- Please put your name and ID number on your blue book.
- CLOSED BOOK, but BOTH SIDES of two pages of notes are allowed.
- Calculators are NOT allowed.
- *In a multipart problem, you can do later parts without doing earlier ones.*

1. (21 pts.) Let p and q be distinct primes.
   (a) Let $R = \{0, p, 2p, \ldots, (q-1)p\}$ under addition and multiplication modulo pq. It is a ring. Prove that R has no zero divisors.
   (b) Prove that the ring in (a) is an integral domain.
   (c) Let $S = \{0, p, 2p, \ldots, (p-1)p\}$ under addition and multiplication modulo p^2. It is a ring. Find a zero divisor.

2. (8 pts.) Find the maximal ideals in $\mathbb{Z}_6$. For each maximal ideal M, find a familiar ring that is isomorphic to $\mathbb{Z}_6/M$ and describe the isomorphism.
   Examples of "familiar" rings include $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{Z}_n[x]$.

3. (8 pts.) Prove that the union of a chain $I_1 \subset I_2 \subset \cdots$ of ideals of a commutative ring R is an ideal of R.

4. (21 pts.) Let F be the splitting field of $x^5 - 1$ over $\mathbb{Q}$.
   (a) Explain why $F = \mathbb{Q}(e^{2\pi i/5})$.
   (b) Find $\mathrm{Gal}(F/\mathbb{Q})$.
   (c) Compute $[F : \mathbb{Q}]$.

5. (16 pts.) Suppose E_1 and E_2 are subfields of the field K and that they contain the field F. Let E be the set intersection $E_1 \cap E_2$.
   (a) Prove that E is a field.
   (b) If $[E_1 : F] = 12$ and $[E_2 : F] = 18$, what are the possible values for $[E : F]$? **Explain your reasoning.**

6. (8 pts.) Can the equilateral triangle be squared? That is, given a side of an equilateral triangle, can one construct the side of a square having the same area?
   Give a reason—don't just answer yes or no.

7. (10 pts.) Suppose that C_k is a linear code with Hamming weight k.
   (a) What can C_3 do that C_2 cannot?
   (b) What can C_4 do that C_3 cannot?

8. (8 pts.) Suppose that $[E : \mathbb{Q}]$ is finite. Prove that there only a finite number of fields between E and $\mathbb{Q}$.

END OF EXAM