

Mathematics for Algorithm and System Analysis

for students of computer and computational science

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Preface

Discrete mathematics is an essential tool in almost all subareas of computer science. Interesting and challenging problems in discrete mathematics arise in programming languages, computer architecture, networking, distributed systems, database systems, AI, theoretical computer science, and other areas.

The course. The University of California, San Diego, has a lower-division two-quarter course sequence in discrete mathematics that includes Boolean arithmetic, combinatorics, elementary logic, induction, graph theory and finite probability. These courses are core undergraduate requirements for majors in Computer Science, Computer Engineering, and Mathematics-Computer Science. This text, *Mathematics for Algorithm and System Analysis*, was developed for the second quarter and *A Short Course in Discrete Mathematics* was developed for the first quarter. Because some students transfer into the second quarter of the course without having taken the first quarter, there is some overlap between the two texts and, with appropriate students, this text could be used without the first.

This book consists of four units of study (Counting and Listing—CL; Functions—Fn; Decision Trees and Recursion—DT; and Basic Concepts of Graph Theory—GT), each divided into four sections. Each section contains a representative selection of problems. These vary from basic to more difficult, including proofs for study by mathematics students or honors students. The first three sections in units CL and Fn are primarily a review of material in *A Short Course in Discrete Mathematics* needed for this course.

The review questions. “Multiple Choice Questions for Review” appear at the end of each unit. The explanatory material in this book is directed towards giving students the mathematical language and sophistication to recognize and articulate the ideas behind these questions and to answer questions that are similar in concept and difficulty. Many variations of these questions have been successfully worked on exams by most beginning students using this book at UCSD.

Students who master the ideas and mathematical language needed to understand these review questions gain the ability to formulate, in the neutral language of mathematics, problems that arise in various applications of computer science. This skill greatly facilitates their ability to discuss problems in discrete mathematics with other computer scientists and with mathematicians.

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Asterisks (stars) are used in the text to mark more difficult material that is not needed in later sections.

Unit CL: Basic Counting and Listing

Section 1: Lists with Repetitions	CL-1
set, list, multiset, sequence, word, permutation, k -set, k -list, k -multiset, k -lists with repetition, rule of product, Cartesian product, lexicographic order (lex order), dictionary order, rule of sum, composition of a positive integer	
Section 2: Lists Without Repetition	CL-9
k -lists without repetition, Stirling's formula for approximating $n!$, circular arrangements, words from a collection of letters	
Section 3: Sets	CL-13
set intersection, set union, set difference, set complement, symmetric difference, set product (Cartesian product), binomial coefficients, generating functions, binomial theorem, full house (card hand), two pairs (card hand), rearranging words, multinomial coefficients, card hands and multinomial coefficients, recursions, set partitions, Stirling numbers of the second kind ($S(n, k)$), straight (card hand), Bell numbers B_n	
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sample space, selections done uniformly at random, event, probability function, combining events, Venn diagrams, odds, hypergeometric probabilities, fair dice, geometric probability, principle of inclusion exclusion, birthday problem	
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Unit Fn: Functions

Section 1: Some Basic Terminology	Fn-1
direct product, intersection, union, symmetric difference, domain, range, codomain, one-line notation, surjection, onto, injection, one-to-one, bijection, permutation, relation, functional relation, two-line notation	
Section 2: Permutations	Fn-7
composition, cycle, cycle form of permutation, involution, permutation matrices, derangements	

Section 3: Other Combinatorial Aspects of Functions **Fn-14**
 image, inverse image, coimage, image size and Stirling numbers, strictly increasing,
 strictly decreasing, weakly increasing, weakly decreasing, monotone, multisets, lists
 without repetition, restricted growth functions and partitions

Section 4: Functions and Probability **Fn-21**
 random variable, probability function, event, probability distribution function, ex-
 pectation, covariance, variance, standard deviation, correlation, independent events,
 independent random variables, product spaces, generating random permutations,
 joint distribution function, marginal distributions, binomial distribution, Poisson
 distribution, normal distribution, standard normal distribution, cumulative distri-
 bution, central limit theorem, normal approximation to binomial, Poisson approx-
 imation to binomial, Tchebycheff's inequality

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Unit DT: Decision Trees and Recursion

Section 1: Basic Concepts of Decision Trees **DT-1**
 decision trees, vertices, root, edges, degree of vertex, down degree, child, parent,
 leaves, internal vertex, height of leaf, path to vertex, traversals of decision tree,
 depth first vertices, depth first edges, breadth first, preorder, postorder, length-
 first lex order, dictionary order, permutations in lex order, partial permutation,
 rank of leaf, direct insertion order for permutations, backtracking, Latin squares,
 domino coverings, strictly decreasing functions, unlabeled balls into boxes, isomorph
 rejection

Section 2: Recursive Algorithms **DT-15**
 recursive algorithm, simplest case reduction, recursive algorithm for 0-1 sequences,
 sorting by recursive merging, recursive approach, recursive solutions, local descrip-
 tion for permutations in lex order, recursive description of Towers of Hanoi, decision
 tree for Towers of Hanoi, recursion and stacks, configuration analysis of Towers of
 Hanoi, abandoned leaves and RANK, characteristic functions and subsets, Gray
 code for subsets, decision tree for Gray code for subsets, local description of Gray
 code, Towers of Hanoi with four poles

Section 3: Decision Trees and Conditional Probability **DT-27**
 conditional probability, independent events, Venn diagrams, probabilities of leaves,
 probabilities of edges, probabilistic decision trees, decision trees and Bayesian meth-
 ods, Bayes' theorem, multiplication theorem for conditional probabilities, sequen-
 tial sampling, the SAT problem, first moment method, tournaments, gambler's ruin
 problem

Section 4: Inductive Proofs and Recursive Equations **DT-40**
 induction, recursive equations, induction hypothesis, inductive step, base case,
 prime factorization, sum of first n integers, local description, recurrence relation,

binomial coefficients $C(n, k)$, Stirling numbers $S(n, k)$, guessing solutions to recurrences, linear two term recurrence, constant coefficients, characteristic equation, two real roots, one real root, complex roots, recursion for derangements, Fibonacci recurrence relation, recurrence relation for derangements

Multiple Choice Questions for Review **DT-52**

Unit GT: Basic Concepts in Graph Theory

Section 1: What is a Graph? **GT-1**

computer network example, simple graph, graph, vertices, edges, set theoretic description of graph, pictorial description of a graph, incidence function, vertices joined by an edge, adjacent vertices, edge incident on a vertex, simple graphs are graphs, form of a graph, equivalence relations, equivalence classes, blocks, binary relations, reflexive, symmetric, transitive, equivalent forms, isomorphism of graphs, graph isomorphism as an equivalence relation, degree of a vertex, loops, parallel edges, isolated vertices, degree sequences and isomorphism, random graphs

Section 2: Digraphs, Paths, and Subgraphs **GT-13**

flow of commodities, directed graph, digraph, simple digraph, simple graphs as simple digraphs, directed loops, digraphs and binary relations, symmetric binary relations and simple graphs with loops, complete simple graphs, path, trail, walk, vertex sequence, walk implies path, restrictions of incidence functions, subgraphs, subgraph induced by edges, subgraph induced by vertices, cycles, connected graphs, connected components and equivalence classes, connectivity in digraphs, Eulerian trail, Eulerian circuit, Hamiltonian cycle, Hamiltonian graph, bicomponents of graphs, bipartite graphs, oriented simple graphs, antisymmetry, order relations, Hasse diagrams, covering relations, counting trees

Section 3: Trees **GT-24**

tree, alternative definitions of a tree, rooted graph, rooted tree, parent, child, sibling, leaf, internal vertices, unique paths in trees, rooted plane tree, RP-tree, traversing RP-trees, depth first sequences, breadth first sequences, spanning trees, minimum spanning trees, greedy algorithms, Prim’s algorithm, Kruskal’s algorithm, lineal or depth-first spanning trees, algorithm for depth-first spanning trees, bipartite graphs and depth first spanning trees, degree sequences of trees, binary trees, full binary trees, height and leaf restrictions in binary trees

Section 4: Rates of Growth and Analysis of Algorithms **GT-37**

comparing algorithms, machine independence, example of finding the maximum, Θ notation, O notation, properties of Θ and O , Θ as an equivalence relation, sufficiently large, eventually positive, asymptotic, “little oh” notation, using Θ to compare polynomial evaluation algorithms, average running time, tractable, intractable, graph coloring problem, traveling salesman problem, clique problem, NP -complete problems, NP -hard, NP -easy, chromatic number of a graph, almost good algorithms, almost correct algorithms, close algorithms, polynomial time, exponential time, Θ and series, Θ and logs

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