Unit IS

Induction, Sequences and Series

Section 1: Induction

Suppose $\mathcal{A}(n)$ is an assertion that depends on n. We use *induction* to prove that $\mathcal{A}(n)$ is true when we show that

- it's true for the smallest value of n and
- if it's true for everything less than n, then it's true for n.

In this section, we will review the idea of proof by induction and give some examples. Here is a formal statement of proof by induction:

Theorem 1 (Induction) *Let $\mathcal{A}(m)$ be an assertion, the nature of which is dependent on the integer m. Suppose that we have proved $\mathcal{A}(n_0)$ and the statement*

"If $n > n_0$ and $\mathcal{A}(k)$ is true for all k such that $n_0 \le k < n$, then $\mathcal{A}(n)$ is true."

Then $\mathcal{A}(m)$ is true for all $m \ge n_0$.[1]

Proof: We now prove the theorem. Suppose that $\mathcal{A}(n)$ is false for some $n \ge n_0$. Let m be the least such n. We cannot have $m = n_0$ because one of our hypotheses is that $\mathcal{A}(n_0)$ is true. On the other hand, since m is as small as possible, $\mathcal{A}(k)$ is true for $n_0 \le k < m$. By the inductive step, $\mathcal{A}(m)$ is also true, a contradiction. Hence our assumption that $\mathcal{A}(n)$ is false for some n is itself false; in other words, $\mathcal{A}(n)$ is never false. This completes the proof. $\square$

Definition 1 (Induction terminology) *"$\mathcal{A}(k)$ is true for all k such that $n_0 \le k < n$" is called the* induction assumption *or* induction hypothesis *and proving that this implies $\mathcal{A}(n)$ is called the* inductive step*. $\mathcal{A}(n_0)$ is called the* base case *or* simplest case*.*

[1] This form of induction is sometimes called *strong* induction. The term "strong" comes from the assumption "$\mathcal{A}(k)$ is true for all k such that $n_0 \le k < n$." This is replaced by a more restrictive assumption "$\mathcal{A}(k)$ is true for $k = n-1$" in *simple* induction. Actually, there are many intermediate variations on the nature of this assumption, some of which we shall explore in the exercises (e.g., "$\mathcal{A}(k)$ is true for $k = n-1$ and $k = n-2$, " "$\mathcal{A}(k)$ is true for $k = n-1$, $k = n-2$, and $k = n-3$, " etc.).

Example 1 (Every integer is a product of primes) A positive integer $n > 1$ is called a *prime* if its only divisors are 1 and n. The first few primes are 2, 3, 5, 7, 11, 13, 17, 19, 23. In another unit, we proved that every integer $n > 1$ is a product of primes. We now redo the proof, being careful with the induction.

We adopt the terminology that a single prime p is a product of one prime, itself. We shall prove $\mathcal{A}(n)$:

"Every integer $n \geq 2$ is a product of primes."

Our proof that $\mathcal{A}(n)$ is true for all $n \geq 2$ will be by induction. We start with $n_0 = 2$, which is a prime and hence a product of primes. The induction hypothesis is the following:

"Suppose that for some $n > 2$, $\mathcal{A}(k)$ is true for all k such that $2 \leq k < n$."

Assume the induction hypothesis and consider $\mathcal{A}(n)$. If n is a prime, then it is a product of primes (itself). Otherwise, $n = st$ where $1 < s < n$ and $1 < t < n$. By the induction hypothesis, s and t are each a product of primes, hence $n = st$ is a product of primes. This completes the proof of $\mathcal{A}(n)$; that is, we've done the inductive step. Hence $\mathcal{A}(n)$ is true for all $n \geq 2$. $\square$

In the example just given, we needed the induction hypothesis "for all k such that $2 \leq k < n$." In the next example we have the more common situation where we only need "for $k = n - 1$." We can still make the stronger assumption "for all k such that $1 \leq k < n$" and the proof is valid.

Example 2 (Sum of first n integers) We would like a formula for the sum of the first n integers. Let us write $S(n) = 1 + 2 + \ldots + n$ for the value of the sum. By a little calculation,

$$S(1) = 1, \quad S(2) = 3, \quad S(3) = 6, \quad S(4) = 10, \quad S(5) = 15, \quad S(6) = 21.$$

What is the general pattern? It turns out that $S(n) = \frac{n(n+1)}{2}$ is correct for $1 \leq n \leq 6$. Is it true in general? This is a perfect candidate for an induction proof with

$$n_0 = 1 \qquad \text{and} \qquad \mathcal{A}(n): \quad \text{“}S(n) = \tfrac{n(n+1)}{2}.\text{”}$$

Let's prove it. We have shown that A(1) is true. In this case we need only the restricted induction hypothesis; that is, we will prove the formula for $S(n)$ by assuming the formula for for $k = n - 1$. Thus, we assume only $S(n-1)$ is true. Here it is (the inductive step):

$$\begin{aligned} S(n) &= 1 + 2 + \cdots + n \\ &= \Big(1 + 2 + \cdots + (n-1)\Big) + n \\ &= S(n-1) + n \\ &= \frac{(n-1)\big((n-1)+1\big)}{2} + n && \text{by } \mathcal{A}(n-1), \\ &= \frac{n(n+1)}{2} && \text{by algebra.} \end{aligned}$$

This completes the proof. $\square$

Example 3 (Intuition behind the sum of first n integers) Whenever you prove something by induction you should try to gain an intuitive understanding of why the result is true. Sometimes a proof by induction will obscure such an understanding. In the following array, you will find one 1, two 2's, three 3's, etc. The total number of entries is $1 + 2 + \cdots + 8$. On the other hand, the array is a rectangle with $4 \times 9 = 36$ entries. This verifies that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ is correct for $n = 8$. The same way of laying out the integers works for any n (if n is odd, it is laid out along the bottom row, if n is even, it is laid out in the last two columns).

$$\begin{array}{ccccccccc}
1 & 2 & 2 & 4 & 4 & 6 & 6 & 8 & 8 \\
3 & 3 & 3 & 4 & 4 & 6 & 6 & 8 & 8 \\
5 & 5 & 5 & 5 & 5 & 6 & 6 & 8 & 8 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 8 & 8
\end{array}$$

This argument, devised by a fourth-grade girl, has all of the features of a powerful intuitive image.

Here is another proof based on adding columns

$$\begin{array}{rccccc}
S(n) = & 1 & + & 2 & + \cdots + & n \\
S(n) = & n & + & (n-1) & + \cdots + & 1 \\
\hline
2S(n) = & (n+1) & + & (n+1) & + \cdots + & (n+1) \\
= & n(n+1) & & & &
\end{array}$$

Here is geometric view of this approach for $n = 8$.

$$\begin{array}{ccccccccc}
O & X & X & X & X & X & X & X & X \\
O & O & X & X & X & X & X & X & X \\
O & O & O & X & X & X & X & X & X \\
O & O & O & O & X & X & X & X & X \\
O & O & O & O & O & X & X & X & X \\
O & O & O & O & O & O & X & X & X \\
O & O & O & O & O & O & O & X & X \\
O & O & O & O & O & O & O & O & X
\end{array}$$

$\square$

Example 4 (Bounding the terms of a recursion) Consider the recursion

$$f_k = f_{k-1} + 2f_{k-2} + f_{k-3},\ k \geq 3, \text{ with } f_0 = 1,\ f_1 = 2,\ f_2 = 4.$$

We would like to obtain a bound on the f_k, namely $f_k \leq r^k$ for all $k \geq 0$. Thus there are two problems: (a) what is the best (smallest) value we can find for r and (b) how can we prove the result?

Since the recursion tells us how to compute f_k from previous values, we expect to give a proof by induction. The inequality $f_k \leq r^k$ tells us that $f_1 \leq r^1 = r$. Since $f_1 = 2$, maybe $r = 2$ will work. Let's try giving a proof with $r = 2$. Thus $\mathcal{A}(n)$ is the statement "$f_n \leq 2^n$" and $n_0 = 0$. In order to use the recursion for f_n, we need $n \geq 3$. Thus we must treat $n = 0, 1, 2$ separately

- Since $f_0 = 1$ and $2^0 = 1$, we've done $n = 0$.

- We've already done $n = 1$.
- Since $f_2 = 4 = 2^2$, we've done $n = 2$.
- Suppose $n \geq 3$. By our induction hypothesis, $f_{n-1} \leq 2^{n-1}$, $f_{n-2} \leq 2^{n-2}$, and $f_{n-3} \leq 2^{n-3}$. Thus

$$f_n = f_{n-1} + 2f_{n-2} + f_{n-3} \leq 2^{n-1} + 2 \times 2^{n-2} + 2^{n-3} = 2^n + 2^{n-3}.$$

  This won't work because we wanted to conclude that $f_n \leq 2^n$.

What is wrong? Either our guess that $f_n \leq 2^n$ wrong or our guess is right and we need to look for another way to prove it. Since it's easier to compute values of f_n than it is to find proofs, let's compute. We have $f_3 = f_2 + 2f_1 + f_0 = 4 + 2 \times 2 + 1 = 9$. Thus $f_3 \leq 2^3$ is false! This illustrates an important idea: Often computing a few values can save a lot of time.

Since 2 won't work, what will? Let's pretend we know the answer and call it r. We already know that we need to have $r > 2$.

- Since $r > 2$, $f_n \leq r^n$ for $n = 0, 1, 2$.
- Suppose $n \geq 3$. Working just as we did for the case $r = 2$, we have

$$f_n \leq r^{n-1} + 2r^{n-2} + r^{n-3}.$$

  We want this to be less than r^n; that is, we want $r^{n-1} + 2r^{n-2} + r^{n-3} \leq r^n$. Dividing both sides by r^{n-3}, we see that we want $r^2 + 2r + 1 \leq r^3$. The smallest $r \geq 2$ that satisfies this inequality is an irrational number which is approximately 2.148.

  For practice, you should go back and write a formal induction proof when $r = 2.2$. $\square$

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*More Advanced Examples of Induction

The next two examples are related, first because they both deal with polynomials, and second because the theorem in one is used in the other. They also illustrate a point about proof by induction that is sometimes missed: Because exercises on proof by induction are chosen to give experience with the inductive step, students frequently assume that the inductive step will be the hard part of the proof. The next example fits this stereotype — the inductive step is the hard part of the proof. In contrast, the base case is difficult and the inductive step is nearly trivial in the second example. A word of caution: these examples are more complicated than the preceding ones.

Example 5 (Sum of k^{th} powers of integers) Let $S_k(n)$ be the sum of the first n k^{th} powers of integers. In other words,

$$S_k(n) = 1^k + 2^k + \cdots + n^k \quad \text{for } n \text{ a positive integer.}$$

In particular $S_k(0) = 0$ (since there is nothing to add up) and $S_k(1) = 1$ (since $1^k = 1$) for all k. We have

$$S_0(n) = 1^0 + 2^0 + \cdots + n^0 = 1 + 1 + \cdots + 1 = n.$$

In Example 2 we showed that $S_1(n) = n(n+1)/2$. Can we observe any patterns here? Well, it looks like $S_k(n)$ might be $\frac{n(n+1)\cdots(n+k)}{k+1}$ A little checking shows that this is wrong since $S_2(2) = 5$. Well, maybe we shouldn't be so specific. If you're familiar with integration, you might notice that $S_k(n)$ is a Riemann sum for $\int_0^n x^k\,dx = n^{k+1}/(k+1)$. Maybe $S_k(n)$ behaves something like $n^{k+1}/(k+1)$. That's rather vague. We'll prove

Theorem 2 (Sum of k^{th} powers) *If $k \geq 0$ is an integer, then $S_k(n)$ is a polynomial in n of degree $k+1$. The constant term is zero and the coefficient of n^{k+1} is $1/(k+1)$.*

Two questions may come to mind. First, how can we prove this since there is no formula to prove? Second, what good is the theorem since it doesn't give us a formula for $S_k(n)$?

Let's start with second question. We can use the theorem to find $S_k(n)$ for any particular k. To illustrate, suppose we don't know what $S_1(n)$ is. According to the theorem $S_1(n) = n^2/2 + An$ for some A since it says that $S_1(n)$ is a polynomial of degree two with no constant term and leading term $n^2/2$. With $n = 1$ we have $S_1(1) = 1^2/2 + A \times 1 = 1/2 + A$. Since $S_1(1) = 1^1 = 1$, it follows that $A = 1/2$. We have our formula: $S_1(n) = n^2/2 + n/2$.

Let's find $S_2(n)$. By the theorem $S_2(n) = n^3/3 + An^2 + Bn$. With $n = 1$ and $n = 2$ we get

n	direct calculation	polynomial
1	$S_2(1) = 1^2 = 1$	$S_2(1) = 1^3/3 + A \times 1^2 + B \times 1$
2	$S_2(2) = 1^2 + 2^2 = 5$	$S_2(2) = 2^3/3 + A \times 2^2 + B \times 2$

After a little algebra, we obtain the two equations

$$\begin{aligned} n = 1: &\quad A + B = 2/3 \\ n = 2: &\quad 4A + 2B = 7/3 \end{aligned}$$

Solving these equations, we find that $A = 1/2$ and $B = 1/6$. Thus $S_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$.

Okay, enough examples — on with the proof! We are going to use induction on k and a couple of tricks. The assertion we want to prove is

$$\mathcal{A}(k) \quad = \quad \begin{array}{l} S_k(n) \text{ is a polynomial in } n \text{ of degree } k+1 \\ \text{with constant term zero and leading term } \frac{1}{k+1}. \end{array}$$

The base case, $k = 0$ is easy: $1^0 + 2^0 + \cdots + n^0 = 1 + 1 + \cdots + 1 = n$, which has no constant term and has leading coefficient $\frac{1}{0+1} = 1$.

Now for the inductive step. We want to prove $\mathcal{A}(k)$. To do so, we will need $\mathcal{A}(t)$ for $0 \le t < k$.

The first trick uses the binomial theorem $(x+y)^m = \sum_{t=0}^{m} \binom{m}{t} x^t y^{m-t}$ with $m = k+1$, $x = j$ and $y = -1$: We have

$$j^{k+1} - (j-1)^{k+1} = j^{k+1} - \sum_{t=0}^{k+1} \binom{k+1}{t} j^t (-1)^{k+1-t} = -\sum_{t=0}^{k} \binom{k+1}{t} j^t (-1)^{k+1-t}.$$

Sum both sides over $1 \le j \le n$. When we sum the right side over j we get

$$-\sum_{t=0}^{k} \binom{k+1}{t} S_t(n)(-1)^{k+1-t}.$$

The second trick is what happens when we sum $j^{k+1} - (j-1)^{k+1}$ over j: Almost all the terms cancel:

$$\begin{aligned}(1^{k+1} - 0^{k+1}) + (2^{k+1} - 1^{k+1}) + \cdots + ((n-1)^{k+1} - (n-2)^{k+1}) + (n^{k+1} - (n-1)^{k+1})\\ = -0^{k+1} + n^{n+1} = n^{k+1}.\end{aligned}$$

Thus we have

$$\begin{aligned} n^{k+1} &= -\sum_{t=0}^{k} \binom{k+1}{t} S_t(n)(-1)^{k+1-t} \\ &= -\binom{k+1}{k} S_k(n)(-1)^{k+1-k} - \sum_{t=0}^{k-1} \binom{k+1}{t} S_t(n)(-1)^{k+1-t} \\ &= (k+1)S_k(n) - \sum_{t=0}^{k-1} \binom{k+1}{t} S_t(n)(-1)^{k+1-t}. \end{aligned}$$

We can solve this equation for $S_k(n)$:

$$S_k(n) = \frac{n^{k+1}}{k+1} + \sum_{t=0}^{k-1} \frac{1}{k+1} \binom{k+1}{t} (-1)^{k+1-t} S_t(n).$$

By the induction hypothesis, $S_t(n)$ is a polynomial in n with no constant term and degree $t+1$. Since $0 \le t \le k-1$, it follows that each term in the messy sum is a polynomial in n with no constant term and degree at most k Thus the same is true of the entire sum. We have proved that

$$S_k(n) = \frac{n^{k+1}}{k+1} + P_k(n),$$

where $P_k(n)$ is a polynomial in n with no constant term and degree at most k. This completes the proof of the theorem. $\square$

Definition 2 (Forward difference) *Suppose $S : \mathbb{N} \to \mathbb{R}$. The forward difference of S is another function denoted by ΔS and defined by $\Delta S(n) = S(n+1) - S(n)$. In this context, Δ is called a difference operator.*

We can iterate Δ. For example, $\Delta^2 S = \Delta(\Delta S)$. If we let $T = \Delta S$, then $T(n) = S(n+1) - S(n)$ and

$$\begin{aligned}\Delta^2 S(n) = \Delta T(n) &= T(n+1) - T(n)\\ &= \big(S(n+2) - S(n+1)\big) - \big(S(n+1) - S(n)\big)\\ &= S(n+2) - 2S(n+1) + S(n).\end{aligned}$$

The operator Δ has properties similar to the derivative operator $\mathrm{d}/\mathrm{d}x$. For example $\Delta(S + T) = \Delta S + \Delta T$. In some subjects, "differences" of functions play the role that derivatives play in other subjects. Derivatives arise in the study of rates of change in continuous situations. Differences arise in the study of rates of change in discrete situations. Although there is only one type of ordinary derivative, there are three common types of differences: backward, central and forward.

The next example gives another property of the difference operator that is like the derivative. You may know that the general solution of the differential equation $f^{(k)}(x) =$ constant is a polynomial of degree $k + 1$. In the next example we prove that the same is true for the difference equation $\Delta^k f(x) =$ constant.

Example 6 (Differences of polynomials) Suppose $S(n) = an + b$ for some constants a and b. You should be able to check that $\Delta S(n) = a$, a constant. With a little more work, you can check that $\Delta^2(an^2 + bn + c) = 2a$. We now state and prove a general converse of these results.

Theorem 3 (Polynomial differences) *If $\Delta^k S$ is a polynomial of degree j, then $S(n)$ is a polynomial of degree $j + k$ in n.*

We'll prove this by induction on k. $\mathcal{A}(k)$ is simply the statement of the theorem.

We now do the base case. Suppose $k = 1$. Let $T = \Delta S$. We want to show that, if T is a polynomial of degree j, then S is a polynomial of degree $j + 1$. We have

$$\begin{aligned}S(n+1) =&\Big(S(n+1) - S(n)\Big) + \Big(S(n) - S(n-1)\Big) + \Big(S(n-1) - S(n-2)\Big)\\ &+ \cdots + \Big(S(2) - S(1)\Big) + S(1)\\ =&T(n) + T(n-1) + T(n-2) + \cdots + T(1) + S(1)\\ =&\sum_{t=1}^{n} T(t) + S(1).\end{aligned}$$

What have we gained by this manipulation? We've expressed an unknown function $S(n+1)$ as the sum of a constant $S(1)$ and the sum of a function T which is known to be a polynomial of degree j. Now we need to make use of our knowledge of T to say something about $\sum T(t)$.

By assumption, T is a polynomial of degree j. Let $T(n) = a_j n^j + \cdots + a_1 n + a_0$, where $a_0, \ldots, a_j$ are constants. Then

$$\begin{aligned}\sum_{t=1}^{n} T(t) &= \sum_{t=1}^{n} (a_j t^j + \cdots + a_1 t + a_0) \\ &= \sum_{t=1}^{n} a_j t^j + \cdots + \sum_{t=1}^{n} a_1 t + \sum_{t=1}^{n} a_0 \\ &= a_j \sum_{t=1}^{n} t^j + \cdots + a_1 \sum_{t=1}^{n} t + n a_0.\end{aligned}$$

By Theorem 2,

$$\begin{array}{ll}\displaystyle\sum_{t=1}^{n} t^j & \text{is a polynomial of degree } j+1, \\ \displaystyle\sum_{t=1}^{n} t^{j-1} & \text{is a polynomial of degree less than } j+1, \\ \cdots & \\ \displaystyle\sum_{t=1}^{n} t & \text{is a polynomial of degree less than } j+1.\end{array}$$

Thus $\sum_{t=1}^{n} T(t)$ is a polynomial of degree $j+1$. Since $S(n+1) = \sum_{t=1}^{n} T(t) + S(1)$, it is a polynomial of degree $j+1$ in n.

Let's see where we are with the base case. We've proved that $S(n+1)$ is a polynomial of degree $j+1$ in n. But we want to prove that $S(n)$ is a polynomial of degree $j+1$ in n, so we have a bit more work.

We can write $S(n+1) = b_{j+1} n^{j+1} + b_j n^j + \cdots + b_1 n + b_0$. Replace n by $n-1$:

$$S(n) = b_{j+1}(n-1)^{j+1} + b_j(n-1)^j + \cdots + b_1(n-1) + b_0.$$

Using the binomial theorem in the form $(n-1)^k = \sum_{i=0}^{k} \binom{k}{i} n^i (-1)^{k-i}$, you should be able to see that $(n-1)^k$ is a polynomial of degree k in n. Using this in the displayed equation, you can see that $S(n)$ is a polynomial of degree $j+1$ in n. The base case is done. Whew!

The induction step is easy: We are given that $\Delta^k S$ is a polynomial of degree j. We want to show that S is a polynomial of degree $j+k$. By definition, $\Delta^k S = \Delta(\Delta^{k-1} S)$. Let $T = \Delta^{k-1} S$. We now take three simple steps.

- By the definition of T, $\Delta T = \Delta^k S$, which is a polynomial of degree j by the hypothesis of $\mathcal{A}(k)$.
- By $\mathcal{A}(1)$, T is a polynomial of degree $j+1$; that is, $\Delta^{k-1} S$ is a polynomial of degree $j+1$.
- By $\mathcal{A}(k-1)$ with j replaced by $j+1$, it now follows that S is a polynomial of degree $(j+1) + (k-1) = j+k$.

The proof is done. $\square$

The best way, perhaps the only way, to understand induction and inductive proof technique is to work lots of problems. That we now do!

Exercises for Section 1

1.1. In each case, express the given infinite series or product in summation or product notation.

(a) $1^2 - 2^2 + 3^2 - 4^2 \cdots$

(b) $(1^3 - 1) + (2^3 + 1) + (3^3 - 1) \cdots$

(c) $(2^2 - 1)(3^2 + 1)(4^2 - 1) \cdots$

(d) $(1 - r)(1 - r^3)(1 - r^5) \cdots$

(e) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots$

(f) $n + \frac{n-1}{2!} + \frac{n-2}{3!} + \cdots$

1.2. In each case give a formula for the n^{th} term of the indicated sequence. Be sure to specify the starting value for n.

(a) $1 - \frac{1}{2},\ \frac{1}{2} - \frac{1}{3},\ \frac{1}{3} - \frac{3}{4},\ \ldots$

(b) $\frac{1}{4},\ \frac{2}{9},\ \frac{3}{16},\ \ldots$

(c) $\frac{1}{2},\ -\frac{2}{3},\ \frac{3}{4},\ \ldots$

(d) $2, 6, 12, 20, 30, 42, \ldots$

(e) $0, 0, 1, 1, 2, 2, 3, 3, \ldots$

1.3. In each case make the change of variable $j = i - 1$.

(a) $\displaystyle\prod_{i=2}^{n+1} \frac{(i-1)^2}{i}$

(b) $\displaystyle\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$

(c) $\displaystyle\prod_{i=n}^{2n} \frac{n-i+1}{i}$

(d) $\displaystyle\prod_{i=1}^{n} \frac{i}{i+1} \prod_{i=1}^{n} \frac{i+1}{i+2}$

1.4. Prove by induction that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$.

1.5. Prove twice, once using Theorem 2 and once by induction, that $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ for $n \geq 1$.

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1.6. Prove by induction that $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for $n \geq 1$.

1.7. Prove by induction that $\displaystyle\sum_{i=1}^{n+1} i2^i = n2^{n+2} + 2$ for $n \geq 0$.

1.8. Prove by induction that $\displaystyle\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$ for $n \geq 2$.

1.9. Prove by induction that $\displaystyle\sum_{i=1}^{n} i\, i! = (n+1)! - 1$ for $n \geq 1$.

1.10. Prove by induction that $\displaystyle\prod_{i=0}^{n} \frac{1}{2i+1} \frac{1}{2i+2} = \frac{1}{(2n+2)!}$ for $n \geq 0$.

1.11. Prove without using induction that $\displaystyle\sum_{k=1}^{n} 5k = 2.5n(n+1)$.

1.12. Prove that, for $a \neq 1$ and $n \geq t$,

$$\sum_{k=t}^{n} a^k = a^t \left(\frac{a^{n-t+1} - 1}{a - 1}\right).$$

1.13. Prove twice, once with induction and once without induction, that $3 \mid (n^3 - 10n + 9)$ for all integers $n \geq 0$; that is, $n^3 - 10n + 9$ is a multiple of 3.

1.14. Prove by induction that $(x - y) \mid (x^n - y^n)$ where $x \neq y$ are integers, $n > 0$.

1.15. Prove twice, once with induction and once without induction, that $6 \mid n(n^2 + 5)$ for all $n \geq 1$.

1.16. Prove by induction that $n^2 \leq 2^n$ for all $n \geq 0$, $n \neq 3$.

1.17. Prove by induction that

$$\sqrt{n} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \quad \text{for } n \geq 2.$$

1.18. Consider the Fibonacci recursion $f_k = f_{k-2} + f_{k-1}$, $k \geq 2$, with $f_0 = 3$ and $f_1 = 6$. Prove by induction that $3 \mid f_k$ for all $k \geq 0$.

1.19. Consider the recursion $F_k = F_{k-1} + F_{k-2}$, $k \geq 2$, with $F_0 = 0$ and $F_1 = 1$. Prove that F_k is even if and only if $3 \mid k$. In other words, prove that, modulo 2, $F_{3t} = 0$, $F_{3t+1} = 1$, and $F_{3t+2} = 1$ for $t \geq 0$.

1.20. Consider the recursion $f_k = 2f_{\lfloor \frac{k}{2} \rfloor}$, $k \geq 2$, with $f_1 = 1$. Prove by induction that $f_k \leq k$ for all $k \geq 1$.

1.21. We wish to prove by induction that for any real number $r > 0$, and every integer $n \geq 0$, $r^n = 1$. For $n = 0$, we have $r^n = 1$ for all $r > 0$. This is the base case. Assume that for $k > 0$, we have that, for $0 \leq j \leq k$, $r^j = 1$ for all $r > 0$. We must show that for $0 \leq j \leq k + 1$, $r^j = 1$ for all $r > 0$. Write $r^{k+1} = r^s r^t$ where $0 \leq s \leq k$ and $0 \leq t \leq k$. By the induction hypothesis, $r^s = 1$ and $r^t = 1$ for all $r > 0$. Thus, $r^{k+1} = r^s r^t = 1$ for all $r > 0$. Combining this with the induction hypothesis gives that for $0 \leq j \leq k + 1$, $r^j = 1$ for all $r > 0$. Thus the theorem is proved by induction. What is wrong?

1.22. We wish to prove by induction the proposition $\mathcal{A}(n)$ that all positive integers j, $1 \leq j \leq n$, are equal. The case $\mathcal{A}(1)$ is true. Assume that, for some $k \geq 1$, $\mathcal{A}(k)$ is true. Show that this implies that $\mathcal{A}(k+1)$ is true. Suppose that p and q are positive integers less than or equal to $k + 1$. By the induction hypothesis, $p - 1 = q - 1$. Thus, $p = q$. Thus $\mathcal{A}(n)$ is proved by induction. What is wrong?

***1.23.** Let $a \in \mathbb{R}$, $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$. Prove the following.

(a) $\Delta(af) = a\Delta f$; that is, for all $n \in \mathbb{N}$, the function $\Delta(af)$ evaluated at n equals a times the function Δf evaluated at n.

(b) $\Delta(f + g) = \Delta f + \Delta g$.

(c) $\Delta(fg) = f\Delta g + g\Delta f + (\Delta f)(\Delta g)$; that is, for all $n \in \mathbb{N}$,
$(\Delta(fg))(n) = f(n)(\Delta g)(n) + g(n)(\Delta f)(n) + (\Delta f)(n)\ (Deltag)(n)$.

***1.24.** Prove by induction on k that, for $k \geq 1$,

$$(\Delta^k f)(n) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(n + j).$$

Hint: You may find it useful to recall that $\binom{k-1}{j-1} + \binom{k-1}{j} = \binom{k}{j}$ for $k \geq j > 0$.

Section 2: Infinite Sequences

Our purpose in this section and the next is to present the intuition behind infinite sequences and series. It is our experience, however, that the development of this intuition is greatly aided by an exposure to a small amount of the precise formalism that lies behind the mathematical study of sequences and series. This exposure takes away much of the mystery of the subject and focuses the intuition on what really matters.

Recall that a function f with domain D and range (codomain) R is a rule which, to every $x \in D$ assigns a unique element $f(x) \in R$. Sequences are a special class of functions.

Definition 3 (Infinite sequence) *Let $n_0 \in \mathbb{N} = \{0, 1, 2, \ldots\}$ A function f whose domain is $D = \mathbb{N} + n_0 = \{n \mid n \in \mathbb{N} \text{ and } n \geq n_0\}$ and whose range is the set $\mathbb{R}$ of real numbers is called an infinite sequence.*

An infinite sequence is often written in subscript notation; for example, $a_2, a_3, a_4, \ldots$ corresponds to a function f with domain $\mathbb{N} + 2$, $f(2) = a_2$, $f(3) = a_3$ and so on.

Each value of the function is a term of the sequence. Thus $f(4)$ is a term in functional notation and a_7 is a term in subscript notation.

If f is an infinite sequence with domain $\mathbb{N} + n_0$ and $k \geq n_0$, the f restricted to $\mathbb{N} + k$ is called a tail of f. For example, $a_7, a_8, \ldots$ is a tail of $a_2, a_3, \ldots$.

Example 7 (Specifying sequences) People specify infinite sequences in various ways. The function is usually given by subscript notation rather than parenthetic notation; that is, a_n instead of $f(n)$. Let's look at some examples of sequence specification.

- "Consider the sequence $1/n$ for $n \geq 1$." This is a perfectly good specification of the function. Since the sequence starts at $n = 1$, we have $n_0 = 1$ and $a_n = 1/n$.
- "Consider the sequence $1/n$." Since the domain of n has not been specified this is not a function; however, specifying a sequence in this manner is common. What should the domain be? The convention is that $n_0 \geq 0$ be chosen as small as possible. Since $1/0$ is not defined, $n_0 = 1$.
- "Consider the sequence $1/1,\ 1/2,\ 1/3,\ 1/4, \ldots$" It's clear what the terms of this sequence are, however no domain has been specified. There are an infinite number of possibilities. Here are three.

  $$n_0 = 0 \text{ and } a_n = \tfrac{1}{n+1} \qquad n_0 = 1 \text{ and } a_n = \tfrac{1}{n} \qquad n_0 = 37 \text{ and } a_n = \tfrac{1}{n-36}$$

  The first choice makes n_0 as small as possible. The second choice makes a_n as simple as possible, which may be convenient. The third choice is because we like the number 37. Which is correct? They all are — but use one of the first two approaches since the third only confuses people. Since we haven't specified a function by saying $1/1,\ 1/2,\ 1/3,\ 1/4, \ldots$, why do we consider this to be a sequence? Often it's the terms in the sequence that are important, so any way you make it into a function is okay.

People sometimes define an infinite sequence to be an infinite list.

Sometimes, we will specify an infinite sequence that way, too.

- "Given the sequence a_n, consider the sequence $a_0, a_2, a_4, \ldots$ of the even terms." As just discussed, $a_0, a_2, a_4, \ldots$ specifies a sequence from the list point of view. We should have said "the terms with even subscripts" rather than "the even terms;" however, people seldom do that. □

The next definition may sound strange at first, but you will get used to it.

Definition 4 (Limit of a sequence) *Let a_n, $n \geq n_0$, be an infinite sequence. We say that a real number A is the limit of a_n as n goes to infinity and write*

$$\lim_{n\to\infty} a_n = A$$

if, for every real number $\epsilon > 0$, there exists N_ϵ such that for all $n \geq N_\epsilon$, $|a_n - A| \leq \epsilon$.

We often omit "as n goes to infinity and simply say "A is the limit of the sequence a_n."

If a sequence a_n has a real number A as a limit, we say that the sequence converges to A. If a sequence does not converge, we say that it diverges.

Since Definition 4 refers only to a_n with $n \geq N_\epsilon$ and since N_ϵ can be as large as we wish, we only need to look at tails of sequences. We state this as a theorem and omit the proof.

Theorem 4 (Convergence and tails) *Let a_n, $n \geq n_0$, be an infinite sequence. The following are equivalent*

- *The sequence a_n converges.*
- *Every tail of the sequence a_n converges.*
- *Some tail of the sequence a_n converges.*

The theorem tells us that we can ignore any "inconvenient" terms at the beginning of a sequence when we are checking for convergence.

Example 8 (What does the Definition 4 mean?) It helps to have some intuitive feel for the definition of the limit of a sequence. We'll explore it here and in the next example.

The definition says a_n will be as close as you want to A if n is large enough. Note that the definition does not say that A is unique — perhaps a sequence could have two limits A and A^*. Since a_n will be as close as you want to A and also to A^* at the same time if n is large, we must have $A = A^*$. (If you don't see this, draw a picture where a_n is within $|A - A^*|/3$ of both A and A^*.) Since $A = A^*$ whenever A and A^* are limits of the same sequence, the limit is unique. We state this as a theorem:

Theorem 5 (The limit is unique) *An infinite sequence has at most one limit. In other words, if the limit of an infinite sequence exists, it is unique.*

Here's another way to picture the limit of an infinite sequence. Imagine that you are in a room sitting at a desk. You have with you a sequence a_n, $n = 0, 1, 2, \ldots$, that you have announced converges to a number A. Every now and then, there is a knock on the door and someone enters the room and gives you positive real number ϵ (like $\epsilon = 0.001$). You must give that person an integer $N_\epsilon > 0$ such that for all $n \geq N_\epsilon$, $|a_n - A| \leq \epsilon$. If you can do that, the person will go away contented. If you are able to convincingly prove that for any such $\epsilon > 0$ there is such an N_ϵ, then they will leave you alone because you are right in asserting that A is the limit of the sequence a_n, as n goes to infinity.

We can phrase the condition for A to be the limit of the sequence in logic notation:

$$\forall\, \epsilon > 0,\ \exists\, N_\epsilon,\ \forall\, n \geq N_\epsilon,\ |a_n - A| \leq \epsilon. \qquad \square$$

Suppose we know that a sequence $a_0, a_1, \ldots$ has a limit A and we want to estimate A. We can do this by computing a_n for large values of n. Of course, estimating the limit A only makes sense if we know the sequence has a limit. How can we know that the sequence has a limit? By Definition 4 of course! Unfortunately, Definition 4 requires that we know the value of A.

What can we do about this? We'd like to know that a limit exists without knowing the value of that limit. How can that be? Let's look at it intuitively. The definition says all the values of a_n are near A when n is large. But if they are all near A, then a_n and a_m must be near each other when n and m are large. (You should be able to see why this is so.) What about the converse; that is, if all the values of a_n and a_m are near each other when n and m are large are they near some A which is the limit of the sequence? We state the following theorem without proof.

Theorem 6 (Second "definition" of a convergent sequence) *Let a_n, $n \geq n_0$, be an infinite sequence.*

The sequence a_n, $n \geq n_0$, converges to some limit A
if and only if
for every real number $\epsilon > 0$ there is an N_ϵ such that for all $n, m \geq N_\epsilon$, $|a_n - a_m| \leq \epsilon$.

In other words, if the terms far out in the sequence are as close together as we wish, then the sequence converges.

Some students misunderstand the definition and think we only need to show that $|a_n - a_{n+1}| \leq \epsilon$ for $n \geq N_\epsilon$. *Don't fall into this trap.* The sequence $a_n = \log n$ shows that we can't do that because $\log n$ grows without limit but $|\log n - \log(n+1)| = \log(1 + 1/n)$ which can be made as close to zero as you want by making n large enough.

Most beginning students have little patience with the formal precision of Definition 4 and Theorem 6. If you look at a particular example such as the sequence $\frac{2n+1}{n+1}$, $n = 0, 1, 2, \ldots$, it is obvious that, as n goes to infinity, this sequence approaches $A = 2$ as a limit. So why confuse the obvious with such formality? The reason is that we need the precise definition of a limit is to enable us to discuss convergent sequences in general, independent of particular examples such as $\frac{2n+1}{n+1}$, $n \geq 0$. Without such formal definitions, we couldn't state general theorems precisely and proofs would be impossible.

Example 9 (Convergence from three viewpoints) Let's take a look at the convergence of $a_n = \frac{2n+1}{n+1}$, $n = 0, 1, 2, \ldots$ from three different points of view.

- First, we can manipulate the terms to see that they converge: Since

$$\frac{2n+1}{n+1} = \frac{2+1/n}{1+1/n}, \quad \lim_{n\to\infty}(2+1/n) = 2 \quad \text{and} \quad \lim_{n\to\infty}(1+1/n) = 1,$$

  we have

$$\lim_{n\to\infty}\frac{2n+1}{n+1} = \lim_{n\to\infty}\frac{2+1/n}{1+1/n} = \frac{\lim_{n\to\infty}(2+1/n)}{\lim_{n\to\infty}(1+1/n)} = 2/1 = 2.$$

- Second, using Definition 4, given $\epsilon > 0$, choose $N_\epsilon = 1/\epsilon$. Then, if $n \geq N_\epsilon$,

$$|a_n - 2| = \left|\frac{2n+1}{n+1} - 2\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N_\epsilon} = \epsilon.$$

- Third, using Theorem 6, given $\epsilon > 0$, choose $N_\epsilon = \frac{2}{\epsilon}$. We have

$$|a_n - a_m| = \left|\frac{2n+1}{n+1} - \frac{2m+1}{m+1}\right| = \left|\left(2 - \frac{1}{n+1}\right) - \left(2 - \frac{1}{m+1}\right)\right| = \left|\frac{1}{m+1} - \frac{1}{n+1}\right|.$$

  But, since $|x - y| \leq |x| + |y|$,

$$\left|\frac{1}{m+1} - \frac{1}{n+1}\right| \leq \frac{1}{m+1} + \frac{1}{n+1} < \frac{1}{N_\epsilon} + \frac{1}{N_\epsilon} = \frac{2}{N_\epsilon} = \epsilon.$$

The easiest method for showing convergence of a particular sequence is usually the first method. You may wonder about our values of N_ϵ in the other two methods:

- *How did we find them?* We found them by working from both ends. To illustrate, consider the third method. Suppose $n \geq N_\epsilon$ and $m \geq N_\epsilon$ but we don't know what to choose for N_ϵ. We found that $|a_n - a_m| < 2/N_\epsilon$. We want to know how to choose N_ϵ so that $|a_n - a_m| \leq \epsilon$. You should be able to see that it will be okay if $2/N_\epsilon \leq \epsilon$. Thus we need $N_\epsilon \geq 2/\epsilon$.
- *Would other values work?* Yes. If someone comes up with a value that works, then any larger value of N_ϵ would also work because it tells us to ignore more of the earlier values in the sequence. □

In Definition 4, we said that, if a sequence a_n, $n \geq n_0$, does not converge then it is said to diverge. So far we haven't looked at any examples. Here are two.

- The infinite sequence is $a_n = (-1)^n$ alternates between $+1$ and -1. It clearly fails our definition and theorem on convergence. For example, the theorem fails with any $0 < \epsilon < 2$. There is no N_ϵ such that for all $m, n \geq N_\epsilon$, $|a_n - a_m| \leq \epsilon$, since $|a_n - a_{n+1}| = 2$ for all $n \geq 0$.
- Another example of a divergent sequence is $b_n = \log n$, $n \geq 1$. Although

$$\lim_{n\to\infty}|b_n - b_{n+1}| \to 0,$$

$|b_n - b_{2n}| = \log 2$ and so the theorem fails for any $\epsilon < \log 2$.

The sequences a_n and b_n of the previous paragraph differ in a fundamental way, as described by the following definition.

Definition 5 (Bounded sequence) *A sequence a_n, $n = 0, 1, 2, \ldots$ is bounded if there exists a positive number B such that $|a_n| \le B$ for $n = 0, 1, 2, \ldots$.*

The sequence $a_n = (-1)^n$ is an example of a bounded divergent sequence. The sequence $b_n = \log n$ is an example of an unbounded divergent sequence. All the convergent sequences we have looked at are bounded. The next theorem shows that there are no unbounded convergent sequences.

Theorem 7 (Boundedness) *Convergent sequences are bounded.*

Proof: Let a_n, $n \ge n_0$, be convergent with limit A. Take $\epsilon = 1$. Then there is an N_1 such that for all $n \ge N_1$, $|a_n - A| \le 1$. Since a_n is within 1 of A, it follows that $|a_n| \le |A| + 1$ for all $n \ge N_1$. Let B be the maximum of $|a_{n_0}|, |a_{n_0+1}|, |a_{n_0+2}|, \ldots, |a_{N_1-1}|$, and $|A| + 1$. Then, $|a_n| \le B$ for $n \ge n_0$. $\square$

The converse of the previous theorem is, "Bounded sequences are convergent." This statement is false ($a_n = (-1)^n$ for example).

The next theorem gives some elementary rules for working with sequences.

Theorem 8 (Algebraic rules for sequences) *Suppose that a_n, $n \ge n_0$ and b_n, $n \ge n_0$ are convergent sequences and that*

$$\lim_{n\to\infty} a_n = A \quad \text{and} \quad \lim_{n\to\infty} b_n = B.$$

Define sequences t_n,r_n, s_n, p_n and q_n, $n \ge n_0$, by

$$\begin{array}{ll} t_n = \alpha a_n + \beta, \quad \alpha, \beta \in \mathbb{R}; & s_n = a_n + b_n; \\ p_n = a_n b_n; & \text{and, if } b_n \ne 0 \text{ for all } n \ge n_0, \quad q_n = a_n / b_n. \end{array}$$

Then

$$\lim_{n\to\infty} t_n = \alpha A + \beta, \quad \lim_{n\to\infty} s_n = A + B, \quad \lim_{n\to\infty} p_n = AB$$

and, if $B \ne 0$, $\lim_{n\to\infty} q_n = A/B$.

Proof: All we are given is that the sequences a_n and b_n converge. This means that $|a_n - A|$ and $|b_n - B|$ are small when n is large. The proof technique is to use that fact to show that other values are small. We illustrate the technique by proving the assertion about p_n. We omit the proofs for t_n, s_n and q_n.

We must show that we can make $|a_n b_n - AB|$ small. Thus, we need to relate $a_n - A$ and $b_n - B$ to $a_n b_n - AB$. An obvious idea is to try multiplying $a_n - A$ and $b_n - B$.

Unfortunately, the product is not of the right form, so we need to be more clever. After some experimentation, you might notice that

$$a_n b_n - AB = a_n(b_n - B) + B(a_n - A)$$

and that the parenthesized expressions are small. This is the key! We have

$$|a_n b_n - AB| = |a_n(b_n - B) + B(a_n - A)| \le |a_n||b_n - B| + |B||a_n - A|.$$

By Theorem 7, there is a constant A^* such that $|a_n| \le A^*$ for all n. Thus

$$|a_n b_n - AB| \le A^*|b_n - B| + |B||a_n - A|.$$

This says that, for all large n, $|a_n b_n - AB|$ is at most a constant (A^*) times a small number ($|b_n - B|$) plus a constant times another small number. If we were being informal in our proof, we could stop here. However, a formal proof requires that we tell how to compute N_ϵ for the sequence $a_n b_n$.

We find the rule for N_ϵ by, in effect, working backwards. For $\delta > 0$, let N^*_δ be such that $|a_n - A| \le \delta$ and $|b_n - B| \le \delta$ for all $n \ge N^*_\delta$. We can do this because a_n and b_n converge. Now we have

$$|a_n b_n - AB| \le A^*|b_n - B| + |B||a_n - A| \le A^*\delta + |B|\delta = (A^* + |B|)\delta.$$

Since we want this to be at most epsilon, we define δ by $(A^* + |B|)\delta = \epsilon$. Thus
$\delta = \epsilon/(A^* + |B|)$ and so $N_\epsilon = N^*_{\epsilon/(A^*+|B|)}$. □

An important class of sequences are those which are "eventually monotone," a concept we now define.

Definition 6 (Monotone sequence) *A sequence a_n, $n \ge n_0$, is*

- *increasing if $a_{n_0} < a_{n_0+1} < a_{n_0+2} < \cdots$,*
- *decreasing if $a_{n_0} > a_{n_0+1} > a_{n_0+2} > \cdots$,*
- *nondecreasing if $a_{n_0} \le a_{n_0+1} \le a_{n_0+2} \le \cdots$,*
- *nonincreasing if $a_{n_0} \ge a_{n_0+1} \ge a_{n_0+2} \ge \cdots$,*
- *monotone if it is either nonincreasing or nondecreasing.*

If a tail of the sequence is monotone, we say the sequence is eventually monotone. We define "eventually increasing" and so on similarly.

Nonincreasing is also called "weakly decreasing" and nondecreasing is also called "weakly increasing." If you understand the definition, you should see the reason for this terminology.

Eventually monotone sequences are fairly common and have nice properties. The following theorem gives one property.

Theorem 9 (Convergence of bounded monotone sequences) *If an infinite sequence is bounded and eventually monotone, then it converges.*

We won't prove this theorem. It is, in a very basic sense, a fundamental property of real numbers. We leave the understanding of this theorem to your intuition. The power of the theorem is in its generality so that it can be applied in discussing sequences in general as well as to discussing specific examples.

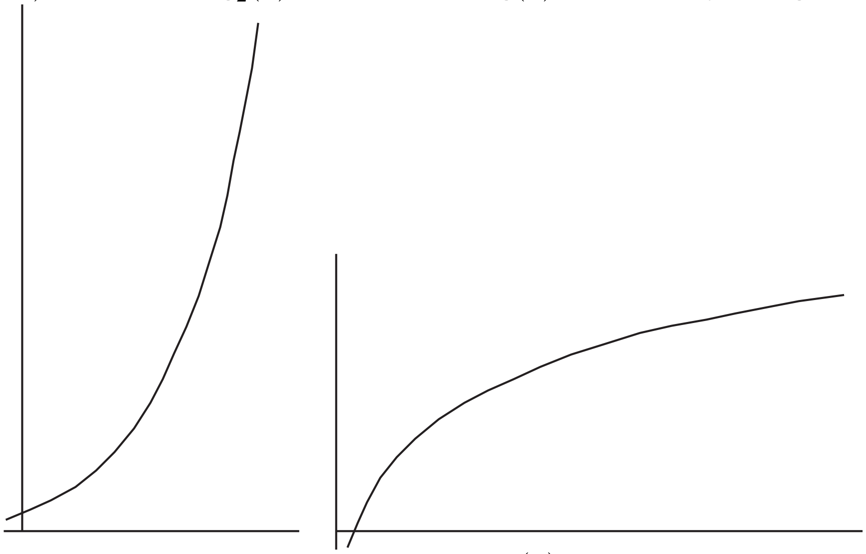
We now study three common classes of eventually monotone functions and their relative rates of growth.

Example 10 (Polynomials, exponentials and logarithms) Consider the sequence a_n, $n = 0, 1, 2, \ldots$, where $a_n = n/1.1^n$. It is a fact that you probably learned in high school, and certainly learned if you have had a course in calculus, that any exponential function $f(x) = b^x$, $b > 1$, "grows faster" than any polynomial function $g(x) = c_k x^k + \ldots + c_1 x + c_0$. By this we mean that

$$\lim_{x\to\infty} g(x)/f(x) = 0 \quad \text{when} \quad g(x) = c_k x^k + \ldots + c_1 x + c_0, \quad f(x) = b^x \quad \text{and} \quad b > 1.$$

If for example, we take the sequence a_n, $n = 0, 1, 2, \ldots$, where $a_n = n^3/2^n$, we get $a_0 = 0$, $a_1 = 1/5$, $a_2 = 2.25$, $a_3 = 3.375$, $a_4 = 4$, and $a_5 = 3.90625$. Some calculations may convince you that $a_4 > a_5 > a_6 > \cdots$, and so the sequence is eventually decreasing.

Recall from high school that the inverse function of the function b^x is the function $\log_b(x)$. That these functions are inverses of each other means that $b^{\log_b(x)} = \log_b(b^x) = x$ for all $x > 0$. It is particularly important that all computer science students understand the case $b = 2$ as well as the usual $b = e$ (the "natural log") and $b = 10$. You should graph 2^x, for $-1 \le x \le 5$ and $\log_2(x)$ for $0.5 \le x \le 32$. You can compute $\log_2(x)$ on your calculator using the LN key: $\log_2(x) = \mathrm{LN}(x)/\mathrm{LN}(2)$ (or you can use the LOG key instead of LN). Note that $\log_2(x)$ is also written $\lg(x)$. Here are typical graphs for $b > 1$.



b^x $\qquad\qquad$ $\log_b(x)$

Notice that, although both b^x and $\log_b(x)$ get arbitrarily large as x gets arbitrarily large, b^x grows *much* more rapidly than $\log_b(x)$. In fact, $\log_b(x)$ grows so slowly that, for any $\alpha > 0$

$$\lim_{x\to\infty} \log_b(x)/x^\alpha = 0.$$

For example,

$$\lim_{x\to\infty} \log_b(x)/x^{0.01} = 0.$$

For those of you who have had some calculus, you can prove the above limit is correct by using l'Hospital's Rule. If you haven't had calculus, you can do some computations with your computer or calculator to get a feeling for this limit. For example, if $b = 2$ then

$$\log_2(2^{10})/2^{0.10} = 9.33033 \quad \log_2(2^{100})/2^{1.0} = 50 \quad \log_2(2^{1000})/2^{10.0} = 0.976563\,.$$

If for example, we take the sequence a_n, $n = 0, 1, 2, \ldots$, where $a_n = \log_2(n)/n^{0.01}$, we will find that the sequence increases at first. But, starting at some (rather large) m, we have $a_m > a_{m+1} > a_{m+2} > \cdots$. These terms will continue to get smaller and smaller and approach zero as a limit. The sequence is eventually decreasing.

These are examples of general results such as:

> If $b > 1$, $c > 0$ and $d > 0$, then n^c/b^{n^d} and $(\log_b(n^d))/n^c$ are eventually monotonic sequences that converge to zero.

We omit the proof. One can replace n^c and n^d by more general functions of n.

People may write log without specifying a base as in $\log_b$. What do they mean? Some people mean $b = 10$ and others mean $b = e$. Still others mean that it doesn't matter what value you choose for b as long as it's the same throughout the discussion. That's what we mean — if there's no base on the logarithm, choose your favorite $b > 1$. □

We conclude our discussion of sequences with a discussion of "converges to infinity."

In Definition 4, we defined what it means for a sequence to have a real number A as its limit. We also find in many mathematical discussions, the statement that "a_n, $n = 0, 1, 2, \ldots$ has limit $+\infty$" or "a_n, $n = 0, 1, 2, \ldots$ has limit $-\infty$." Alternatively, one sees "a_n, $n = 0, 1, 2, \ldots$ tends to $+\infty$, converges to $+\infty$, or diverges to $+\infty$. In symbols,

$$\lim_{n\to\infty} a_n = +\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = -\infty.$$

This use of "limit" is really an abuse of the term. Such sequences are actually divergent sequences, but they diverge with a certain consistency. Thus, $a_n = n$, $n = 0, 1, 2, \ldots$ or $a_n = -n$, $n = 0, 1, 2, \ldots$, though divergent, are said to "have limit $+\infty$" or "have limit $-\infty$, " respectively. Compare this with the divergent sequence $a_n = (-1)^n n$, $n = 0, 1, 2, \ldots$, which hops around between ever increasing positive and negative values. Here is a formal definition.

Definition 7 (Diverges to infinity) *Let a_n, $n \ge n_0$ be an infinite sequence. We say that the sequence converges to $+\infty$ or that it diverges to $+\infty$ and write*

$$\lim_{n\to\infty} a_n = +\infty$$

if, for every real number $r > 0$, there exists N_r such that for all $n \ge N_r$, $a_n \ge r$.

Similarly, we say that the sequence converges to $-\infty$ or that it diverges to $-\infty$ and write

$$\lim_{n\to\infty} a_n = -\infty$$

if, for every real number $r < 0$, there exists N_r such that for all $n \ge N_r$, $a_n \le r$.

Exercises for Section 2

2.1. *For each of the following sequences, answer the following questions.*

- *Is the sequence bounded?*
- *Is the sequence monotonic?*
- *Is the sequence eventually monotonic?*

(a) $a_n = n$ *for all* $n \geq 0$.

(b) $a_n = 1$ *for all* $n \geq 0$.

(c) $a_n = 2n + (-1)^n$ *for all* $n \geq 0$.

(d) $a_n = n + (-1)^n 2$ *for all* $n \geq 0$.

(e) $a_n = 2^n - 10n$ *for all* $n \geq 0$.

(f) $a_n = 10 - 2^{-n}$ *for all* $n \geq 0$.

2.2. *Discuss the convergence or divergence of the following sequences:*

(a) $\frac{2n^3+3n+1}{3n^3+2}$, $n = 0, 1, 2, \ldots$

(b) $\frac{-n^3+1}{2n^2+3}$, $n = 0, 1, 2, \ldots$

(c) $\frac{(-n)^n+1}{n^n+1}$, $n = 0, 1, 2, \ldots$

(d) $\frac{n^n}{(n/2)^{2n}}$, $n = 1, 2, \ldots$

2.3. *Discuss the convergence or divergence of the following sequences:*

(a) $\frac{\log_2(n)}{\log_3(n)}$, $n = 1, 2, \ldots$

(b) $\frac{\log_2(\log_2(n))}{\log_2(n)}$, $n = 2, 3, \ldots$

*Section 3: Infinite Series

We now look at infinite series. Every infinite series is associated with two infinite sequences. Thus the study of infinite series can be thought of as the study of sequences. However, the viewpoint is different.

Definition 8 (Infinite series) *Let* a_n, $n \geq n_0$, *be an infinite sequence. Define a new sequence* s_n, $n \geq n_0$, *by*

$$s_n = a_{n_0} + a_{n_0+1} + \cdots + a_n = \sum_{k=n_0}^{n} a_k.$$

The infinite sequence s_n is called the sequence of partial sums of the sequence a_n. We call a_n a term of the series.

If $\lim_{n\to\infty} s_n$ exists, we write

$$\sum_{k=n_0}^{\infty} a_k = \lim_{n\to\infty} s_n.$$

We call $\sum_{k=n_0}^{\infty} a_k$ the infinite series whose terms are the a_k and whose sum is $\lim_{n\to\infty} s_n$. We say the infinite series converges to $\lim_{n\to\infty} s_n$.

If $\lim_{n\to\infty} s_n$ does not exist, we still speak of the infinite series $\sum_{k=n_0}^{\infty} a_k$, but now we say that the series diverges and that it has no sum. If s_n diverges to $+\infty$ or to $-\infty$, we say that the infinite series diverges to $+\infty$ or to $-\infty$.

The infinite series associated with a tail of a sequence, is a tail of the infinite series associated with the sequence. In this case, mathematical notation is clearer than words: If $t \ge n_0$, then

$$\sum_{k=t}^{\infty} a_k \quad \textit{is a tail of} \quad \sum_{k=n_0}^{\infty} a_k.$$

So where are we? Given an infinite sequence a_n, $n \ge n_0$, we can ask whether the infinite series $\sum_{k=n_0}^{\infty} a_k$ converges. This is the same as asking whether the sequence of partial sums converges. So what's new? There are often situations where we know something about the terms a_n and are interested in the sum of the series. For example, what can be said about the value of $\sum_{k=1}^{\infty} 1/k$? the value of $\sum_{k=0}^{\infty}(-1)^k/k!$? We get to see the terms, but we're interested in the sum. Thus, we want to use information about the infinite sequence a_n to say something about the infinite sequence s_n of partial sums. This presence of *two* sequences is what makes the study of infinite series different from the study of a single sequence. Here's a simple example of that interplay:

Theorem 10 (Terms are small) *If the infinite series $\sum_{n=n_0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof: We are given that the infinite series converges, which means that the sequence $s_n = \sum_{k=n_0}^{n} a_n$ converges. We use Theorem 6 with $m = n-1$ and a_n in the theorem replaced by s_n. By Theorem 6, whenever n is large enough

$$\epsilon \ge |s_n - s_m| = |s_n - s_{n-1}| = |a_n| = |a_n - 0|.$$

Since ϵ can be made as close to zero as we wish, this proves that $\lim_{n\to\infty} |a_n - 0| = 0$. Therefore a_n converges to zero. $\square$

Example 11 (Geometric series) For $r \in \mathbb{R}$, let $a_n = r^n$, $n \geq 0$. The partial sum s_n associated with a_n is called a *geometric series*. Note that, from high school mathematics,

$$s_n = \sum_{k=0}^{n} r^k = \begin{cases} \frac{r^{n+1}-1}{r-1} & \text{if } r \neq 1, \\ n+1 & \text{if } r = 1. \end{cases}$$

If $|r| \geq 1$, the infinite series $\sum_{k=0}^{\infty} r^k$ diverges by Theorem 10. If $|r| < 1$ then

$$\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

For example, when $r = 2/3$, we have $\sum_{k=0}^{\infty}(2/3)^k = 3$. $\square$

Example 12 (Harmonic series) A basic infinite series, denoted by H_n, is the one that is associated with the sequence $a_n = 1/n$, $n = 1, 2, \ldots$. Let $H_n = a_1 + \cdots + a_n$ denote the partial sums of this series. The sequence H_n, $n = 1, 2, \ldots$, is called the *harmonic series* (for reasons that any of you who have studied music will know). In infinite series notation, this series can be represented by

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We can visualize this series by grouping its terms as follows:

$$\underbrace{\frac{1}{1}}_{b_0} + \underbrace{\frac{1}{2} + \frac{1}{3}}_{b_1} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}_{b_2} + \underbrace{\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}}_{b_3} + \cdots.$$

Note that b_k contains the terms

$$\frac{1}{2^k} \quad \frac{1}{2^k+1} \quad \cdots \quad \frac{1}{2^{k+1}-1} = \frac{1}{2^k + (2^k - 1)}$$

and so contains 2^k terms. Which b_k is $\frac{1}{11}$ in? Easy. Just take $\lfloor \log_2(11) \rfloor = 3$ and you get the answer, b_3. In general, $\frac{1}{n}$ is in b_k where $k = \lfloor \log_2(n) \rfloor$.

What is a lower bound for the sum of all the numbers in b_3? Easy. They are all bigger than $\frac{1}{16}$, the first number in b_4. There are 8 numbers in b_3, all bigger than $\frac{1}{16}$, so a lower bound is $b_3 > 8 \times \frac{1}{16} = \frac{1}{2}$. You can do this calculation in general for group b_k, getting $b_k > \frac{1}{2^{k+1}} \times 2^k = \frac{1}{2}$. Now that you are getting a feeling for this grouping, you can see that an upper bound for the sum of the terms in b_k is $\frac{1}{2^k} \times 2^k = 1$. Thus

$$\frac{1}{2} \leq b_k \leq 1.$$

Now suppose you pick an integer n and want to get an estimate on the size of H_n. To get a lower bound just find the k such that b_k contains the term $1/n$. (By our earlier work, $k = \lfloor \log_2(n) \rfloor$.) Then

$$H_n > b_0 + b_1 + \cdots + b_{k-1} > k/2 \quad \text{and} \quad H_n \leq b_0 + b_1 + \cdots + b_k \leq k+1.$$

Using our value for k and the fact that $x - 1 < \lfloor x \rfloor \leq x$, we have

$$\frac{\log_2(n) - 1}{2} < H_n \leq \log_2(n) + 1.$$

We learned in Example 10 that $\log_2(n)$ is a very slowly growing function of n. But it does get arbitrarily large (has limit $+\infty$). Thus, H_n grows very slowly and diverges.

There is more to the story of the harmonic series. Although the derivations are beyond the scope of our study, the results are worth knowing. Here is a very interesting way of representing H_n:

$$H_n = \ln(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\epsilon_n}{120n^4} \quad \text{where} \quad 0 < \epsilon_n < 1.$$

The "ln" refers to the natural logarithm. It is a special function key on all scientific calculators. To ten decimal places, $\gamma = 0.5772156649$.

Your first reaction might be, "What good is this formula, we don't know ϵ_n exactly?" Since $\epsilon_n > 0$, we'll get a number that is less than H_n if we throw away $\epsilon_n/120n^4$. Since $\epsilon_n < 1$, we'll get a number that is greater than H_n if we replace $\epsilon_n/120n^4$ with $1/120n^4$. These upper and lower bounds for H_n are quite close together — they differ by $1/120n^4$. With $n = 10$ we have upper and lower bounds that differ by only $1/1200000 = 0.0000008333\ldots$. For example, by adding up the terms we get $H_{10} = 2.928968254$ to nine decimal places. The lower bound gotten with $\epsilon_{10} = 0$ is 2.928967425 and the upper bound gotten with $\epsilon_{10} = 1$ is 2.928968258. Get the idea? No matter what value ϵ_n takes in the interval from 0 to 1, the denominator $120n^4$ grows rapidly with n, so the error is small. □

Example 13 (Alternating harmonic series) Let h_n be the sequence of partial sums associated with the sequence $(-1)^{n-1}/n$ for $n \geq 1$. The series h_n is called the *alternating harmonic* series. What about the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}?$$

It converges. To see why, imagine that you are standing in a room with your back against the wall. Imagine that you step forward 1 meter, then backwards 1/2 meter, then forwards 1/3 meter, etc. After n such steps, your distance from the wall is h_n meters. By the time you are stepping backwards one millimeter, forwards 0.99 millimeter, etc., an observer in the room (who by now has decided that you are crazy) would conclude that you are standing still. In other words, you have converged. It turns out your position doesn't converge to infinity because your forward and backward motions practically cancel each other out. How can we see this? Each pair of forward–backward steps moves you a little further from the wall; e.g., $1 - \frac{1}{2} = \frac{1}{2}$, $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. Thus you never have to step through the wall. (All partial sums are positive.) On the other hand, after first stepping forward 1 meter, each following pair of backward–forward steps moves you a little closer to the the wall; e.g., $-\frac{1}{2} + \frac{1}{3} = \frac{-1}{6}$, $-\frac{1}{4} + \frac{1}{5} = \frac{-1}{20}$. Thus you are never further than 1 meter from the wall.

This argument works just as well for any size steps as long as they are decreasing in size towards zero and are alternating forward and backwards. In the case of the alternating harmonic series, your distance from the wall will converge to $\ln(2)$, meters, where "ln" is the natural logarithm. We won't prove this fact, as it is best proved using calculus. You can check this out on your calculator or computer by adding up a lot of terms in the series. $\square$

A series is called *alternating* if the terms alternate in sign; that is, the sign pattern of terms is $+ - + - \cdots$ or $- + - + \cdots$.

Example 14 (Some particular alternating series and variations) By taking particular sequences a_n that converge monotonically to zero, you get particular alternating series. Here are some examples of alternating convergent series:

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}} \qquad \sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln(n)} \qquad \sum_{n=3}^{\infty}(-1)^n\frac{1}{\ln(\ln(n))}.$$

It is an interesting fact about such series that the sequence $(-1)^n$ in the above examples can be replaced by any sequence b_n which has bounded partial sums. Of course, $(-1)^n$, $n = k, k+1, \ldots$, has bounded partial sums for any starting value k (bounded by $B = 1$). For example, it can be shown that $b_n = \sin(n)$ and $b_n = \cos(n)$ are sequences with bounded partial sums.[2] Thus,

$$\sum_{n=1}^{\infty}\sin(n)\frac{1}{n} \quad \text{and} \quad \sum_{n=0}^{\infty}\cos(n)\frac{1}{\ln(n)}$$

are convergent generalized "alternating" series. The fact that these generalized "alternating" series converge is proved in more advanced courses and called *Dirichlet's Theorem.* $\square$

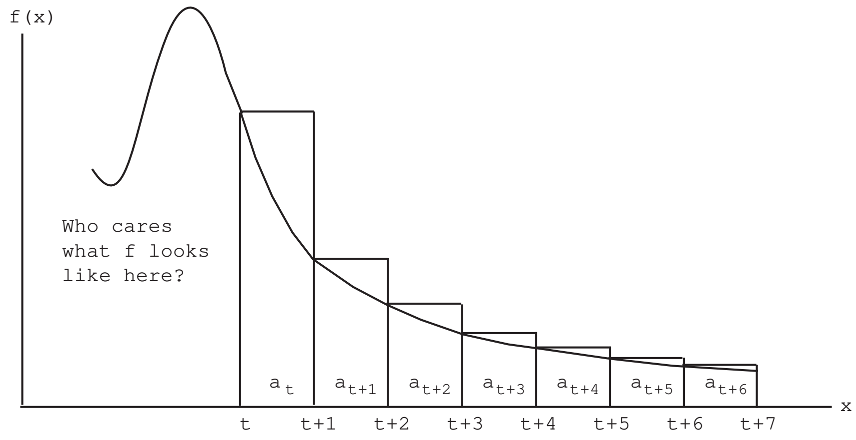
Example 15 (Series and the integral test) Suppose we have a function $f(x)$ that is defined for all $x \geq m$ where $m \geq 0$ is an integer. Then we can associate with $f(x)$ a sequence $a_n = f(n)$, $n \geq m$. In summation notation, $\sum_{n=m}^{\infty} a_n$ is an infinite series, and we are interested in the divergence or convergence of this series. Suppose that $f(x)$ is weakly decreasing for all $x \geq t$ where $t \geq m$. Study the pictures shown below. If the area under the curve is infinite, as intended in the first picture, then the summation $\sum_{k=t}^{\infty} a_k$, which represents the sum of the areas of the rectangles, must also be infinite.

If the area under the curve is finite, as in the second picture, then the summation $\sum_{k=t}^{\infty} a_k$, which represents the sum of the areas of the rectangles, must also be finite.

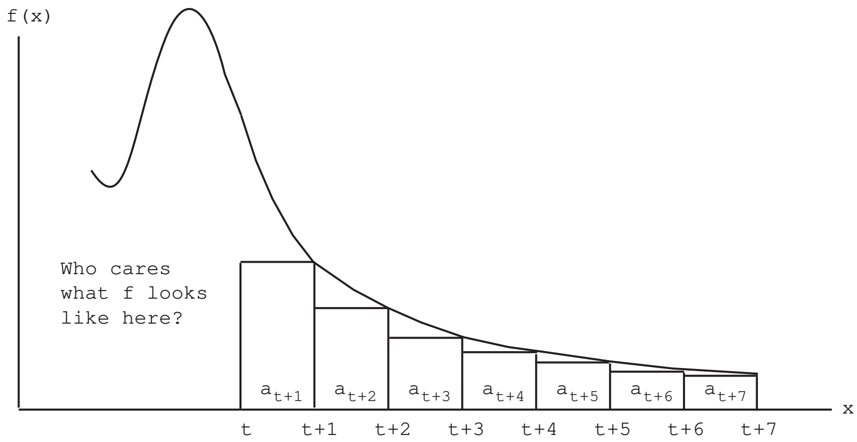
[2] Here is how it's done for those of you who are familiar with complex numbers and Euler's relation. From Euler's relation, $\cos(n) = \Re(e^{in})$ and so

$$\sum_{n=0}^{N}\cos(n) = \Re\left(\sum_{n=0}^{N}(e^i)^n\right) = \Re\left(\frac{e^{i(N+1)}-1}{e^i-1}\right).$$

Since the numerator is bounded and the denominator is constant, this is bounded.



$$\text{If } a_k = f(k),\ k \geq t,\ \text{then } \int_t^{\infty} f(x)\,\mathrm{d}x = +\infty \quad \text{implies} \quad \sum_{k=t}^{\infty} a_k \text{ diverges.}$$



$$\text{If } a_k = f(k),\ k \geq t,\ \text{then } \int_t^{\infty} f(x)\,\mathrm{d}x < +\infty \quad \text{implies} \quad \sum_{k=t}^{\infty} a_k \text{ converges.}$$

In one or the other of the two cases, we conclude that a tail of the given series diverges or converges and, thus, that the given series diverges or converges.

This way of checking for convergence and/or divergence is called the *integral test*. □

Example 16 (General harmonic series) We can extend the harmonic series H_n with terms $\frac{1}{n}$ to a series $H_n^{(r)}$ based on the sequence $\frac{1}{n^r}$, where r is a real number. We call the series the *general harmonic series with parameter* r. In summation notation, this series is

$$\sum_{n=1}^{\infty} \frac{1}{n^r}.$$

If $r \le 0$ then it is obvious that $H_n^{(r)}$, $n = 1, 2, \ldots$, diverges. For example, $r = -1$ gives the series

$$\sum_{n=1}^{\infty} n,$$

which diverges. If $r > 0$ then the function $f_r(x) = \frac{1}{x^r}$ is strictly decreasing for $x \ge 1$. This means that we can apply the integral test with $t = 1$.

From calculus, it is known that $\int_1^\infty (1/x^r)\,\mathrm{d}x = +\infty$ if $r \le 1$. It is also known that $\int_1^\infty (1/x^r))\,\mathrm{d}x = \frac{1}{r-1}$ if $r > 1$. Thus, by the integral test,

$$\sum_{n=1}^{\infty} \frac{1}{n^r} \quad \text{diverges if } 0 < r \le 1 \text{ and converges if } r > 1. \qquad \square$$

The integral test can produce some surprises. The harmonic series H_n, based on $\frac{1}{n}$, $n = 1, 2, \ldots$, diverges. But what about the series s_n, $n = 2, 3, \ldots$, based on $\frac{1}{n\ln(n)}$? The terms of that series get smaller faster, so maybe it converges? Applying the integral test gives

$$\int \frac{1}{x\ln(x)}\,\mathrm{d}x = \ln(\ln(x)) + C \quad \text{so} \quad \int_2^\infty \frac{1}{x\ln(x)}\,\mathrm{d}x = +\infty.$$

Thus

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)} \quad \text{diverges.}$$

It looks like $\ln(n)$ just doesn't grow fast enough to help make the terms $1/n\ln(n)$ small enough for convergence. So using $\ln(n)$ twice probably won't help. It gives the series s_n, $n = 2, 3, \ldots$, based on $\frac{1}{n(\ln(n))^2}$. We have

$$\int \frac{1}{x(\ln(x))^2}\,\mathrm{d}x = \frac{-1}{\ln(x)} + C \quad \text{so} \quad \int_2^\infty \frac{1}{x(\ln(x))^2}\,\mathrm{d}x < \frac{1}{\ln(2)}.$$

Thus

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2} \quad \text{converges!}$$

In fact, if $\delta > 0$, then

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^{1+\delta}} \quad \text{converges.}$$

You should prove this by using the integral test.

Definition 9 (Absolute convergence) *Let s_n, $n = 0, 1, 2, \ldots$, be a series based on the sequence a_n, $n = 0, 1, 2, \ldots$. Let t_n, $n = 0, 1, 2, \ldots$, be a series based on the sequence $|a_n|$, $n = 0, 1, 2, \ldots$. If the series t_n converges then the series s_n is said to converge absolutely or to be absolutely convergent. In other words,*

$$\sum_{n=0}^{\infty} a_n \quad \text{converges absolutely if} \quad \sum_{n=0}^{\infty} |a_n| \quad \text{converges.}$$

If a series is convergent, but not absolutely convergent, then it is called conditionally convergent.

Any geometric series with $|r| < 1$ is absolutely convergent. The alternating harmonic series is convergent but not absolutely convergent (since the harmonic series diverges).

Theorem 11 (Absolute convergence and bounded sequences) *Suppose that s_n, $n \geq n_0$ is an absolutely convergent series based on the sequence a_n, $n \geq n_0$. Let b_n, $n \geq n_0$ be a bounded sequence. Then the series p_n, $n \geq n_0$, based on the sequence $a_n b_n$, $n \geq n_0$, is absolutely convergent. In other words,*

$$\sum_{n=n_0}^{\infty} a_n \text{ converges absolutely and } b_n \text{ bounded} \quad \text{implies} \sum_{n=n_0}^{\infty} a_n b_n \text{ converges absolutely.}$$

Proof: Let $M > 0$ be a bound for b_n. Thus, $M \geq |b_n|$, $n \geq n_0$. Since a_n is absolutely convergent, given $\epsilon > 0$, there exists $N_{\epsilon/M}$ such that for all $i \geq j \geq N_{\epsilon/M}$, $|a_{j+1}| + |a_{j+2}| + \cdots |a_i| \leq \epsilon/M$. Given $\epsilon > 0$, let $N_\epsilon = N_{\epsilon/M}$. Then, for all $i \geq j \geq N_\epsilon$,

$$|a_{j+1}||b_{j+1}| + |a_{j+2}||b_{j+2}| + \cdots |a_i||b_i| \leq (|a_{j+1}| + |a_{j+2}| + \cdots |a_i|)M \leq (\epsilon/M)M = \epsilon.$$

This shows that p_n is absolutely convergent. $\square$

Example 17 (Series convergence and using your intuition) Based on the ideas we have studied thus far, you can develop some very powerful intuitive ideas that will correctly tell you whether or not a series converges. We discuss these without proof. The basic idea is to look at a constant C times a convergent or divergent series: $C\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} C\, a_n$. Then think about what conditions on a sequence b_n, will allow you to replace the constant C on the right hand side by b_n to get $\sum_{n=0}^{\infty} b_n a_n$ without changing the convergence or divergence of the series. Here are some specific examples:

(1) Suppose the series $\sum_{n=0}^{\infty} a_n$ converges absolutely. An example is $a_n = r^n$, $0 \leq r < 1$ (i.e., the geometric series). If you have a bounded sequence b_n, $n = 0, 1, 2, \ldots$ then you can replace C to get $\sum_{n=0}^{\infty} b_n a_n$ and still retain absolute convergence. This was proved in Theorem 11. An example is

$$\sum_{n=0}^{\infty}(1 + \sin(n))r^n\,, 0 \leq r < 1.$$

Note that, b_n can be any convergent sequence (which is necessarily bounded). One way this situation arises in practice is that you are given a series such as

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^3+1}.$$

You notice that the terms $\frac{2n+1}{n^3+1}$ can be written $\frac{2+1/n}{n^2+1/n}$ and thus, for large n, the original series should be very similar to the terms of the series

$$\sum_{n=1}^{\infty} \frac{2}{n^2},$$

which converges absolutely (general harmonic series with parameter 2). Thus, the original series with terms $\frac{2n+1}{n^3+1}$ converges absolutely. Here is an explanation based on absolute convergence and bounded sequences. Start with the absolutely convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Here, $a_n = n^{-2}$. Let $c_n = (2n+1)/(n^3+1)$. The $\lim_{n\to\infty} c_n/a_n = 2$. By our previous discussion, with $b_n = c_n/a_n$,

$$\sum_{n=0}^{\infty} b_n a_n = \sum_{n=1}^{\infty} \frac{2n+1}{n^3+1}$$

converges absolutely.

(2) Suppose the series $\sum_{n=n_0}^{\infty} a_n$ converges (but perhaps only conditionally). In that case, you can replace the constant C by any eventually monotonic convergent sequence b_n. In this case, $\sum_{n=n_0}^{\infty} a_n b_n$ converges. This result is proved in more advanced courses and called *Abel's Theorem.* For example, take the alternating series

$$\sum_{n=1}^{\infty} C\,\frac{(-1)^n}{\sqrt{n}},$$

which converges by Example 14. Replace C with $b_n = (1 + 1/\sqrt{n})$ which is weakly decreasing, converging to 1:

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right) \frac{(-1)^n}{\sqrt{n}}.$$

The monotonicity of b_n is *important.* If we replace C with $b_n = (1 + (-1)^n/\sqrt{n})$ which converges to 1 but is not monotonic. We obtain

$$S = \sum_{n=1}^{\infty} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right) \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right).$$

Since

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{converges and} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges,}$$

S diverges. □

We conclude this section by looking at the question "How common are primes?" What does this mean? Suppose the primes are called p_n so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$ and so on. We might ask for an estimate of p_n. It turns out that p_n is approximately $n \ln n$. In fact, the *Prime Number Theorem*, states that the ratio of p_n and $n \ln n$ approaches 1 as n goes to infinity. The proof of the theorem requires much more background in number theory and much more time than is available in this course.

It might be easier to look at $p_1 + \cdots + p_n$. Indeed it is, but it is still too hard for this course.

It turns out that things are easier if we can work with an infinite sum. Of course $p_1 + \cdots + p_n + \cdots$ diverges to infinity because there are an infinite number of primes, so that sum is no help. What about summing the reciprocals:

$$\sum_{n=1}^{\infty} \frac{1}{p_n}?$$

Now we're onto something useful that is within our abilities!

- If the primes are not very common, we might expect $p_n \geq Cn^{1+\delta}$ for some $\delta > 0$ and some C. In that case, $\sum 1/p_n$ converges because $\sum 1/p_n \leq C \sum 1/n^{1+\delta}$ and this general harmonic series converges by Example 16.
- On the other hand, if the primes are fairly common, then $\sum 1/p_n$ might diverge because $\sum 1/n$ diverges.[3]

How can we study $\sum 1/p_n$? The key is unique factorization.

Imagine that it made sense to talk about the infinite sum $1+2+3+4+\cdots$. We claim that then

$$1+2+3+4+\cdots = (1+2+2^2+2^3+\cdots)(1+3+3^2+\cdots)(1+5+5^2+\cdots)\cdots,$$

where the factors on the right are sums of powers of primes. Why is this? Imagine that you multiply this out using the distributive law. Let's look at some number, say $300 = 2^2 \times 3 \times 5^2$. We get it by taking 2 from $1+2+2^2+\cdots$, 3^2 from $1+3+3^2+\cdots$, 5^2 from $1+5+5^2+\cdots$ and 1 from each of the remaining factors $1+p+p^2+\cdots$ for $p = 7,\ 11, \ldots$. This is the only way to get 300 as a product. In fact, by unique factorization, each positive integer is obtained exactly once this way.

Instead, suppose we do this with the reciprocals, remembering that $p^0 = 1$. We have

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \left(\frac{1}{p_1^0}+\frac{1}{p_1^1}+\frac{1}{p_1^2}+\cdots\right)\left(\frac{1}{p_2^0}+\frac{1}{p_2^1}+\frac{1}{p_2^2}+\cdots\right)\cdots.$$

Each of the series in parentheses is a geometric series and so it can be summed. In fact

$$\frac{1}{p_n^0}+\frac{1}{p_n^1}+\frac{1}{p_n^2}+\cdots = \frac{1}{1-1/p_n} = \frac{p_n}{p_n-1} = 1+\frac{1}{p_n-1}.$$

We can't give a proof in this way because the series we started with is the harmonic series, which diverges, and we don't have tools for dealing with divergent series. As a result, we work backwards and give a proof by contradiction.

Suppose that $\sum 1/p_n$ converges. Since the terms are positive, it converges absolutely. We now introduce a mysterious sequence b_n. (Actually the values of b_n were found by continuing with the incorrect approach in the previous paragraph.) Let

$$b_n = p_n \log\left(1+\frac{1}{p_n-1}\right).$$

[3] In fact, $\sum 1/p_n$ diverges because p_n behaves like $n \ln n$ (which we can't prove) and $\sum 1/n \ln n$ diverges by Example 16.

We claim b_n is bounded. This can be proved easily by l'Hôpital's Rule, but we omit the proof since we have not discussed l'Hôpital's Rule. Let $a_n = 1/p_n$. Remember that we are assuming $\sum 1/p_n$ converges. By Theorem 11, $\sum a_n b_n$ converges. By the previous paragraph, $a_n b_n$ is the logarithm of

$$\frac{1}{p_n^0} + \frac{1}{p_n^1} + \frac{1}{p_n^2} + \cdots .$$

Hence, again by the previous paragraph, $\sum a_n b_n$ is the logarithm of the harmonic series. Since $\sum a_n b_n$ converges, so does the harmonic series. This is a contradiction.

Since we reached a contradiction by assuming that $\sum 1/p_n$ converges, it follows that $\sum 1/p_n$ diverges and so the primes are fairly common. How close are we to the Prime Number Theorem (p_n behaves like $n \ln n$)? If p_n grew much faster than this, say $p_n > Cn(\ln n)^{1+\delta}$ for some C and some $\delta > 0$, then $\sum 1/p_n$ would converge because $\sum 1/n(\ln n)^{1+\delta}$ converges by Example 16. But we've just shown that $\sum 1/p_n$ diverges.

Exercises for Section 3

3.1. Discuss the convergence or divergence of the following series:

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{2^{n/2}}{n^2+n+1} \qquad\qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{n+1}{2n+1}$$

3.2. Discuss the convergence or divergence of the following series:

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{n^5}{5^n} \qquad\qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{1}{n^2-150}$$

3.3. Discuss the convergence or divergence of the following series:

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{1}{(n^3-n^2-1)^{1/2}} \qquad\qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{(n+1)^{1/2}-(n-1)^{1/2}}{n}$$

3.4. Discuss the convergence or divergence of the following series:

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right)$$

$$\text{(b)}\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

3.5. Discuss the convergence or divergence of the following series:

$$\text{(a)}\ \sum_{n=0}^{\infty} \frac{\sin(n)}{|n-99.5|} \qquad\qquad \text{(b)}\ \sum_{n=0}^{\infty} (-1)^n \frac{-9n^2-5}{n^3+1}$$

Multiple Choice Questions for Review

In each case there is one correct answer (given at the end of the problem set). Try to work the problem first without looking at the answer. Understand both why the correct answer is correct and why the other answers are wrong.

1. Which of the following sequences is described, as far as it goes, by an explicit formula ($n \ge 0$) of the form $g_n = \lfloor \frac{n}{k} \rfloor$?

(a) 0000111122222

(b) 001112223333

(c) 000111222333

(d) 0000011112222

(e) 0001122233444

2. Given that $k > 1$, which of the following sum or product representations is **WRONG**?

(a) $(2^2+1)(3^2+1)\cdots(k^2+1) = \prod_{j=2}^{k}[(j+1)^2 - 2j]$

(b) $(1^3-1)+(2^3-2)+\cdots+(k^3-k) = \sum_{j=1}^{k-1}[(k-j)^3 - (k-j)]$

(c) $(1-r)(1-r^2)(1-r^3)\cdots(1-r^k) = \prod_{j=0}^{k-1}(1-r^{k-j})$

(d) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k-1}{k!} = \sum_{j=2}^{k} \frac{j-1}{j!}$

(e) $n + (n-1) + (n-2) + \cdots + (n-k) = \sum_{j=1}^{k+1}(n-j+1)$

3. Which of the following sums is gotten from $\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$ by the change of variable $j = i+1$?

(a) $\sum_{j=2}^{n} \frac{j-1}{(n-j+1)^2}$

(b) $\sum_{j=2}^{n} \frac{j-1}{(n-j-1)^2}$

(c) $\sum_{j=2}^{n} \frac{j}{(n-j+1)^2}$

(d) $\sum_{j=2}^{n} \frac{j}{(n-j-1)^2}$

(e) $\sum_{j=2}^{n} \frac{j+1}{(n-j+1)^2}$

4. We are going to prove by induction that $\sum_{i=1}^{n} Q(i) = n^2(n+1)$. For which choice of $Q(i)$ will induction work?

(a) $3i^2 - 2$ (b) $2i^2$ (c) $3i^3 - i$ (d) $i(3i-1)$ (e) $3i^3 - 7i$

5. The sum $\sum_{k=1}^{n}(1+2+3+\cdots+k)$ is a polynomial in n of degree

(a) 3 (b) 1 (c) 2 (d) 4 (e) 5

6. We are going to prove by induction that for all integers $k \ge 1$,

$$\sqrt{k} \le \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}.$$

Clearly this is true for $k = 1$. Assume the Induction Hypothesis (IH) that $\sqrt{n} \le \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n}}$. Which is a correct way of concluding this proof by induction?

(a) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n+1}} \ge \sqrt{n} + \frac{1}{\sqrt{n+1}} = \sqrt{n+1} + 1 \ge \sqrt{n+1}$.

(b) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n+1}} \ge \sqrt{n+1} + \frac{1}{\sqrt{n+1}} \ge \sqrt{n+1}$.

(c) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n+1}} \ge \sqrt{n} + 1 \ge \sqrt{n+1}$.

(d) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n+1}} \ge \sqrt{n} + \frac{1}{\sqrt{n}} \ge \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{n}} \ge \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$.

(e) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n+1}} \ge \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}} \ge \frac{\sqrt{n}\sqrt{n}+1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$.

7. Suppose $b_1, b_2, b_3, \cdots$ is a sequence defined by $b_1 = 3$, $b_2 = 6$, $b_k = b_{k-2} + b_{k-1}$ for $k \ge 3$. Prove that b_n is divisible by 3 for all integers $n \ge 1$. Regarding the induction hypothesis, which is true?

(a) Assuming this statement is true for $k \le n$ is enough to show that it is true for $n+1$ and no weaker assumption will do since this proof is an example of "strong induction."

(b) Assuming this statement is true for n and $n-1$ is enough to show that it is true for $n+1$.

(c) Assuming this statement is true for n, $n-1$, and $n-3$ is enough to show that it is true for $n+1$ and no weaker assumption will do since you need three consecutive integers to insure divisibility by 3.

(d) Assuming this statement is true for n is enough to show that it is true for $n+1$.

(e) Assuming this statement is true for n and $n-3$ is enough to show that it is true for $n+1$ since 3 divides n if and only if 3 divides $n-3$.

8. Evaluate $\displaystyle\lim_{n\to\infty} \frac{(-1)^{n^3} n^3 + 1}{2n^3 + 3}$.

(a) $-\infty$ (b) $+\infty$ (c) Does not exist. (d) $+1$ (e) -1

9. Evaluate $\displaystyle\lim_{n\to\infty} \frac{\log_5(n)}{\log_9(n)}$.

(a) $\ln(9)/\ln(5)$ (b) $\ln(5)/\ln(9)$ (c) $5/9$ (d) $9/5$ (e) 0

10. Evaluate $\displaystyle\lim_{n\to\infty} \frac{\cos(n)}{\log_2(n)}$.

(a) Does not exist. (b) 0 (c) $+1$ (d) -1 (e) $+\infty$

***11.** The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^{500}}{(1.0001)^n}$

(a) converges absolutely.

(b) converges conditionally, but not absolutely.

(c) converges to $+\infty$

(d) converges to $-\infty$

(e) is bounded but divergent.

***12.** The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\left(1+\frac{1}{n^2}\right)$.

(a) is bounded but divergent.

(b) converges absolutely.

(c) converges to $+\infty$

(d) converges to $-\infty$

(e) converges conditionally, but not absolutely.

Answers: **1** (c), **2** (b), **3** (a), **4** (d), **5** (a), **6** (e), **7** (b), **8** (c), **9** (a), **10** (b), **11** (a), **12** (e).