in S$, if $x|y$ and $x \neq y$, then y does not divide x. For all $x,y,z \in S$, $x|y$, $y|z$ implies that $x|z$.

(b) The covering relation is

$$\begin{aligned} H = \{&(2,4),(2,6),(2,10),(2,14),(3,6),(3,9),(3,15),\\ &(4,8),(4,12),(5,10),(5,15),(6,12),(7,14)\}. \end{aligned}$$

We leave it to you to draw the picture!

GT-3.1 (a) Suppose G is a connected graph with v vertices and v edges. A connected graph is a tree if and only if the number of vertices is one more than the number of edges. Thus G is not a tree and must have at least one cycle. This proves the base case, $n = 0$. Suppose $n > 0$ and G is a graph with v vertices and $v + n$ edges. We know that the graph is not a tree and thus has a cycle. We know that removing an edge from a cycle does not disconnect the graph. However, removing the edge destroys any cycles that contain it. Hence the new graph G' contains one less edge and at least one less cycle than G. By the induction hypothesis, G' has at least n cycles. Thus G has at least $n+1$ cycles.

(b) Let G be a graph with components $G_1,\ldots,G_k$. With subscripts denoting components, G_i has v_i vertices, $e_i = v_i + n_i$ edges and at least $n_i + 1$ cycles. From the last two formulas, G_i has at least $1 + e_i - v_i$ cycles. Now sum over i.

(c) For each n we wish to construct a simple graph that has n more edges than vertices but has only $n+1$ cycles. There are many possibilities. Here's one solution. The vertices are v and, for $0 \le i \le n$, x_i and y_i. The edges are $\{v,x_i\}$, $\{v,y_i\}$, and $\{x_i,y_i\}$. (This gives $n+1$ triangles joined at v.) There are $1+2(n+1)$ vertices, $3(n+1)$ edges, and $n+1$ cycles.

GT-3.2 (a) $\sum_{v\in V} d(v) = 2|E|$. For a tree, $|E| = |V| - 1$. Since $2|V| = \sum_{v\in V} 2$,

$$2 = 2|V| - 2|E| = \sum_{v\in V}(2 - d(v)).$$

(b) Suppose that T is more than just a single vertex. Since T is connected, $d(v) \neq 0$ for all v. Let n_k be the number of vertices of T of degree k. By the previous result, $\sum_{k\geq 1}(2-k)n_k = 2$. Rearranging gives $n_1 = 2 + \sum_{k\geq 2}(k-2)n_k$. If $n_m \geq 1$, the sum is at least $m-2$.

(c) Let the vertices be u and v_i for $1 \leq i \leq m$. Let the edges be $\{u, v_i\}$ for $1 \leq i \leq m$.

GT-3.3 (a) No such tree exists. A tree with six vertices must have five edges.

(b) No such tree exists. Such a tree must have at least one vertex of degree three or more and hence at least three vertices of degree one.

(c) A graph with two connected components, each a tree, each with five vertices will have this property.

(d) No such graph exists.

(e) No such tree exists.

(f) Such a graph must have at least $c + e - v = 1 + 6 - 4 = 3$ cycles.

(g) No such graph exists. If the graph has no cycles, then each component is a tree. In such a graph, the number of vertices is strictly greater than the number of edges for each component and hence for the whole graph.

GT-3.4 (a) The idea is that for a rooted planar tree of height h, having at most 2 children for each non-leaf, the tree with the most leaves occurs when each non-leaf vertex has exactly 2 children. You should sketch some cases and make sure you understand this point. For this case $l = 2^h$ and so $\log_2(l) = h$. Any other rooted planar tree of height h, having most 2 children for each non-leaf, is a subtree (with the same root) of this maximal-leaf binary tree and thus has fewer leaves.

(b) Knowing the number of leaves does not bound the height of a tree — it can be arbitrarily large.

(c) The maximum height is $h = l - 1$. One leaf has height 1, one height 2, etc., one of height $l - 2$ and, finally, two of height $l - 1$.

(d) (answers (e) as well) $\lceil \log_2(l) \rceil$ is a lower bound for the height of *any* binary tree with l leaves. It is easy to see that you can construct a full binary tree with l leaves and height $\lceil \log_2(l) \rceil$.

GT-3.5 (a) A binary tree with 35 leaves and height 100 is possible.

(b) A full binary tree with 21 leaves can have height at most 20. So such a tree of height 21 is impossible.

(c) A binary tree of height 5 can have at most 32 leaves. So one with 33 leaves is impossible.

(d) No way! The total number of vertices is

$$\sum_{i=0}^{5} 3^5 = \frac{3^6 - 1}{2} = 364\,.$$

GT-3.6 (a) For *(1)* there are four spanning trees. For *(2)* there are 8 spanning trees. Note that there are $\binom{5}{3} = 10$ ways to choose three edges. Eight of these 10 choices result in spanning trees, the other two choices result in cycles (with vertex sequences (A, B, D) and (B, C, D)). For *(3)* there are 16 spanning trees.

(b) For *(1)* there is one. For *(2)* there are two. For *(3)* there are two.

(c) For *(1)* there are two. For *(2)* there are four. For *(3)* there are six.

(d) For *(1)* there are two. For *(2)* there are three. For *(3)* there are six.

GT-3.7 (a) For *(1)* there are three minimum spanning trees. For *(2)* there are two minimum spanning trees. For *(3)* there is one minimum spanning tree.

(b) For *(1)* there is one minimum spanning tree up to isomorphism. For *(2)* there are two. For *(3)* there is one.

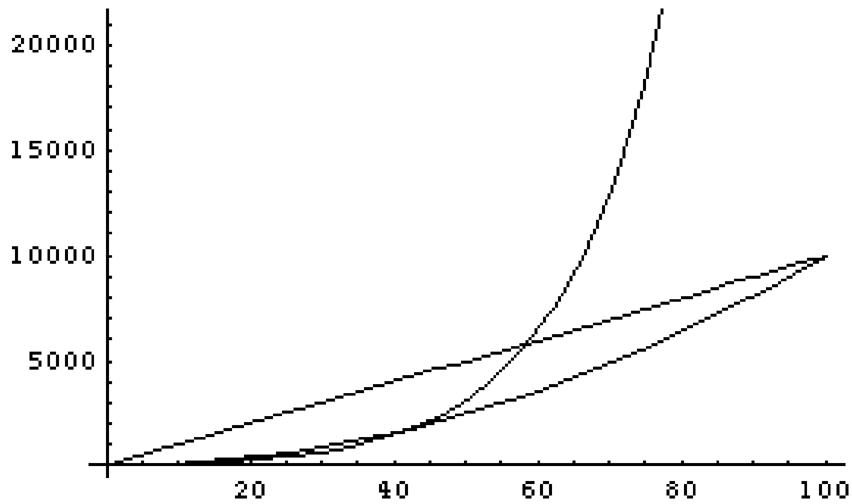
(c) For *(1)* there is one. For *(2)* there is one. For *(3)* there are four.

(d) For *(1)* there are two. For *(2)* there is one. For *(3)* there are four.

GT-3.8 (a) (and (b)) There are 21 vertices, so the minimum spanning tree has 20 edges. Its weight is 30. We omit details.

(c) Note that K is a the only vertex in common to the two bicomponents of this graph. Whenever this happens (two bicomponents, common vertex), the depth-first spanning tree rooted at that common vertex has exactly two "principal subtrees" at the root. In other words, the root of the depth-first spanning tree has down-degree two (two children). The two children of K can be taken to be P and L. P is the root of a subtree consisting of 5 vertices, 4 with one child, one leaf. L is the root of a subtree consisting of 15 vertices, 14 with one child, one leaf.

GT-4.1 (a) The algorithm that has running time $100n$ is better than the one with running time n^2 for $n > 100$. $100n$ is better than $(2^{n/10} - 1)100$ for $n \geq 60$. For $1 \leq n < 10$, $(2^{n/10} - 1)100$ is worse than n^2. At $n = 10$ they are the same. For $10 < n < 43$, n^2 is worse than $(2^{n/10} - 1)100$. For $n \geq 43$, $(2^{n/10} - 1)100$ is worse than n^2. Here are the graphs:



(b) When n is very large, B is fastest and C is slowest. This is because, of two polynomials the one with the lower degree is eventually faster and an exponential function grows faster than any polynomial.

Solutions for Basic Concepts in Graph Theory

GT-4.2 (a) The most direct way to prove this is to use Example 23. additional observations on Θ and O.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = C > 0 \quad \text{implies} \quad g(n) \quad \text{is} \quad \Theta(f(n))$$

Let $p(n) = \sum_{i=0}^{k} b_i n^i$ with $b_k > 0$. Take $f(n) = p(n)$, $g(n) = n^k$ and $C = b_k > 0$. Thus, $p(n)$ is $\Theta(n^k)$, hence the equivalence class of each is the same set: $\Theta(p(n))$ is $\Theta(n^k)$.

(b) $\mathrm{O}(p(n))$ is $\mathrm{O}(n^k)$ follows from (a).

(c) $\lim_{n\to\infty} p(n)/a^n = 0$. This requires some calculus. By applying l'Hospital's Rule k times, we see that the limit is $\lim_{n\to\infty} (k!/(\log(a))^k)/a^n$, which is 0. Any algorithm with exponential running time is eventually much slower than a polynomial time algorithm.

(d) For $p(n)$ to be $\Theta(a^{Cn^k})$, we must have positive constants A and B such that $A \le a^{p(n)}/a^{Cn^k} \le B$. Taking logarithms gives us $\log_a A \le p(n) - Cn^k \le \log_a B$. The center of this expression is a polynomial which is not constant unless $p(n) = Cn^k + D$ for some constant D, the case which is ruled out. Thus $p(n) - Cn^k$ is a nonconstant polynomial and so is unbounded.

GT-4.3 Here is a general method of working this type of problem:

Let $p(n) = \sum_{i=0}^{k} b_i n^i$ with $b_k > 0$. Show using definition that $\Theta(p(n))$ is $\Theta(n^k)$. Let $s = \sum_{i=0}^{k-1} |b_i|$ and assume that $n \ge 2s/b_k$. We have

$$|p(n) - b_k n^k| \le \left| \sum_{i=0}^{k-1} b_i n^i \right| \le \sum_{i=0}^{k-1} |b_i| n^i \le \sum_{i=0}^{k-1} |b_i| n^{k-1} = s n^{k-1} \le b_k n^k/2.$$

Thus $|p(n)| \ge b_k n^k - b_k n^k/2 \ge (b_k/2) n^k$ and also $|p(n)| \le b_k n^k + b_k n^k/2 \le (3b_k/2) n^k$.

The definition is satisfied with $N = 2s/b_k$, $A = (b_k/2)$ and $B = (3b_k/2)$. If you want to show, using the definition, that $\Theta(p(n))$ is $\Theta(Kn^k)$ for some $K > 0$, replace A with $A' = A/K$ and B with $B' = B/K$.

In our particular cases we can be sloppy and it gets easier. Take (a) as an example.

(a) For $g(n) = n^3 + 5n^2 + 10$, choose N such that $n^3 > 5n^2 + 10$ for $n > N$. You can be ridiculous in the choice of N. $N^3 > 5N^2 + 10$ is valid if $1 > 5/N + 10/N^3$. $N = 10$ is plenty big enough. If $n^3 > 5n^2 + 10$ then $n^3 < g(n) < 2n^3$. So taking $A = 1$ and $B = 2$ works for the definition: $An^3 < g(n) < Bn^3$ showing g is $\Theta(n^3)$. If you want to use $f(n) = 20n^3$ as the problem calls for, replace these constants by $A' = A/20$ and $B' = B/20$. Thus, $A'(20n^3) < g(n) < B'(20n^3)$ for $n > N$.

This problem should make you appreciate the much easier approach of Example 23.

GT-4.4 (a) There is an explicit formula for the sum of the squares of integers.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

This is a polynomial of degree 3, hence the sum is $\Theta(n^3)$.

(b) There is an explicit formula for the sum of the cubes of integers.

$$\sum_{i=1}^{n} i^3 = \Big(\frac{n(n+1))}{2}\Big)^2 .$$

This is a polynomial of degree 4, hence the sum is $\Theta(n^4)$.

(c) To show the $\sum_{i=1}^{n} i^{1/2}$ is $\Theta(n^{3/2})$ it helps to know a little calculus. You can interpret the integral as upper and lower Riemann sum approximations to the integral of $f(x) = x^{1/2}$ with $\Delta x = 1$:

$$\int_0^n f(x)\,dx < \sum_{i=1}^{n} i^{1/2} = \sum_{i=1}^{n-1} i^{1/2} + n^{1/2} < \int_1^n f(x)\,dx + n^{1/2}.$$

Since $\int x^{1/2}\,dx = 2x^{3/2}/3 + C$. You can fill in the details to get $\Theta(n^{3/2})$.

The method used in (c) will also work for (a) and (b). The idea works in general: Suppose $f(x) \geq 0$ and $f'(x) > 0$. Let $F(x)$ be the antiderivative of $f(x)$. If $f(n)$ is $\mathrm{O}(F(n))$, then $\sum_{i=0}^{n} f(n)$ is $\Theta(F(n))$. There is a similar result if $f'(x) < 0$: replace "$f(n)$ is $\mathrm{O}(F(n))$" with "$f(1)$ is $\mathrm{O}(F(n))$."

GT-4.5 (a) To show $\sum_{i=1}^{n} i^{-1}$ is $\Theta(\log_b(n))$ for any base $b > 1$ use the Riemann sum trick from the previous exercise. $\int_1^n x^{-1}\,dx = \ln(x)$. This shows that $\sum_{i=1}^{n} i^{-1}$ is $\Theta(\log_e(n))$. But, $\log_e(x) = \log_e(b)\log_b(x)$ (as we learned in high school). Thus, $\log_e(x)$ and $\log_b(x)$ belong to the same Θ equivalence class as they differ by a positive constant multiple $\log_e(b)$ (recall $b > 1$).

(b) First you need to note that $\log_b(n!) = \sum_{i=1}^{n} \log_b(i)$. Use the Riemann sum trick again.

$$\int_1^n \log_b(x)\,dx = \log_b(e)\int_1^n \log_e(x)\,dx = \log_b(e)\big(n\ln(n) - n + 1\big) .$$

Thus, the sum is $\Theta(n\ln(n) - n + 1)$ which is $\Theta(n\ln(n))$ which is $\Theta(n\log_b(n))$.

(c) Use Stirling's approximation for $n!$, $n!$ is asymptotic to $(n/e)^n(2\pi n)^{1/2}$. Thus, $n!$ is $\Theta((n/e)^n(2\pi n)^{1/2})$, by Example 23. Do a little algebra to rearrange the latter expression to get $\Theta((n/e)^{n+1/2})$.

GT-4.6 A single execution of "C(i,j) = C(i,j) + A(i,k)*B(k,j)" takes a constant amount of time and so its time is $\Theta(1)$.
The loop on k is done n times and so its time is $n\Theta(1)$, which is $\Theta(n)$.
The loop on j is done n times and each time requires work that is $\Theta(n)$. Thus its time is $n\Theta(n)$, which is $\Theta(n^2)$.
The loop on i is done n times and so its time is $n\Theta(n^2)$, which is $\Theta(n^3)$.

Alternatively, you could notice that innermost loops take the most time and "C(i,j) = C(i,j) + A(i,k)*B(k,j)" is executed once for each value of i, j, and k. Thus it is done n^3 times and so the time for the algorithm is $\Theta(n^3)$.

GT-4.7 We use the Master Theorem. Since there is just one recursive call, $w = 1$ and $s_1(n) = q$. Since $0 \leq n/2 - q \leq 1/2$, $c = 1/2$. We have $T(n) = a_n + T(s_1(n))$ where a_n is 1 or 2. Thus a_n is $\Theta(n^0)$. In summary, $w = 1$, $c = 1/2$ and $b = 0$. Thus $d = -\log(1)/\log(1/2) = 0$ and so $T(n)$ is $\Theta(\log n)$.