WCATSS 2016:
Chromatic homotopy theory
(notes from talks given at the workshop)

Notes taken by Eva Belmont

Last updated: August 12, 2016

Disclaimer

These are notes I took during the 2016 West Coast Algebraic Topology Summer School. I, not the speakers, bear responsibility for mistakes. If you do find any errors, please report them to:

Eva Belmont
`ekbelmont at gmail.com`

Contents

Talks were prepared in teams of two. An asterisk (*) indicates the team member(s) who delivered the lecture.

Talk 1: Introduction (Mike Hill)

Mike Hill: I'll write theorems without hypotheses, so be aware that they should be more like "slogans".

Formal groups provide a bridge between algebraic topology and number theory. If you've ever seen a Lie group (or if you haven't seen one, just the circle); it is a smooth (analytic) manifolds. The fact that it's analytic means that the multiplication can be written as a power series in local coordinates (in a really small neighborhood). In the number theory camp, I would describe it as algebraic groups, and look at what the multiplication is in a small neighborhood.

Theorem 1.1. *I have a functor sending $R \to \{$formal group laws$/R\}$ (for a commutative ring R), and a functor sending $R \to \{$isos of f.g.l.$/R\}$, and these are representable. There is a universal formal group law, and the ring is the Lazard ring.*

Theorem 1.2. *For nice cohomology theories E,*

(1) $E^0\mathbb{C}\mathrm{P}^\infty$ has a formal group law

(2) We have a theory of Chern classes for complex vector bundles.

The prototype of such a theory is MU: this gives manifolds together with a complex structure on the normal bundle. MU is nice and carries the univeral formal group law.

In particular, given a formal group law over R, you have a map from $L \to R$, and so E_* has a map to $MU_* = L$. All of the big theorems in the field use this connection.

Question: how do we reverse this connection? Given a formal group law over R, can I find a cohomology theory that has this formal group law?

Theorem 1.3 (Landweber)**.** *Given a ring R and a formal group law F over R, there are purely algebraic conditions which ensure that there is a cohomology theory R that is nice and whose formal group law is F. (There is a sequence of elements of R naturally associated to the formal group law, and you have to check that these form a regular sequence.)*

Let's find some formal group laws and see what the associated cohomology theories are. As you vary the formal group, how do the cohomology theories vary?

There is a couple of examples of Lie groups you might be thinking of: maybe $(\mathbb{R}, +)$, or $(\mathbb{R}^\times, \cdot)$, or $(\mathbb{C}, +)$, or $(\mathbb{C}^\times, \cdot)$. Examples:

(1) The additive group

(2) The multiplicative group

(3) Elliptic curves (think of this as a torus $\mathbb{C}/\mathbb{Z}^2$)

When I talk about elliptic curves, I want to remember the embedding. This ends the list of the 1-dimensional abelian group schemes. An elliptic curve has an associated formal group in the same way a Lie group has a formal group (what happens infinitesimally around the origin). We can ask if these satisfy the Landweber exact functor theorem, and when they do, you get an elliptic cohomology theory.

Look at the ways you can embed $\mathbb{Z}^3$ into $\mathbb{C}$; these form an orbifold. I can have an elliptic curve over a given base, and these might vary. How can I describe the collection of all possible elliptic curves? How can I describe the functor that assigns to a given base a collection of elliptic curves? This is a moduli problem, and it gives rise to a stack.

The final question is: how do families of elliptic curves show up in cohomology theories? I.e. we want an analogue of LEFT for something like projective space, which is only locally built out of rings. This was a hugely motivating problem that was solved by Hopkins and Miller, and refined by Hopkins and Goerss.

Theorem 1.4. *There is a sheaf of commutative ring spectra on the moduli of elliptic curves.*

The only word I used before in saying this is "elliptic curves". I have to take several jumps to get here. The first thing to observe is that cohomology theories are hard to get a handle on because we're working in a homotopy theory. If I'm woring with gluing, I need to know about limits, and those are badly behaved in a homotopy category: this is part of why we like model categories. Spectra help you do this. The other key observation is that if you add conditions, it becomes harder and harder for this theorem to not hold. There are lots of maps between spectra. Instead, instead of considering spectra, consider spectra together with a multiplication which is super coherently defined, and only consider maps that commute with the structure. This cuts down the collection of maps to something we can understand.

As I said, we've exhausted the list of 1-dimensional formal groups that come from 1-dimensional abelian varieties. This is the end of a story, as it answers the question about families of elliptic curves. To go further, we need to understand this in other cases.

Theorem 1.5 (Hopkins-Miller)**.** *Given a formal group F over a perfect field k of characteristic p, the set of deformations is representable.*

There's a functor lurking in the background – if we have a deformation of k (a local ring R that sits over k with nilpotent maximal ideal and quotient $R/\mathfrak{m} = k$, e.g. $k = \mathbb{F}_p$ and $R = \mathbb{Z}/p^2$), they show we can get a commutative ring spectrum. From the algebraic geometry perspective, I haven't gone anywhere – I'm in a tangent direction, so I've turned and noticed that I'm deformed. The limit of all these rings $\mathbb{F}/p^n$ is the p-adics. The maximal ideal is not nilpotent, but is topologically nilpotent.

Theorem 1.6 (Hopkins-Miller)**.** *There is an essentially unique commutative ring spectrum associated to the universal deformation of F over k. They look at the category of perfect fields with deformations, and show that this is functorial.*

Lurie proves a vast generalization of this.

Talk 2: Formal group laws 1 (Dileep Menon and Ningchuan Zhang*)

2.1. Definitions and examples. First consider a 1-dimensional real Lie group G with identity e and multiplication μ. A Lie group is analytic, so pick an analytic coordinate neighborhood (U,φ) with $\varphi : U \to \mathbb{R}$ such that $\varphi(e) = 0$. A priori, this is not closed under multiplication (it never is, unless it is a component). Let $U' \subset U$ be an open subset such that the product of elements in U' is contained in U. So we have a map $\mu : \varphi(U') \times \varphi(U') \to \varphi(U)$. The source is an open neighborhood of $(0,0) \in \mathbb{R}^2$, and the target is an open neighborhood of $0 \in \mathbb{R}$. Since the group is analytic, μ is equal to its Taylor expansion, i.e. it's a formal power series $F(x,y)$. Since F is induced from the group structure on G, it has to satisfy the following conditions:

(1) (Unital) $F(x,0) = x$ and $F(0,y) = y$

(2) (Associative) $F(x,F(y,z)) = F(F(x,y),z)$

(3) (Symmetric) $F(x,y) = F(y,x)$ (because 1-dimensional Lie groups are commutative)

Definition 2.1. A *formal group law* F over a ring R is a formal power series $F \in R[[x,y]]$ satisfying the three conditions.

You might object that we should be calling this a *commutative* formal group law. But when R has no nonzero nilpotents, the last condition is implied by the first two (this is nontrivial).

Let's now switch to the language of algebraic geometry by letting (G,e,μ) be a smooth 1-dimensional group variety over a field k. Now we have a map $\mu : G \times_k G \to G$. Analogously to the situation for a Lie group, we can take a formal neighborhood by completing G at e. We get a formal scheme $\widehat{G}$. In this case, since this is a smooth 1-dimensional group variety, we have $\widehat{G} \cong \operatorname{Spf} k[[t]]$. Here $\operatorname{Spf} k[[t]]$ has underlying space $\{e\}$, with ring $k[[t]]$. This is not the same as $\operatorname{Spec} k[[t]]$, which is different topologically,

We get $\widehat{\mu} : \widehat{G} \times_k \widehat{G} \to \widehat{G}$. This can be written as a map $\operatorname{Spf} k[[x,y]] \to \operatorname{Spf} k[[t]]$. It is induced by a map $f : k[[t]] \to k[[x,y]]$; I claim that the power series $f(t)$ is a formal group law.

When k is algebraically closed and G is connected, then G is isomorphic to:

(1) the additive group $(\mathbb{A}^1, +)$,

(2) the multiplicative group $(\mathbb{A}^1 - \{9\}, \times)$, or

(3) an elliptic curve.

In these three cases, we can apply the above recipe to get three basic examples of formal group laws:

(1) the additive formal group law $G_a(x,y) = x + y$,

(2) the multiplicative formal group law $G_m(x,y) = x + y - xy$ (this could also be $x + y + xy$ but here I will use $x + y - xy$), and

(3) the elliptic formal group law.

Remark 2.2. Formal groups and formal group laws are closely related but different. The slogan is a formal group = a formal group + a local coordinate. For Lie groups, we took a neighborhood of e, and a choice of coordinate function. For group schemes, I took a noncanonical isomorphism $\widehat{G} \cong \operatorname{Spf} k[[t]]$. Here, t is a coordinate.

2.2. Maps between formal group laws. Let $\alpha : G \to H$ be a morphism of group schemes. This takes $e_G \mapsto e_H$. If we complete this map at the identity element, we get $\widehat{\alpha} : \widehat{G} \to \widehat{G}$, which is the same as $\operatorname{Spf} k[[t]] \to \operatorname{Spf} k[[t]]$. Let $f := \widehat{\alpha}(t)$. The fact that $\alpha(e_G) = e_H$ says that $f(0) = 0$. From the definition fo morphism of group schemes, we have a commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu_G} & G \\ \downarrow{\scriptstyle \alpha\times\alpha} & & \downarrow{\scriptstyle \alpha} \\ H \times H & \xrightarrow{\mu_H} & H \end{array}$$

We can complete at the identity and choose local coordinates, which induces

$$\begin{array}{ccc} k[[x,y]] & \xleftarrow{F_G} & k[[t]] \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle F_G} \\ k[[x,y]] & \xleftarrow[F_H]{} & k[[t]] \end{array}$$

This diagram shows that $f(F_G(x,y)) = F_H(f(x), f(y))$. This motivates the definition of a morphism of formal group laws.

Definition 2.3. A map $f : F_G \to F_H$ of formal group laws is a formal power series $f(t) \in R[[t]]$ such that

(1) $f(0) = 0$

(2) $f(F_G(x,y)) = F_H(f(x), f(y))$

Note that f does not have a constant term: you can write $f(t) = \sum b_n t^{n+1}$. I claim that when $b_0 \in R^\times$, there exists another power series $g \in R[[t]]$ such that $f(g(t)) = g(f(t)) = t$.

Definition 2.4. A map $f : F_G \to F_H$ of formal group laws is called an isomorphism if $f'(0) \in R^\times$. This map is called a strict isomorphism if $f'(0) = 1$.

This corresponds to a change of coordinates. A strict isomorphism preserves the trivialization of Lie algebras.

Definition 2.5 (Change of base)**.** Let f be a formal group law over R. Let φ be a ring homomorphism $R \to SS$. Let $F(x,y) = \sum a_{ij} x^i y^j$. Define $\varphi^* F(x,y) = \sum \varphi(a_{ij}) x^i y^j$. (Despite the notation, really this is pullback in the algebraic geometry sense, since φ is a morphism $\operatorname{Spec} S \to \operatorname{Spec} R$.)

2.3. Classification over $\mathbb{Q}$. More generally, this is a classification over $\mathbb{Q}$-algebras.

Claim 2.6. *Formal group laws over $\mathbb{Q}$ (or $\mathbb{Q}$-algebras) are strictly isomorphic to the additive formal group law.*

PROOF. Let's construct a strict isomorphism $f : F \to G_a$. By the definition of strict isomorphisms, we know:

(1) $f'(0) = 1$
(2) $f(F(x, y)) = f(x) + f(y)$

Now solve for f by taking partial derivates w.r.t. y at $y = 0$. On the LHS, by chain rule you get $F_y(x, 0)f'(F(x, 0)) = F_y(x, 0) \cdot f'(x)$, and on the RHS you get 1. So $f'(x) = \frac{1}{F_y(x,y)}$ and $f(x) = \int_0^x \frac{dt}{F_y(t,0)}$. Here we are using the fact that we are working over $\mathbb{Q}$ to take the integral, because there might be arbitrary (integer) denominators. Check that $f(0) = 0$, $f'(t) = F_y(t, 0)$ so $f(t) = \int_0^t \frac{d}{F_y(x,0)}$. One can check that f is a strict isomorphism $F \to G_a$. □

For example, the isomorphism $G_m \to G_a$ over $\mathbb{Q}$ is

$$f(t) = \int_0^t \frac{dx}{1-x} = -\log(1-t) = \sum_{n\geq 1} \frac{t^n}{n}$$

Definition 2.7. The logarithm of F is the formal power series $f(t) = \int_0^t \frac{dx}{F_y(x,0)} \in R \otimes \mathbb{Q}[[t]]$.

The integrand $\frac{dx}{F_y(x,0)}$ is called the *invariant differential*; note that it is always defined over R.

To justify this I have to introduce elliptic curves, so I'll leave that to a later talk.

2.4. The universal formal group law. Recall that a formal group law is a formal power series $F(x, y) = \sum a_{ij}x^i y^j \in R[[x, y]]$ such that $F(x, 0) = x$, $F(0, y) = y$, $F(F(x, y), z) = F(x, F(y, z))$, and $F(x, y) = F(y, x)$. From here we see that the formal group law is determined by the a_{ij}, but we can't randomly pick a_{ij}'s because they have to satisfy these conditions.

Let $L = \mathbb{Z}[a_{ij}]/\sim$, where the relations are: $a_{0,j} = a_{i,0} = 0$ for $i, j \neq 0$, $a_{0,1} = a_{1,0} = 1$ (the power series always starts with $x + y + \dots$), $a_{i,j} = a_{j,i}$, and another condition corresponding to associativity which is harder to write down. (Set $F(F(x, y), z) = \sum b_{i,j,k}x^i y^j z^k$ and $F(x, F(y, z)) = \sum b'_{i,j,k}x^i y^j z^k$.)

Proposition 2.8. *There is a correspondence*

$$\mathrm{Hom}_{Ring}(L, R) = \{\textit{fgl's over } R\}$$

sending $\varphi \mapsto \sum \varphi(a_{i,j})x^i y^j$. There exists a universal formal group law over L, namely $F_{univ}(x, y) = \sum a_{i,j}x^i y^j$. Note the first map is $\varphi \mapsto \varphi^* F_{univ}$.

Let $W = L[b_1, b_2, \dots]$. There is a correspondence

$$\mathrm{Hom}_{Ring}(W, R) = \{\textit{strict isomorphisms } f : F_1 \to F_2 \textit{ over } R\}.$$

Where does this send φ? The source is $F_1 = \varphi^* F_{univ} = \sum \varphi(a_{i,j})x^i y^j$, and the target satisfies $F_2(f(x), f(y)) = fF_1(x, y)$, so $F_2 = fF_1(f^{-1}(x), f^{-1}(y))$. The morphism f is given by $f = t + \sum_{i\geq 1} f(b_i)t^{i+1}$.

This gives rise to a functor Rings $\to$ Groupoids sending $R \mapsto (\mathrm{FGL}(R), \mathrm{StrictIso}(R))$ (i.e. the objects are formal group laws over R, and the morphisms are strict isomorphisms between them). This functor is represented by a pair (L, W). This implies (L, W) is a Hopf algebroid.

$$L \underset{\eta_R}{\overset{\eta_L}{\rightleftarrows}} W \xrightarrow{\Delta} W \otimes_L W \qquad (\varepsilon : W \to L)$$

The left counit η_L is inclusion, the right counit η_R is the coaction, the coaugmentation ε is the quotient by b_i, the antipode c corresponds to inversion of strict isomorphism, and the diagonal is composition of strict isomorphisms. Furthermore, (L, W) is a graded Hopf algebroid: if we write $F_{\text{univ}}(x, y) \in L[[x, y]]$, set $|x| = |y| = -1$ (the reason for this will be explained in a later talk). We want F to be homogeneous of degree -2, so this forces us to set $|a_{i,j}| = 2i + 2j - 2$. The universal strict isomorphism $f(t) = \sum b_i t^{i+1}$ should have $|t| = -2$ and $|b_i| = 2i$.

2.5. $\mathrm{Aut}(G_a)$ **over** $\mathbb{Z}/p$**.** Strict isomorphisms $f(t) : G_a \to G_a$ have $f(t) = \sum_{i=0} b_i t^{i+1}$ and b_0. Plugging in $f(x + y) = f(x) + f(y)$ gives $\sum b_i(x + y)^{i+1} = \sum b_i(x^{i+1} + y^{u+1})$. Since we're working over $\mathbb{F}_p$, we get $b_i = 0$ unless $i + 1 = p^k$. Let's relabel the coefficients $c_k = b_{p^k-1}$ (so $|c_k| = 2(p^k - 1)$). So

$$\mathrm{Aut}\, G_a = \mathrm{Spec}\, \mathbb{Z}/p[c_1, c_2, \dots].$$

Let $f(t) = \sum c_k t^{p^k}$ and $g(t) = \sum d_k t^{p^k}$ be strict isomorphisms $G_a \to G_a$. The composition law is given by

$$\begin{aligned} f(g(t)) &= \sum c_i \big(\sum d_j t^{p^j}\big)^{p^i} \\ &= \sum_k \sum_{i+j=k} c_i d_j^{p^i} t^{p^k} \\ \Delta(c_k) &= \sum_{i+j=k} c_i^{p^j} \otimes c_j \end{aligned}$$

This is the formula for the diagonal on the dual Steenrod algebra!

Remark 2.9. This argument recovers a sub-Hopf algebra of the dual Steenrod algebra $P_* \subset A_*$, where

$$P_* = \begin{cases} P(\xi_1^2, \xi_2^2, \dots) & \text{when } p = 2 \\ P(\xi_1, \xi_2, \dots) & \text{when } p \text{ is odd.} \end{cases}$$

(The ξ's are squared because of the grading.)

Recall $L = \mathbb{Z}[a_{ij}]/\sim$.

Theorem 2.10 (Lazard)**.** *There is an isomorphism $L \cong \mathbb{Z}[x_1, x_2, \dots]$ where $|x_i| = 2i$.*

Define polynomials $C_n(x,y)$, called symmetric 2-cocycles:

$$C_n(x,y) = \gamma_n((x+y)^n - x^n - y^n) \qquad \text{where } \gamma_n = \begin{cases} p \text{ if } n = p^k \\ 1 & \text{otherwise.} \end{cases}$$

The precise statement of Lazard's theorem is:

Theorem 2.11. *Over $L' = \mathbb{Z}[x_1, x_2, \dots]$ there exists a formal group law $F(x,y)$ such that*

$$F(x,y) \equiv \sum x_n C_{n+1}(x,y) \pmod{(x_1, x_2, \dots)^2}.$$

Then the map $L \to L'$ that classifies this formal group law is an isomorphism.

Talk 3: Stable homotopy theory (Tyler Lawson*)

The goal of stable homotopy theory is to construct and classify maps between topological spaces. We have homotopy groups π_n and (co)homology groups $H_n(X)$. These are designed to go together: in order to understand maps $X \to Y$ up to deformations, if we know information about $\pi_n Y$ and $H_n(X)$ we can put those together to find out information about maps $X \to Y$. Homology tells you about how something is built, and homotopy tells you about how something is mapped into.

Ordinary homology starts off in the following way. We have a category of spaces, and a functor $H_* : \text{Spaces} \to \text{(graded) abelian groups}$. Out of a space, we strip out a sequence of algebraic information. Unfortunately, the abelian groups by themselves are hard to work with unless you have the information

$$\begin{array}{ccc} & \nearrow & \text{Chain complexes}/\mathbb{Z} \\ & & \downarrow \\ \text{Spaces} & \xrightarrow{H_*} & \text{(Graded) abelian groups} \end{array}$$

C_* takes disjoint unions to direct sums, takes pushouts to pushouts (in some derived sense), and takes Cartesian products of spaces to tensor products of chain complexes. Spaces and chain complexes have a different notion of equivalence:

$$\begin{array}{ccc} \text{Spaces} & \xrightarrow{C_*} & \text{Chain complexes} \\ {\scriptstyle \text{mod by } \sim}\downarrow & & \downarrow{\scriptstyle \text{mod by qis.}} \\ \text{Homotopy category} & \longrightarrow & \text{Derived category} \end{array}$$

The derived category $D(\mathbb{Z})$ of chain complexes over $\mathbb{Z}$ is abelian-ish.

There are several equivalent ways to get C_* or something like it. One of them is called the singular complex (this is what you see in Hatcher's textbook); it is very robust. There are also more delicate answers that turn out to be more useful sometimes.

One of the more unusual way to get homology groups is the following. Associated to the space X, we can construct a new space $AG(X)$, the free topological abelian group on X. Imagine that we can take formal sums and differences of elements of X, and figure out what topology to put on this for it to make any sense. This is due to Dold and Thom.

Theorem 3.1. *For nice X, $H_n(X) \cong \pi_n(AG(X))$.*

Think of homology as stripping out the abelian information. We can think of C_* as an abelianization functor that lands in an abelian category.

From this chain functor C_*, we can construct new homology theories. For example, we can take homology with coefficients $H_n(X; A) = H_n(C_*(X) \otimes A)$, or cohomology with coefficients $H^n(X; A) =$ homotopy classes of chain maps $C_*(X) \to A[n] = \Sigma^n A$ (here $A[n]$ is the shift of A thought of as a complex concentrated in degree n). (Warning: Weibel has the opposite convention $A[-n]$ for this.)

This makes C_* universal among these abelian-type homology theories. But there are many other geometric theories that are not recovered this way. There is K-theory (studying real (or complex) vector bundles on a space), or looking at different ways a manifold can map to your space. Many of these are very natural, but you can't recover them from the singular chain complex.

Is there a way to capture these other theories by moving to something more refined than just abelianization? Stable homotopy theory adjusts and captures these others. There is a very similar picture:

$$\begin{array}{ccc} \text{Spaces} & \xrightarrow{\Sigma^\infty} & \text{“Spectra”} \\ {\scriptstyle \text{mod by } \sim}\big\downarrow & & \big\downarrow{\scriptstyle \text{mod by } \sim} \\ \text{Homotopy category} & \longrightarrow & \text{Stable homotopy category} \end{array}$$

Just like the chain functor (which took derived pushouts to derived pushouts, etc.), Σ^∞ takes (derived) colimits to (derived) colimits, and it takes a product $\times$ in spaces to a product (called the smash product) in spectra; the target (the stable homotopy category) is a tensor triangulated category. This has mapping cones, so you get long exact sequences, etc.

If E is any object in the stable homotopy category, you get a new homology and cohomology theory

$$\begin{aligned} E_n(X) &= \pi_*(E \wedge \Sigma^\infty X) \\ E^n(X) &= \mathrm{Map}_{\mathrm{SHC}}(\Sigma^\infty X, E[n]) \end{aligned}$$

If we know about E, this tells us more about E_* and E^*. For example, there's a notion called being a ring spectrum: this means that you have maps in the stable homotopy category $\mu : E \wedge E \to E$ and the unit $\eta : S \to E$ (where S is the sphere spectrum, i.e. $\Sigma^\infty(*)$). One can ask for it to be associative or commutative by asking for certain diagrams to commute. This is a notion that exists in the homotopy category of spectra. The corresponding “ring” notion in just Spectra is a strictly/ coherently commutative ring spectrum. (The meaning of “ring” depends on what category you're talking about.)

Aside: there's always a fold map $E \sqcup E \to E$, and there's always a diagonal $E \xrightarrow{\Delta} E \times E$, so you can compose these to compose maps.

If E is a ring spectrum, then E^*X naturally has a multiplication $E^p(X) \to E^q(X) \to E^{p+q}(X)$. If it has an associative property then this map is multiplicative, etc.

Every cohomology theory is represented in the stable homotopy category. This is the "universal" place where all homology and cohomology theories live. This is good: this gives us legitimate information about the stable homotopy category (everything else I said would have been satisfied by a functor to the zero category). We need to get more information, and the way to do that is by calculating: we need to find what makes the stable homotopy category tick and what makes it different from other categories like $D(\mathbb{Z})$.

Write $[X,Y] = \mathrm{Map}_{\mathrm{SHC}}(X,Y)$. The first tool developed for this purpose is obstruction theory. Spectra have homotopy and (co)homology, and there's a method (spectral sequence) for putting those together: $H^s(X,\pi_t(Y)) \implies [X,\Sigma^{t-s}Y]$. There are a couple problems: first you need to calculate π_*Y, which is the information we wanted out; also, this has different perspectives on the source and target (know X by its homology and Y by its homotopy). We would like to get a more uniform approach.

The cellular method is a little more sophisticated. If we know a set of building blocks (e.g. the spheres S^n), and we know all the maps $S^n \to S^n$, all the maps $S^n \to Y$, and we know how hard it is to build X out of S^n, then we can et information about maps $X \to Y$. The information of the maps $S^n \to S^n$ is completely encapsulated by knowing π_*S, a graded commutative ring. Knowing maps $S^n \to Y$ is completely encapsulated by knowing π_*Y. If we know these, and know how hard it is to construct a projective resolution over π_*S, then you can calculate the spectral sequence

$$\mathrm{Ext}^s_{\pi_*S}(\pi_*X, \pi_*Y[t]) \implies [X, Y[t-s]].$$

This says that the stable homotopy category looks a lot like the algebraic category $D(\pi_*S\text{-mod})$. Problem: even worse, we need to know π_*, and π_*S is, to put it mildly, unpleasant. This serves as sort of a goal, but it's not useful for doing calculations.

Homology and cohomology are a lot more calculable than homotopy groups. The "flipped" cellular method due to Adams starts with a different set of building blocks: we start with $H\mathbb{F}_p$ (Eilenberg-Maclane spectrum), which is constructed so that it has only one nonzero homotopy group. Then we look at possible shifts of this. Instead of knowing maps between spheres, and maps from spheres into Y, we try to find if we can find all maps $H\mathbb{F}_p[n] \to H\mathbb{F}_p[m]$, and look at how hard it is to build the target Y out of them. This is an algebra, and it's known: it's called the mod p Steenrod algebra A^*. It has certain generators and relations involving binomial coefficients. We also need to know $X \to H\mathbb{F}_p[n] = H^*X$ as a module over A^*, and need to know how hard it is to build Y out of $H\mathbb{F}_p[n]$'s (resolving $H^*(Y)$ over A^*). So the resulting spectral sequence is

$$\mathrm{Ext}^s_{A^*}(H^*Y, H^*X[t]) \implies [X, Y[t-s]]$$

(actually, it only recovers the completion of the RHS). This is called the Adams spectral sequence. It is very effective. Renee Thom used this to classify manifolds modulo bordism.

One weird thing about this is the variance shift. It tends to turn disjoint unions into products, which are much bigger and you have to fend off the axiom of choice. We would rather dualize this: A^* has a dual A_*, and we can dualize the property of being an A^*-module $A^* \otimes M \to M$ by the property of being an A_*-comodule $M \to A_* \otimes M$. A_* is a commutative ring (analyzed by Milnor) that looks like $\mathbb{F}_p[\xi_i] \otimes \Lambda[\tau_i]$ for $p > 2$ and $\mathbb{F}_p[\xi_i]$ for $p = 2$, and there's a coproduct that we saw in the last talk.

A_* is some kind of affine group scheme, and H_*X is a representation of A_*. We can instead recast this:

$$\mathrm{Ext}_{A_*\text{-rep}}(H_*X, H_*Y) \implies [X, Y]$$

(completed, again). This is a little more uniform, and it generalizes. If E is a ring spectrum (which this makes E_* a ring) such that $\pi_*(E \wedge E)$ is a flat right π_*E-module, then the pair $(\pi_*E, \pi_*(E \wedge E))$ form a Hopf algebroid. (Hopf algebroids are a construction opposite to being a group object in rings.) $E_*(X)$ is a representation of it, and we get the Adams-Novikov spectral sequence:

$$\mathrm{Ext}_{(E_*, E_*E)\text{-rep}}(E_*X, E_*Y[t]) \implies [X, Y[t-s]].$$

This gives us a lot more possibilities: we started with two tools, homology and homotopy. This says that, actually, there's a ton of other objects, and each of them provides a lens into what the stable homotopy category might look at. They also can interpolate between homotopy, which is hard to compute but contains a lot of information, and homology, which is easy to compute but loses a lot of information. This is saying that the stable homotopy category tries to be like the derived category of (E_*, E_*E)-representations, for all E.

Talk 4: Complex orientable cohomology theories (Janis Lazovskis and Maximilien Holmberg-Peroux*)

Recall: let L be the Lazard ring, and (L, W) the Hopf algebroid we talked about earlier. The goal is to give a homotopical description of this. Quillen's theorem says that $L \cong \pi_*MU$.

4.1. Complex orientation.

Definition 4.1. A spectrum E is a collection $(E_n, \sigma_n)_{n \in \mathbb{Z}}$ where $E_n \in \mathrm{Top}_*$ and σ_n are maps $\Sigma E_n \to E_{n+1}$.

Recall that $\mathbb{C}\mathrm{P}^\infty \simeq BU(1) \simeq K(\mathbb{Z}, 2)$. This means that $H_*(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}$ generated by the map $S^2 \cong \mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{C}\mathrm{P}^\infty$. This can be regarded as a map $\Sigma^\infty S^2 \to \Sigma^\infty \mathbb{C}\mathrm{P}^\infty$, i.e. a map $i : S \to \Sigma^{\infty-2}\mathbb{C}\mathrm{P}^\infty$.

Definition 4.2. A ring spectrum E is *complex orientable* if the map $i^* : \widetilde{E}^2(\mathbb{C}\mathrm{P}^\infty) \hookrightarrow \widetilde{E}^2(S^2) \cong \widetilde{E}^0(S^0) = \pi_0(E) = [S, E]$ is surjective.

Saying that i^* is surjective means that there is an element $x^E \in \widetilde{E}^2(\mathbb{C}\mathrm{P}^\infty)$ such that $i^*(x^E) = 1 \in \pi_0(E)$. Such a class x^E is called a *complex orientation.*

Recall: $\widetilde{E}^k(X) = \mathrm{colim}_n[\Sigma^n X, E_{k+n}]$.

The element $x^E \in \widetilde{E}^2(\mathbb{C}\mathrm{P}^\infty)$ corresponds to a map $\Sigma^\infty \mathbb{C}\mathrm{P}^\infty \to E$. There is a diagram

$$\begin{array}{ccc} \Sigma^\infty \mathbb{C}\mathrm{P}^\infty & \xrightarrow{x^E} & E \\ {\scriptstyle i}\uparrow & \nearrow_{\eta} & \\ S & & \end{array}$$

For example, suppose $E = H\mathbb{Z}$. In that case, $i^* : H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \to H^2(S^2) \cong \mathbb{Z}$ is an isomorphism. Here x^E is the first universal Chern class for E.

Example 4.3. Let $E = KU$, complex K-theory. By Bott periodicity, $\widetilde{KU}^2(\mathbb{C}\mathrm{P}^\infty) \cong \widetilde{KU}^0(\mathbb{C}\mathrm{P}^\infty)$. Then $x^{KU} = [\ell] - 1_{\mathbb{C}}$, where ℓ is the universal complex line bundle.

4.2. Formal group law for x^E.

Recall $H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}[[c_1, c_2, \dots]]$ where c_i are the Chern classes.

Proposition 4.4. *E is complex orientable with complex orientation $x^E = x$.*

(1) $E^*(\mathbb{C}\mathrm{P}^\infty) \cong \pi_* E[[x]]$

(2) $E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \pi_* E[[x_1, x_2]]$

Proof. Use the Atiyah-Hirzebruch sequence, which says

$$H^p(\mathbb{C}\mathrm{P}^n, \pi_q(E)) \implies E^{p+q(\mathbb{C}\mathrm{P}^n)}$$

for $n \geq 1$. This spectral sequence degenerates at page 2. Then use the fact that $\mathbb{C}\mathrm{P}^n$ is a colimit of $\mathbb{C}\mathrm{P}^n$'s. □

So there exists a multiplication m for $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$, given by a map $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^{n+m+1}$, which classifies the tensor product of line bundles.

We have

$$m^* : \pi_* E[[x]] \cong E^*(\mathbb{C}\mathrm{P}^\infty) \to E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \pi_* E[[x_1, x_2]].$$

Proposition 4.5. *Let $\mu^E(x_1, x_2)$ be the image of x in the map above. Then $\mu^E(x_1, x_2)$ is a formal group law over $\pi_* E$.*

For example, suppose $E = H\mathbb{Z}$. Then $\mu^{H\mathbb{Z}}(x_1, x_2) = m^*(G) = x_1 + x_2$ (the additive formal group law). If $E = KU$, then $x^{KU} = [\ell] - 1_{\mathbb{C}}$. Then $m^*(x) + 1 = m^*(x+1) = m^*(\ell) = \ell_1 \ell_2 = (x_1 + 1)(x_2 + 1)$. So $\mu^{KU}(x_1, x_2) = x_1 + x_2 + x_1 x_2$, the multiplicative formal group law.

4.3. Thom spaces.

Definition 4.6. Let $p : E \to B$ be a real vector bundle of rank r. Then the Thom space $\mathrm{Thom}(p) = D(E)/S(E)$ where $D(E)$ is the disc bundle (on each fiber take the vectors of length ≤ 1) and $S(E)$ is the sphere bundle (on each fiber take the vectors of length $= 1$). You need a metric to do this; it works for paracompact E.

You can think of the Thom space as being a twisted suspension of the space B, just as a vector bundle is a twist of the product.

Let $p : B \times \mathbb{R}^r \to B$ be the trivial projection. Then $\mathrm{Thom}(P) = B \times D^r / B \times S^{r-1} = \Sigma^r B_+$. If you don't believe this, take the case where $r = 1$: this is just the cylinder on B modulo the caps. But in general, p is not simply a projection, which is why I said this is the twisted suspension.

Finally, Thom is functorial: it is a functor from $\mathbb{R}$-vector bundles to Top_*.

4.4. Constructing MU. Define $MU(n) = \mathrm{Thom}(\xi^n)$, where $\xi^n : l \to BU(n)E(n)$ is the universal vector bundle of rank r. (You can think of this as an $\mathbb{R}$-vector bundle by using the fact that $\mathbb{C} \cong \mathbb{R}^2$.) There is a canonical map $BU(n) \to BU(n+1)$, which is induced by the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ sending $x \mapsto (x, 0)$, which gives rise to a map $U(n) \to U(n+1)$.

Consider the pullback

$$\begin{array}{ccc} E(n) \oplus \mathbb{C} & \longrightarrow & E(n+1) \\ {\scriptstyle \xi^n \oplus 1}\downarrow & & \downarrow{\scriptstyle n+1} \\ BU(n) & \longrightarrow & BU(n+1) \end{array}$$

This is because nothing happens on the $(n+1)$st coordinate. So Thom $\mathrm{Thom}(\xi^n) = \mathrm{Thom}(\xi^n \oplus 1^2_{\mathbb{R}}) \to \mathrm{Thom}(\xi^{n+1}) = MU(n+1)$. Since $MU(n) = \mathrm{Thom}(\xi^n)$, this gives a map $\Sigma^2 MU(n) \to MU(n+1)$.

We want to define the spaces in the spectrum MU as $MU_n = MU(n)$, but this isn't quite the right thing to do, because our suspension maps suspend by 2, not 1.

Definition 4.7. Define $(MU)_{2n} = MU(n)$ and $(MU)_{2n+1} = \Sigma MU(n)$.

Remark 4.8. Originally, when Thom studied this spectrum, it had a very deep geometrical meaning: $\pi_* MU = \Omega^U_*(*)$ where Ω^* is the cobordism ring.

This is a ring spectrum: we have maps $MU \wedge MU \to MU$ induced by $BU(n) \times BU(m) \to BU(n+m)$ (take an $n \times n$ matrix and an $m \times m$ matrix and put them together in a block diagonal to get a $(n+m) \times (n+m)$ matrix). We also have $S \xrightarrow{\eta} MU$ induced by $S^0 \to MU(0)$. Note that $\mathrm{Thom}(\xi^0) = *_+ = S^0$, so $S^0 \to MU(0)$ is the identity.

Why is it complex orientable? This is because $MU(n) \simeq BU(n)/BU(n-1)$ for $n \geq 1$.

We want $x^{MU} \in \widetilde{MU}^2(\mathbb{CP}^\infty) = \mathrm{colim}[\Sigma^n \mathbb{CP}^\infty, (MU)_{2n+1}]$. This is induced by a map $\mathbb{CP}^\infty \to (MU)_2 = MU(1)$. But $MU(1) = BU(1)/BU(0) \simeq BU(1)$, and $\mathbb{CP}^\infty \simeq BU(1) \xrightarrow{\simeq} MU(1)$ is our complex orientation. I still have to check that $i^*(x^{MU}) = 1 \in \pi_0(MU)$.

To see this, note that $\Sigma^2 S^0 = \Sigma^2 MU(0) \to MU(1) = \mathbb{C}\mathrm{P}^\infty$ is just i. So you get a formal group law $\mu^{MU}(x_1, x_2)$ on $\pi_*(MU)$.

It turns out that MU is the universal complex orientable cohomology theory: there is a correspondence between complex orientations x^E of E and maps $MU \to E$, where $x^{MU} \mapsto x^E$.

Given $g : MU \to E$, define $x^E : \Sigma^{\infty-2}\mathbb{C}\mathrm{P}^\infty \stackrel{x^{MU}}{\to} MU \stackrel{g}{\to} E$.

Theorem 4.9 (Quillen)**.** *The ring homomorphism $L \to \pi_* MU$ is an isomorphism.*

Recall: a pair of rings (A, Γ) is a Hopf algebroid if $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ is a groupoid object in affine schemes.

Proposition 4.10. *If E is a commutative ring spectrum such that $\pi_*(E \wedge E)$ is a flat left $\pi_* E$-module, $(\pi_* E, \pi_*(E \wedge E))$ is a Hopf algebroid.*

The Hopf algebroid structure is

$$\pi_* E \rightrightarrows \pi_*(E \wedge E) \longrightarrow \pi_*(E \wedge E) \otimes \pi_*(E \wedge E)$$

The two unit maps $\pi_* E \to \pi_* E \wedge E$ are given by $S \wedge E \to E \wedge E$ and $E \wedge S \to E \wedge E$. The others are described in the problem set.

By Quillen's theorem, there is an isomorphism of Hopf algebroids $(L, W) \cong (\pi_* MU, MU_* MU)$.

First, we need to understand $\pi_*(MU \wedge MU)$.

Proposition 4.11. *Let $\{b_i\}$ be the top dual basis of $E^\infty(\mathbb{C}\mathrm{P}^\infty) \cong \pi_* E[[x]]$. Then $\pi_*(E \wedge MU) = \pi_* E[b_1, b_2, \dots]$ and $\pi_*(MU \wedge MU) \cong L \otimes \mathbb{Z}[b_1, b_2, \dots]$, where b_i are the classes with $|b_i| = 2$ from the earlier talk.*

Let E be any complex oriented cohomology theory. Then $E \wedge MU$ is also a complex oriented cohomology theory in two ways, via the maps $\Sigma^{\infty-2}\mathbb{C}\mathrm{P}^\infty \to MU \sim S \wedge MU \to E \wedge MU$ and $\Sigma^{\infty-2}\mathbb{C}\mathrm{P}^\infty \to E \sim E \wedge S \to E \wedge MU$. Call these x^{MU} and x^E, respectively (even though this is sort of abuse of notation).

Let $R = \pi_*(E \wedge MU) = \pi_* E[b_1, b_2, \dots]$. Then $R[[x^{MU}]] \cong (E \wedge MU)^*(\mathbb{C}\mathrm{P}^\infty) \cong R[[x^E]]$. You can write $x^{MU} = f(x^E)$ for some power series f, where $f(1) = t + b_1 t^2 + b_2 t^3 + \dots$. Then $m^*(x^E) = \mu^E(x_1^E, x_2^E)$, $m^*(x^{MU}) = \mu^{MU}(x_1^{MU}, x_2^{MU})$. This also implies that $\mu^{MU}(x, y) = f\mu^E(f^{-1}(x), f^{-1}(y))$. This gives the two unit maps $L \rightrightarrows W$.

Talk 5: Elliptic curves (Cody Gunton* and Chris Kapulkin)

5.1. Elliptic curves over $\mathbb{C}$. Given $\Lambda \subset \mathbb{C}$ a free abelian rank 2 subgroup, I can form the quotient $\mathbb{C}/\Lambda$. Someone else will talk about how to parametrize the Λ's, but for now we only consider one Λ at a time. Associated to a given Λ is

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} + \frac{1}{\lambda^2} \right).$$

This determines an embedding $\mathbb{C}/\Lambda \hookrightarrow \mathbb{P}^2(\mathbb{C})$, which sends $z \mapsto (\wp(x) : \wp'(x) : 1)$. There is a differential equation the $\wp$ satisfies:

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6.$$

As you vary the lattice, G_i are functions; but for now, they're just constants. Conversely, if I start with a genus 1 surface, the universal cover is $\mathbb{C}$, and that gives rise to a lattice.

I have to add a point: the image compactifies to a curve E containing a special point $O = (0 : 1 : 0)$. There's a group law on $\mathbb{C}/\Lambda$, and that transfers to a group law on the image curve, which we describe in different terms. Let $P, Q \in E$, and let L be the line in $\mathbb{P}^2$ through P and Q (if $P = Q$ this is the tangent line). Let R be a third point on $L \cap E$. Let L' be the line through O and R. Define $P + Q$ to be the third point on $L' \cap E$.

Proposition 5.1. *This defines a group structure on E.*

5.2. Elliptic curves over an arbitrary field K. Let R be a (commutative, unital) ring. A *Weierstrass polynomial over R* is defined to be a polynomial of the form

$$f(x, y) = y^2 + c_1xy + c_3y - x^3 + c_2x^2 + c_4x + c_6 \in R[x, y].$$

(Beware: there is no c_5, and $c_1, \dots, c_6$ means without c_5.)

Definition 5.2. The universal Weierstrass polynomial is the polynomial $f(x, y) = y^2 + c_1xy + \dots$ in $\mathbb{Z}[c_1, \dots, c_6][x, y]$.

Definition 5.3. Let K be a field. A Weierstrass curve will be the projective curve determined by a Weierstrass polynomial over K. It still contains the point $O = [0 : 1 : 0]$.

Definition 5.4. An *elliptic curve* over K is a smooth Weierstrass curve over K.

The composition law in terms of lines can be described in terms of a formula, and that formula makes sense on the smooth locus on a Weierstrass curve. This is a little less standard.

Fact 5.5. *Let $C^{\text{reg}}(K)$ be the smooth locus of a curve C over K. If C is a Weierstrass curve over K, then $C^{\text{reg}}(K)$ is a group.*

Theorem 5.6 (Bézout)**.** *Two curves X, Y in $\mathbb{P}^2_K$ with no common irreducible component intersect in $\deg X \deg Y$ points (with multiplicity).*

The degree is a property of how the curves are embedded (given by polynomials).

Fact 5.7. *Let C be a geometrically irreducible*[1] *plane curve. Then every line in $\mathbb{P}^2$ intersects a singular point of C with multiplicity ≥ 2.*

Corollary 5.8.

(1) There is at most one singular point on a Weierstrass curve over K.

(2) The chord-and-tangent law restricts to $C^{\text{reg}}(K)$.

Proof. (1) Suppose there are two singular points P and P'. Then the line between them contradicts Bézout's theorem.

(2) If $P+Q$ is singular, the line from O to $P+Q$ contradicts Bézout's theorem. □

The discriminant Δ is a polynomial in the coefficients of the Weierstrass form that detects the singularity type of the curve. It comes from some general fact about resultants.

If you're not in characteristic 2 or 3, then there is a change of coordinates that gets the Weierstrass form into a polynomial of the form $y^2 = x^3 + c_4 x + c_6$. References for everything so far: Silverman's book. In this case, there is a simple formula for the discriminant.

Proposition 5.9. *Let C be a Weierstrass curve over K. Then one of the following hold:*

(1) C is smooth, $\Delta \neq 0$, and $C^{\text{reg}} = C$ is an elliptic curve;

(2) C is singular, $\Delta = 0$, $c_4 \neq 0$, and $C^{\text{reg}}_{\overline{K}} \cong \mathbb{G}_{m,\overline{K}}$; or

(3) C is singular, $\Delta = 0$, $c_4 = 0$, and $C^{\text{reg}}_{\overline{K}} \cong \mathbb{G}_{a,\overline{K}}$.

Proposition 5.10. *A curve C is elliptic iff it is genus 1, smooth, projective, and geometrically integral*[2]*, with a K-rational point O.*

5.3. Weierstrass curves over any ring R.

Let $S = \operatorname{Spec} R$. Given a Weierstrass polynomial f over R, you can form $\operatorname{Spec} R[x,y]/f$. This will compactify to a projective S-scheme $C \xrightarrow{\pi} S$. This can be recorded as a family of Weierstrass curves over fields: for every $s \in S$, the fiber is $C_s = \operatorname{Spec}(k_s[x,y]/f)$ where $k_s = R_s/sR_s$. (The fiber can also be written as a tensor product $(R[x,y]/f) \otimes_R R_s/sR_s$.)

Definition 5.11. A relative curve of genus 1 is an S-scheme that is finitely presented, proper, and flat such that every fiber has arithmetic genus[3] 1.

A relative elliptic curve is the same thing, where you replace flat with smooth, and with a section $S \to C$.

[1] A curve is *geometrically irreducible* if, when you tensor up to an algebraically closed extension, it's irreducible
[2] I think *geometrically integral* means it's integral when you tensor up to $\overline{K}$.
[3] The arithmetic genus of a curve C is $Pa := h^1(\mathcal{O}_{C_S})$.

This is in Deligne's article "...Formulaire" in the Atwerp 1972(?) conference proceedings (eds. Birche, Kuye).

There exists an S-scheme[4] $C^{\text{reg}} \to S$ such that $(C^{\text{reg}})_S = (C_S)^{\text{reg}}$. One has:

Proposition 5.12 (Deligne-Rapoport, 2.7)**.** *$C^{\text{reg}} \to S$ is a group.*

5.4. Deriving the formal group law associated to $C^{\text{reg}} \xrightarrow{\pi} S$. The identity section $e : S \to C^{\text{reg}}$ is a closed embedding $0 \to \mathcal{I} \to \mathcal{O}_{C^{\text{reg}}} \to e^*\mathcal{O}_S \to 0$. Take the formal completion to have topological space the underlying space of S, and sheaf of rings $\lim_n e^{-1}(\mathcal{O}_{C^{\text{reg}}}))$.

Suppose we have an elliptic curve E over K defined by a Weierstrass polynomial f. We have a map $O : \operatorname{Spec} K \to E$ picking out the origin $O = (0 : 1 : 0)$. Change coordinates to O: send $(x, y) \mapsto (z, w) = (\frac{x}{y}, \frac{1}{-y})$. Assume I have the Weierstrass curve in the following form (although you can still do this with the more complicated form):

$$y^2 = x^3 + c_4 x + c_6.$$

Now $x = z/w$ and $y = -1/w$, so this turns into $\frac{1}{w^2} = \frac{z^3}{w^3} + c_4 \frac{z}{w} + c_6$, or

$$w = z^3 + c_4 z + c_6 w^3.$$

You can recursively plug this into itself: $w = w(z, w)$. The idea is that it starts to stabilize: the w's appearing in higher iterations start to appear further and further out. Eventually, you get $w = z^3(1 + A_1 z + A_2 z^2 + \dots)$.

Work in the formal (z, w)-plane. Use the chord-and-tangent rules formally; I will produce a formal group law. Connect $(z_1, w(z_1))$ and $(z_2, w(z_2))$ to get a line of slope $\lambda(z_1, z_2) := \frac{w(z_2) - w(z_1)}{z_2 - z_1}$. I can expand this as a formal power series. You get a line $W = \lambda Z - \nu$. Now substitute this W into my Weierstrass equation, and divide formal power series to get a third point. I just have to get a formal reflection law. The z-coordinate of the third point is

$$z_3 = -z_1 - z_2 + \frac{c_1 \lambda + c_3 \lambda^2 - c_2 y - 2c_\$ \lambda \nu - 3c_6 \lambda^2 \nu}{1 + a_2 \lambda + a_4 \lambda^2 + a_6 \lambda^3}.$$

The formal group law is

$$F(z_1, z_2) = i(z_3(z_1, z_2))$$

where i is the reflection rule $i(z) = \frac{x(z)}{y(z) + a_1 x(z) + a_3}$.

Talk 6: Recap (Kyle Ormsby*)

In 2.1, we'll talk about Lubin-Tate deformation theory. We define the height of a formal group law (some invariant), and find the universal deformation of a height n formal group law F over a perfect field k. We'll then see how $\operatorname{End}(F)$ act on deformations.

In 2.2, we'll hear about local class field theory. This classifies abelian extensions of a local field F, and classifies these in terms of the subgroups of $F^\times$. This can all be phrased in terms of an Artin reciprocity map $\operatorname{rec}_F : F^\times \to \operatorname{Gal}(F^{ab}/F)$.

[4] An S-scheme is a scheme with a map to S

In 2.3, we'll get to the Landweber exact functor theorem. This answers the following question: suppose F is a graded formal group law over a graded ring R. By the Quillen theorem, this is then classified by a map $L = MU_* \to R$. This gives R an L-module structure. Then we can ask the question: when is $X \mapsto MU_*X \otimes_{MU_*} R$ a generalized homology theory? We need this to preserve exact sequences, so flatness (of R over MU_*) is a natural condition to ask for, but the Landweber exact functor theorem gives a weaker condition than flatness.

Finally, we'll get to modular forms in 2.4. We'll learn that these are functions on the set of elliptic curves satisfying some sort of modularity condition. Depending on what you think of when I say "set of elliptic curves", this could mean various things – it could be the set of elliptic curves over $\mathbb{C}$, or the stack of elliptic curves.

Talk 7: More fun with formal groups (Padma Srinivasan* and Isabel Vogt*)

Yesterday, we fixed a commutative ring R and studied the set of formal group over R:

$$\mathrm{FGL}(R) = \{F \in R[[x,y]] \, : \, F \text{ satisfies } \ldots\}$$

for some conditions.

Alternatively, any power series $F \in R[[x,y]]$ gives rise to a functor $\mathcal{F}_F$: commutative R-algebras $\to$ Sets with a binary operation taking $S \mapsto (N_S, \oplus)$ where N_S is the set of nilpotent elements of S and $n_1 \oplus n_2 = F(n_1, n_2)$. Then a power series F is a formal group law iff $\mathcal{F}_F$ factors through abelian groups. Furthermore, $\mathcal{F}_F$ is "representable" by $(\mathrm{Spf}\, R[[t]], F)$.

We want to understand the functor commutative rings $\to$ sets sending $R \mapsto \mathrm{FGL}(R)$. Yesterday, we said this is representable by the Lazard ring. This is way too big. We want to cut this down so we're studying an object of finite type. So we study formal group laws up to strict isomorphism.

Proposition 7.1. *If R is a $\mathbb{Q}$-algebra, then* $\mathrm{FGL}(R)$ *is a single point.*

This prompts us to ask whether this is a single point for all rings R. The answer is "no". Fix a perfect field of characteristic $p > 0$. We want to understand $\mathrm{FGL}(k)/\sim$. We will also be more brave and study formal group laws over other rings – deformations of this field.

Notation:

- $\mathcal{O}$ is a nilpotent thickening of k (this is an Artinian local ring with $\mathcal{O}/\mathfrak{m} = k$)
- $\mathfrak{m}$ is a maximal ideal which is nilpotent
- $k = \mathcal{O}/\mathfrak{m}$ is a perfect field of characteristic $p > 0$

To show that FGL does not always give a single isomorphism class, we will define an invariant and show that it can take on different values.

Fix $F \in \mathrm{FGL}(k)$ and $n \in \mathbb{Z}$. You get a natural transformation

$$\mathcal{F}_F \xrightarrow{[n]} \mathcal{F}_F$$

induced by the "multiplication-by-n" on groups. This gives rise to a map Spf $R[[t]] \xrightarrow{[n]}$ Spf $R[[t]]$; in the opposite direction, it sends t to $[n](t) = F(t, [n-1]t) = F(t, F(t, [n-2]t)) = \dots$. This is a power series of the form $nt+$ higher order terms.

Claim 7.2. *Either $[p](t) \equiv 0$, or $[p](t) = g(t^{p^h})$ for some integer $h \geq 1$ and some other power series $g(t)$ such that $g(0) = 0$ and $g'(0) \neq 0$.*

Assuming the claim, make the definition:

Definition 7.3. The height of F is ∞ if $[p](t) \equiv 0$, and h otherwise.

This height is invariant under isomorphisms.

Example 7.4.

(1) $\mathbb{G}_a$: this is the formal group law $F(x, y) = x + y$, so $[n]$ is just normal addition, and $[p]$ is 0 mod p. So this has infinite height.

(2) $\mathbb{G}_m$: this is the formal group law $F(x, y) = (1 + x)(1 + y) - 1$. You can check that $[p]t = (1 + t)^p - 1 \equiv t^p \pmod{p}$. So $\mathrm{ht}(\mathbb{G}_m) = 1$.

Proof of Claim. Apply $\frac{\partial}{\partial y}|_{y=0}$ to

$$F([p](x), [p](y)) = [p](F(x, y))$$

and expand using the chain rule to get

$$\begin{aligned} F_2([p](x), 0)[p]'(0) &= [p]'(F(x, 0))F_2(x, 0) \\ &= [p]'(x) \cdot (\text{invertible power series}) \end{aligned}$$

using the fact that $F(x, y) = x + y+$ higher order terms. Also recall $[p](x) = px+$ higher order terms, so this is 0 modulo terms of degree ≥ 2. Since $[p]'(0) = 0$ and $[p](x) = \sum_{i \geq 1} a_i x^i$, we have $[p]'(x) = 0$. So $[p](x) = g(x^p)$. Now repeat this argument with g in place of $[p]$. □

Theorem 7.5 (Lazard)**.** *The map*

$$h : \mathrm{FGL}(k)/\cong \to \mathbb{Z}_{>0} \cup \{\infty\}$$

is a bijection if k is separably closed.

Theorem 7.6. *The fiber of h over $n \in \mathbb{Z}_{>0}$ is in bijection with $H^1(\mathrm{Gal}(k^{sep}/k), \mathrm{Aut}_{k^{sep}}(F))$ for any $F \in \mathrm{FGL}(k)$ of height n.*

So, if k is separably closed, the height completely classifies formal group laws, and if not, we still have a classification.

Definition 7.7. Given $\Phi \in \mathrm{FGL}(k)$, $F, G \in \mathrm{FGL}(\mathcal{O})$ such that $\mathrm{red}(F) = \mathrm{red}(G) = \Phi$, a $\star$-isomorphism between F and G is $f \in \mathrm{Hom}(F, G)$ such that

$$\begin{array}{ccc} F & \longrightarrow & G \\ {\scriptstyle \mathrm{red}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{red}} \\ \Phi & = \!= & \Phi \end{array}$$

commutes.

Definition 7.8. $\mathrm{Def}_\Phi(\mathcal{O}) = \mathrm{red}(\Phi)$ *I think this should be* $\mathrm{Def}_\Phi(\mathcal{O}) =$ *the set of fgl's over* $\mathcal{O}$ *that reduce to* Φ

Here is the main theorem:

Theorem 7.9. *Fix* $\Phi \in \mathrm{FGL}(k)$ *of height* h*. The functor* Def : *nilpotent thickenings of* $k \to$ *Sets sending* $\mathcal{O} \mapsto \mathrm{Def}_\Phi(\mathcal{O})$ *is "representable" by a formal scheme* $(R := \mathrm{Spf}\, W(k)[[t_1, \ldots, t_{n-1}]], \Gamma)$*. That is, for every nilpotent thickening* $(\mathcal{O}, \mathfrak{m})$*, the map*

$$\mathfrak{m}^{n-1} = \mathrm{Hom}_{cts}(R, \mathcal{O}) \to \mathrm{Def}_\Phi(\mathcal{O})$$

sending $\varphi \mapsto \varphi^*(\Gamma)$ *is a bijection.*

Corollary 7.10. $\mathrm{Aut}_k \Phi$ *acts on* $\mathrm{Def}_\Phi(-)$*.*

Key steps of proof of Theorem 7.9. Step 1: construct a specific element in $\mathrm{FGL}(k)$.

Step 2: classify infinitesimal deformation of Φ. These are in bijection with $H^2_k(\Phi)$ (this should be thought of as a tangent space).

Step 3: Show $H^2_k(\Phi) \cong k^{h-1}$.

Step 4: show Γ is universal. ∎

7.1. More on the proof: There is only one way to build this universal deformation, which is to build it degree-by-degree. This is because $J = (x, y) \subset R[[x, y]] = \lim R[[x, y]]/J^n$ (this is just saying it's complete).

Definition 7.11. F is an r-bud if it satisfies the axioms of a formal group law mod degree $r + 1$.

Fact 7.12. *If you have an* r*-bud* F_r*, then there always exists an* $(r + 1)$*-bud* F_{r+1} *extending it, and given any two such, say* F_{r+1} *and* F'_{r+1}*, we have* $(F'_{r+1} - F_{r+1}) = aC_{r+1}(x, y)$ *mod degree* $r + 2$*, where* C_* *is the 2-cocycle defined previously.*

Because Φ has height $h < \infty$, Φ can be put in the form $x + y + aC_{p^h}(x, y)$ mod degree p^{h+1}.

Proposition 7.13. *There exists a power series $\Gamma(t)(x,y) \in R[[x,y]]$, where $R = \operatorname{Spf} W(k)[[t_1,\dots,t_{n-1}]]$, satisfying:*

(1) $\Gamma^\star(0,\dots,0)(x,y) = \Phi(x,y)$

(2) for all i, $\Gamma(0,\dots,0,t_i,\dots,t_{n-1})(x,y) = x+y+t_iC_{p^i}(x,y)$ *mod degree* p^{i+1}.

Look at the series of liftings $k = \mathcal{O}/\mathfrak{m} \leftarrow \mathcal{O}/\mathfrak{m}^2 \leftarrow \mathcal{O}/\mathfrak{m}^3 \leftarrow \dots$. In general, we can look at square-zero extensions $k \hookrightarrow k[\varepsilon]/\varepsilon^2$.

Φ is a lift of itself. If $F(x,y)$ is any other lift, it has the form $F(x,y) = \Phi(\Phi(x,y)+\varepsilon G(x,y))$ for some power series G. From now on, $+$ and $-$ mean $+$ and $-$ in the group law. This simplifies some of the calculations in Lubin and Tate's paper. So $F(x,y) = x+y+\varepsilon G(x,y)$ (with this new notation). Definitions of formal group laws say:

(1) $G(x,0) = G(0,y) = 0$

(2) $G(x,y) = G(y,x)$

(3) $F(F(x,y),z) = F(x,F(y,z))$

In (3), the LHS (mod ε^2) is $x+y+\varepsilon G(x,y)+z+\varepsilon G(x+y,z)$ and the RHS is $x+y+z+\varepsilon G(y,z)+\varepsilon G(x,y+z)$. Now subtract these *in the group law*. Cancel $x+y+z$ on both sides, and get

$$\varepsilon(G(y,z) - G(x+y,z) + G(x,y+z) - G(x,y)) = 0$$

This is now true with the normal $+$ and $-$. This is what is called a symmetric 2-cocycle.

A map $f : F \to F'$ looks like $f(x) = x + \varepsilon g(x)$ with $g(0) = 0$ is a homomorphism if $f(F(x,y)) = F'(f(x), f(y))$. This expands to $\varepsilon(G(x,y) - G'(x,y)) = \varepsilon(g(x+y) - g(x) - g(y))$ (addition in the group law). This is what we call a symmetric 2-coboundary.

So these lifts $k \hookrightarrow k[\varepsilon]/\varepsilon^2$ are classified by 2-cocycles modulo 2-coboundaries. This is the definition of $H^2_k(\Phi)$.

Talk 8: Introduction to local class field theory (Rachel Davis* and Jize Yu)

References: Milne's CFT notes, Jared Weinstein's notes on the geometry of Lubin-Tate spaces

8.1. Definitions and notation.

Definition 8.1. A field K is called a *local field* if it is locally compact w.r.t. the nontrivial valuation.

Our main examples will be finite extensions of $\mathbb{Q}_p$, finite extensions of Laurent series rings $\mathbb{F}_p((T))$, and the archimedean cases ($\mathbb{R}$ or $\mathbb{C}$). We'll focus on the non-archimedean cases. Let K be a finite extension of $\mathbb{Q}_p$, and let $\mathcal{O}_K$ be the ring of integers of K, $\mathfrak{m}_K$ the maximal, and $\mathcal{O}_K^\times = U_K$ the units in $\mathcal{O}_K$. Take π to be a prime generator of $\mathfrak{m}_K$. Every nonzero element $a \in K^\times$ can be written as $a = u\pi^n$ for some unit u. Define the order $\mathrm{ord}_K(a) := n$. We also have a residue field $k = \mathcal{O}_K/\mathfrak{m}_K$ of degree $q = p^f$. We have a valuation defined by $|a| := q^- rd_K(a)$. Let K^{al} be the separable algebraic closure of K, and K^{ab} the union of all abelian extensions of K.[5]

Goals:

(1) Classify all finite degree abelian extensions

(2) Study the structure of K^{ab}

(3) Construct a subfield K_π of K^{ab}

Let L/K be a finite Galois extension of local fields, with Galois group $G = \mathrm{Gal}(L/K)$. Define

$$G_i = \{\tau \in G \,:\, \mathrm{ord}_L(\tau(x) - x) \geq i + 1 \ \forall x \in \mathcal{O}\}.$$

Then there are inclusions $G \supset G_0 \supset G_1 \supset \dots$, and $G_0 = 1$ iff L/K is unramified. (Topological intuition: compare a ramified (branched) cover with an unramified cover (looks trivial). The ramified point is the branch point. Arithmetically, take the uniformizer π and look at how it factors in L into prime ideals. You can write $\pi_K = u\pi_L^e$ for some unit u and exponent e. If $e > 1$, it's ramified; if $e = 1$, it's unramified.)

8.2. Main theorems of local class field theory. Let K be a local field, and L a finite unramified extension of K. Then L/K is Galois. There is an element $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma x \equiv x^q$ for all $x \in \mathcal{O}_L$.

We will call σ the Frobenius element of $\mathrm{Gal}(L/K)$.

Theorem 8.2 (Local reciprocity law)**.** *For any non-archimedean local field, there is a homomorphism $\varphi_K : K^\times \to \mathrm{Gal}(K^{ab}/K)$ such that:*

(1) For any prime $\pi \in \mathfrak{m}_K$ and any finite, unramified extension L/K, $\varphi_K(\pi)$ acts on L as $\mathrm{Frob}_{L/K}$.

(2) For any finite abelian extension L/K, $N(L^\times) \subset \ker(\varphi_K|_L)$.

Combining these, φ_K induces an isomorphism

$$\varphi_{L/K} : K^\times/N_{L/K}(L^\times) \to \mathrm{Gal}(L/K).$$

Call both φ_K and $\varphi_{L/K}$ *local Artin maps.*

Theorem 8.3 (Local existence theorem)**.** *A subgroup N of $K^\times$ is of the form $N_{L/K}(L^\times)$ for some finite, abelian extension L iff it is a finite index open subgroup of $K^\times$.*

Taking inverse limits over abelian extensions $L \supset K$, we get

$$\widehat{\varphi_K}^i : \widehat{K^\times} \to \mathrm{Gal}(K^{ab}/K)$$

[5]An extension L/K is called *abelian* if $\mathrm{Gal}(L/K)$ is an abelian group.

sending $\pi \mapsto \mathrm{Frob}_{L/K}(\pi)$. Note that $K^\times \cong \mathcal{O}_K^\times \times \mathbb{Z}$, so $\widehat{K^\times} = \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}$. Let K_π be the subfield of K^{ab} fixed by $\widehat{\varphi}_K(\pi)$. Let K^{un} be the subfield of K^{ab} fixed by $\varphi_K(\mathcal{O}_K^*)$. So $K^{ab} = K_\pi \cdot K^{un}$.

For example, when K has characteristic p, K^{un} is generated by all roots of unity of order prime to p.

8.3. Lubin-Tate formal group laws and formal $\mathcal{O}_K$-modules. The Lubin-Tate construction is in 1965. Let K be a non-archimedean local field, and A be an $\mathcal{O}_K$-algebra with $L : \mathcal{O}_K \to A$ (not presumed injective).

Definition 8.4. A formal $\mathcal{O}_K$-module law over A is:

(1) a formal group law G over A

(2) a family of power series $[a]_G$ for every $a \in \mathcal{O}_K$

such that

(1) the collection of $[a]_G$'s represent a homomorphism $\mathcal{O}_K \to \mathrm{End}(G)$ sending $a \mapsto [a]_G$, and

(2) $[a]_G(x) = \pi(a)x + O(x^2)$.

Examples:

(1) $\mathbb{G}_a$ is an $\mathcal{O}_K$-module over the $\mathcal{O}_K$-algebra A

(2) For $K = \mathbb{Q}_p$, $\mathbb{G}_m$ becomes a formal $\mathcal{O}_K = \mathbb{Z}_p$-module, because for $a \in \mathbb{Z}/p$, $[a]_{\mathbb{G}_m}(x) = (1+x)^a - 1 = \Sigma_{n=1}^\infty \binom{a}{n} x^n \in \mathbb{Z}_p[[x]]$.

Lubin-Tate start with a choice $[\pi]_G$, and construct G from this. Let $f \in \mathcal{O}_K[[x]]$ be any power series satisfying

(a) $f(x) = \pi x + O(x^2)$

(b) $f(x) = x^q \pmod{\pi}$

Theorem 8.5 (in Weinstein's notes)**.** *there exists a unique formal $\mathcal{O}_K$-module law G_f over $\mathcal{O}_K$ for which $[\pi]_{G_f}(x) = f(x)$. Furthermore, if g is a power series satisfying (a) and (b), then G_f and G_g are isomorphic.*

Theorem 8.6 (Weinstein, Lubin-Tate)**.** *There is a map*

$$C : \mathrm{Gal}(K_\pi/K) \to \mathrm{Aut}(T_\pi(G)) \simeq \mathcal{O}_K^\times$$

which is an isomorphism $K_\pi K^{un} = K^{ab}$. The Artin reciprocity map

$$\varphi_{K_\pi/K} : K^\times \to \mathrm{Gal}(K^{ab}/K)$$

is the unique map which sends π to 1 and $a \in \mathcal{O}_K$ to $C^{-1}(\alpha^{-1})$.

Talk 9: Landweber Exact Functor Theorem (John Berman* and Danny Shi*)

I'm going to start with a bit of history.

Theorem 9.1 (Conner-Floyd, 1966)**.** *For $X \in$ Top,*

$$K_*X \cong MU_*X \otimes_{MU_*} K_*.$$

(There is a complex orientation $MU \to K$ and $MU_* \to K_*$ is the multiplicative formal group law.)

Question: Say F is a formal group law over R, given by a map $MU_* \to R$. Is $MU_*(-) \otimes_{MU_*} R$ a homology theory? If so, it has a cup product $\cup$ which comes from the fact that it comes from a commutative ring, and is complex oriented with formal group law F.

So we've got a formal group law, and are asking if it comes from a homology theory.

We need to check the axioms of a homology theory:

(1) Homotopy functor (this is fine, as base change preserves isomorphisms)

(2) Excision (also OK because base change preserves isomorphisms)

(3) Additivity (also OK, same)

(4) LES of a pair (possibly not OK: true if R is flat but that's unlikely because L is so big)

What we need is a weaker "flatness" condition that still makes this satisfy the LES axiom. This is the Landweber exact functor theorem (LEFT):

Theorem 9.2 (Landweber, 1976)**.** *Say M is an L-module (e.g. a commutative algebra over L, or equivalently a formal group law). If $(p, v_1, v_2, \dots) \in L$ is a regular sequence for M for all p, then $MU_*(-) \otimes_L M$ is a homology theory.*

Remark 9.3. Recall $MU_* = L = \mathbb{Z}[x_1, x_2, \dots]$, and $v_n = x_{p^n-1}$ (so there's a v_i for every *pair* of i, p). A sequence $(x_1, x_2, \dots) \in R$ is *regular* for an R-module M if, for all n, the action of x_n on $M/(x_1, x_2, \dots, x_{n-1})$ has no kernel.

I haven't told you how to compute v_n in practice. Say F is a formal group law, $[p]x$ is the p-series. Then v_n is the coefficient of x^{p^n}. Note that v_n is not invariant under coordinate change, but it is invariant modulo $(p, v_1, v_2, \dots, v_{n-1})$, which is all we care about. If M was a commutative algebra over L, we get a complex orientable homology theory. (If you plug in $\mathbb{C}\mathrm{P}^\infty$, you get a first Chern class.)

If M satisfies the condition in the theorem statement then call it (or the formal group law, or the resulting homology theory) "Landweber exact".

Example 9.4. Consider the additive formal group law over $R = \mathbb{Z} = H\mathbb{Z}_*$. The p-series is $[p]x = px$. We want to check whether $(p, 0, 0, \dots)$ is a regular sequence in $\mathbb{Z}$. It is not, because

0 does not act injectively on $\mathbb{Z}$! So there *is* a homology theory with this formal group law (ordinary homology), but it doesn't come out of MU_*.

Example 9.5. Let $R = \mathbb{Z}[\beta^{\pm}]$ and consider the formal group law $x + y + \beta^{-1}xy = \beta((1 + \beta^{-1}x)(1 + \beta^{-1}y) - 1)$. *What's the point of the extra parameter β? You want the formal group law to be homogeneous, so if you set $|\beta| = 2$ then this works out. (Or maybe -2? I forget.)* Then the p-series is

$$[p]x = \beta((1 + \beta^{-1}x)^p - 1) \equiv \beta(\beta^{-1}x)^p = \beta^{1-p}x^p.$$

Check that $(p, \beta^{1-p}, 0, \dots)$ *is* a regular sequence in $\mathbb{Z}[\beta^{\pm}]$, so you recover K-theory, and the Conner-Floyd theorem.

Example 9.6. $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ is a quotient of $L_{(p)}$, where $v_n = x_{p^n-1}$. There is a map $L \to BP_*$, and we want to check that $(p, v_1, v_2, \dots)$ is a regular sequence – this is true when the p's are the same, because they are polynomial generators. We could also check $(q, v_1, v_2, \dots)$ for $q \neq p$. Landweber exactness then gives BP as a homology theory.

Proof steps for LEFT. (1) Work one prime at a time, and consider a BP_*-module M. Then we have a regular sequence $(p, v_1, v_2, \dots)$ which we want to show gives rise to a homology theory.

(2) Landweber invariant prime ideal theorem (key part): all prime ideals of BP_* that are fixed by coaction of BP_*BP are $(p, v_1, \dots, v_n)$ for $0 \leq n \leq \infty$.

□

I'm going to give an application of this theorem and construct the main players we need for chromatic homotopy theory:

(1) Johnson-Wilson theory $E(n)$

(2) Morava K-theory $K(n)$

(3) Lubin-Tate theories (a.k.a. Morava E-theories) E_n

(1) To start, I need to specify a formal group law over a ring. Fix p. The ring I start with is $\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_{n-1}, v_n^{\pm}]$. There is an obvious map $BP_* \to \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm}]$ (sending $v_i \mapsto v_i$ if $1 \leq i \leq n$ and $v_i \mapsto 0$ otherwise). This classifies a p-typical formal group law.

I claim that the criterion in LEFT is obvious in this case. First, we send p to something non-zero. Now we mod out by p, which turns $\mathbb{Z}_{(p)}$ to $\mathbb{F}_p$, and we check whether v_1 goes to a non-zero divisor. You kill v_1 and check for v_2, and keep doing this. At the last step, you mod out by $v_1, \dots, v_{n-1}$ and check whether v_n is a non-zero divisor – indeed, it's even a unit. After this, there is nothing more to check. So LEFT gives a homology theory which we call Johnson-Wilson theory $E(n)$.

Since v_n is inverted, this means that the height of the associated formal group law can be no larger than n. So intuitively, $E(n)$ encodes information of height $\leq n$. Later on, we'll see that localizing at $E(n)$ corresponds to restricting to the open substack of height $\leq n$ formal group laws.

(2) What about height exactly n? You have to invert v_n, and also get rid of $v_1, \ldots, v_{n-1}$ (otherwise v_i for $i < n$ could be the (first) invertible v_*, and then it would be height i). So we have a map $BP_* \to \mathbb{F}_p[v_n^{\pm}]$ classifying these. Sadly, this doesn't satisfy Landweber exactness. The answer is that you can construct Morava K-theories out of Johnson-Wilson E-theories. Analyze the coefficient rings: $\pi_* E(n) = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm}]$, so you have to make sense of quotienting out the regular sequence $(p, v_1, \ldots, v_{n-1})$ to get $K(n)_*$.

The idea is to take iterative cofibers. Suppose I want to kill v_i. This lives in $\pi_{2(p^i-1)}E(n)$, and so it corresponds to a map $S^{2(p^i-1)} \to E(n)$. To mod out by v_i, make this construction:

$$S^{2(p^i-1)} \wedge E(n) \stackrel{v_i \wedge 1}{\to} E(n) \wedge E(n) \stackrel{\mu}{\to} E(n) \stackrel{\text{cofiber}}{\to} E(n)/v_i.$$

There are a few more details you need... This is a pretty general construction, but things like complex orientations are usually destroyed by this.

$K(n)$'s are really important! Here is a list of facts and important properties.

- When $n = 1$, $K(1)_* = \mathbb{F}_p[v_1^{\pm}]$, where $|v_1| = 2(p-1)$. It turns out that $K(1)$ is one of the $(p-1)$ summands of mod p complex K-theory.
- There is a Künneth isomorphism
$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_* Y$$
and moreover, these and ordinary cohomology theories with coefficients in a field are the only cohomology theories that have a Künneth isomorphism. In this sense, $K(n)$'s behave like fields, and this is part of why they're so important.
- For $p \geq 3$, $K(n)$ (for all n) are homotopy commutative. But for $p = 2$, the $K(n)$'s are no longer homotopy commutative (but they're homotopy associative). In 1989, Robinson used THH to show that, for all p, $K(n)$'s are A^∞ ring spectra.

(3) Start with a perfect field k. We saw earlier that $E(k, \Gamma) = W(k)[[u_1, u_2, \ldots, u_{n-1}]][u^{\pm}]$ has the universal deformation. (The extra $u^{\pm}$ here is again to make things homogeneous.) There is a map $BP_* \to E(k, \Gamma)$ which gives rise to a homology theory $E_{(k,\Gamma)}$. If $k = \mathbb{F}_{p^n}$ and Γ is the Honda formal group law (defined to have the p-series $[p]x = x^{p^n}$), then call the resulting spectrum E_n.

$E(n)$ and E_n are complex orientable. $K(n)$ is not in the traditional sense because it's only homotopy associative, but if you plug nice spaces in you still get something commutative.

Talk 10: Modular forms (Carolyn Yarnall* and Don Larson*)

If you quotient the complex plane by a lattice Λ you get an elliptic curve using the Weierstrass $\wp$ function. Write $\Lambda = \mathbb{Z}\{\omega_1, \omega_2\}$ where $\omega_i \in \mathbb{C}$. Quotienting by this identifies the sides of the parallelogram given by $0, \omega_1, \omega_2, \omega_1 + \omega_2$ in the complex plane, and you get a torus.

Let $\tau = \omega_2/\omega_1 \in \mathcal{H}$ (where $\mathcal{H}$ is the upper half plane). Write $\Lambda_\tau = \mathbb{Z}\{1, \tau\}$, and $C_\tau = \mathbb{C}/\Lambda_\tau$. There is an action of $SL_2(\mathbb{Z})$ on $\mathbb{C}$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

When is there an isomorphism $C_\tau \xrightarrow{\cong} C_{\tau'}$ as Riemann surfaces? This happens exactly when there is some $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma\tau = \tau'$.

If we want to study collections of elliptic curves modulo isomorphisms, we just need to study the upper half plane modulo this action of $SL_2(\mathbb{Z})$. Defining some function on $\mathcal{H}$ is the same as defining some function on elliptic curves. But this definition is not easy to generalize to the context we care about, and hard to keep track of topological data.

The quotient map $\mathbb{C} \to C_\tau$ can be thought of as a map $C_\tau \to \operatorname{Spec} \mathbb{C}$, where now C_τ means a scheme. There is a sheaf Ω of 1-forms, and that is the trivial 1-dimensional line bundle. But if you want to think of all elliptic curves, then you get a nontrivial bundle ω of all Ω_{C_τ} over $\mathcal{H}/SL_2(\mathbb{Z})$. A section $e : \mathcal{H}/SL_2(\mathbb{Z}) \to \omega$ is what we think of as a modular form. *Warning: I was being a little glib; this should probably be over the universal elliptic curve.*

Definition 10.1. A modular form of weight k over $\mathbb{C}$ is a section of $\omega^{\otimes k}$.

For example, in $\mathbb{C}$, z is the natural coordinate, and dz is the corresponding differential.

A modular form g of weight k takes (C_τ, dz) to $f(\tau) \in \mathbb{C}$. So $g(C_\tau) = f(\tau)(dz)^k$. So if you scale z by λ, it scales the modular form by λ^k. If I know what it's doing to C_τ, and I want to know how to pull this back along $C_\tau \to C_{\tau'}$ given by $(c\tau + d)^{-1}$,

$$g(C_{\tau'}) = f(\tau) \left(\frac{dz}{(c\tau + d)} \right)^k = (c\tau + d)^{-k} f(\tau)(dz)^k.$$

10.1. Modular forms over $\mathbb{Z}$ and the Weierstrass Hopf algebroid.

Definition 10.2. A *Weierstrass curve* is a scheme WC over a base scheme S with projection $p : WC \to S$ with a section $e : S \to WC$ corresponding to the identity, where WC is a collection of elliptic curves whose affine equations look like $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. I want to think of the a_i's as depending on where you are in the base scheme, i.e. $a_i \in \mathcal{O}_S$. Actually, the elliptic curves I work with only look like this locally; they are patched together from things like this.

A *generalized elliptic curve* is a scheme C with projection $p : C \to S$ and unit $e : S \to C$, where S is covered by open subsets S_i such that $C_i = C \times_S S_i$ is isomorphic to a Weierstrass curve.

I will look at Weierstrass curves up to Weierstrass equivalence (how we can toy around with the Weierstrass equation and not change anything), and strict isomorphisms – isomorphisms between these curves where the morphism on the formal groups is a strict isomorphism in the way we saw yesterday.

If we allow ourselves to play around with these wigglings, what do they look like? Let $r, s, t \in R$. Strict isos allow you to make the following changes of coordinates:

$$\begin{aligned} &a_1 \mapsto a_1 + 2s \\ &a_2 \mapsto a_2 - a_1 s + 3r - s^2 \\ &\dots \\ &a_6 \mapsto \dots \end{aligned}$$

Weierstrass manipulation allows you to make the change of coordinates

$$\begin{aligned} x &\mapsto x + r_1 \\ y &\mapsto y + sx + t \end{aligned}$$

Form a pair of commutative rings whose job is to keep track of these:

$$\begin{aligned} A &= \mathbb{Z}[a_1, \dots, a_6] && \text{where } |a_i| = 2i \\ \Gamma &= A[r, s, t] && \text{where } |r| = 4, |s| = 2, |t| = 6 \end{aligned}$$

This makes all the expressions in the list homogeneous. The pair (A, Γ) is called the Weierstrass Hopf algebroid.

We have left and right units $\eta_L, \eta_R : A \rightrightarrows \Gamma$, where η_L is the inclusion, and η_R is the map that performs the coordinate changes indicated above on $a_1, \dots, a_6$. What is in the equalizer of this diagram, written $\mathrm{Eq}(\eta_L, \eta_R)$? Here is one element in here:

$$c_4 = a_1^4 + 8a_1^2 a_2 + 16a_2^2 - 48a_4 - 24a_1 a_3$$

Given a generalized elliptic curve $C \to S$, let $\omega_{C/S}$ be the cotangent space of C along S. (Think of S as embedded in C by virtue of the zero section; look at the cotangent space at each point of S.) Let π denote a section of $\omega_{C/S}$. We saw that one way to think of a modular form is as something that scales the invariant differential. Here, π is playing the role of the invariant differential, except we're varying in families, so we're picking a 1-form for each location in S.

Given that $c_4 \in \mathrm{Eq}(\eta_L, \eta_R)$, the section $c_4 \cdot \pi^{\otimes 4}$ of $\omega_{C/S}^{\otimes 4}$ does not depend on x, y, or π.

Definition 10.3. A modular form over $\mathbb{Z}$ of weight k is a rule that associates to each $C \to S$ a section $g(C/S)$ of $\omega_{C/S}^{\otimes k}$ compatible with base change.

Let MF_* denote the graded ring of modular forms over $\mathbb{Z}$. This says that $MF_* = \mathrm{Eq}(\eta_R, \eta_L)$.

Theorem 10.4 (Deligne, Antwerp 1972 (Lecture notes in math 476))**.** *The equalizer can be identified as follows:*

$$\begin{array}{ccccc} \mathbb{Z}[c_4, c_6, \Delta]/(12^3\Delta = c_4^3 - c_6^2) & \longrightarrow & A & \rightrightarrows & \Gamma \\ & & \downarrow & & \downarrow \\ \mathbb{Q}[c_4, c_6] & \longrightarrow & A \otimes_{\mathbb{Q}} & \rightrightarrows & \Gamma \otimes \mathbb{Q} \end{array}$$

Note that the relation is the same relation that turns up in the classical theory of modular forms. If you tensor with $\mathbb{Q}$, 2 and 3 are both invertible and you get a much simpler version of the Weierstrass mechanics. (You could have also just tensored with $\mathbb{Z}[\frac{1}{6}]$.)

There is a map

$$(MU_*, MU_*MU) \to (A, \Gamma)$$

because you can complete at the identity to get a formal group. Take both of these and localize or complete at a prime; the LHS is the input for the ANSS converging to p-primary stable homotopy groups of spheres. On the RHS, create the cobar complex and take cohomology, and that is the input for a machine that approximates the p-primary stable stems.

Talk 11: Recap (Agnès Beaudry*)

Padma and Isabel talked about deformations of formal group laws. They had a formal group Φ of height n over $k = \mathbb{F}_{p^n}$. This means that $[p]_\Phi(x) = x^{p^n} + \dots$, and it turns out that you can find one where $[p](x) = x^{p^n}$. This is called the Honda formal group law. They also explain why there is another formal group law Γ over a bigger ring, R (which in this example is $W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$) which represents deformations (lifts) of Φ. This has residue field $\mathbb{F}_{p^n}$.

In this case, $W(\mathbb{F}_{p^n})$ is the ring of integers in an unramified extension $L/\mathbb{Q}_p$ of degree n (adjoin a p^{n-1}th root of unity). The pair (Γ, R) represents deformations of (Φ, k), i.e. for every formal group law G over a deformation B of k that reduces to Φ/k, there is a $\star$-isomorphism $f^* \to G$ for a map $f : R \to B$.

Suppose you have an automorphism $u : \Phi \to \Phi$. Suppose you have a deformation Γ. Given an automorphism $u \in \mathrm{Aut}(\Phi)$, this gives rise to $f_u \in \mathrm{Aut}(R)$, and $\mathrm{Aut}(\Phi)$ acts on R in this way.

$$\begin{array}{ccccc}
\Gamma & \xrightarrow{u} & u\Gamma u^{-1} & \longrightarrow & f_u^*\Gamma \\
\downarrow & & \downarrow & & \downarrow \\
\Phi & \xrightarrow[\cong]{u} & \Phi & = & \Phi
\end{array}$$

Today, we'll talk about $\mathbb{S}_n = \mathrm{Aut}(\Phi)$. In number theory, this is defined as $\mathbb{S}_n \cong \mathcal{O}_D^\times$ where D is a central division algebra over $\mathbb{Q}_p$ of Hasse invariant $\frac{1}{n}$. In homotopy theory, $\mathbb{S}_n$ is called the Morava stabilizer group.

We saw that if you have a complex oriented cohomology theory, you get a formal group law. The Honda formal group law comes from a spectrum E_n called Morava E-theory (which is Landweber exact). Its coefficient ring is $(E_n)_* = W(\mathbb{R}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$.

Using a slightly different choice than we've made before, $K(n)$ is Morava K-theory with $K(n)_* \cong \mathbb{F}_{p^n}[u^{\pm 1}]$. The universal deformation sends Γ (the formal group law of E_n) to Φ (the formal group law of $K(n)$).

Write $R = (E_n)_0 = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$. Today, Mingcong and Irina will show that the action of $\mathrm{Aut}(\Phi)$ on $(E_n)_0$ can be upgraded to an action of $\mathrm{Aut}(\Phi)$ on $(E_n)_*$, or even on the whole spectrum E_n.

A spectrum X is built from other "simpler" spectra, which people denote $L_{K(n)}X$. The L means localization. These spaces $L_{K(n)}X$ interpolate between $X \otimes \mathbb{Q}$ (rationalization) and $X^{\wedge}_p$ (p-completion). If you want to understand certain spectra, you want to understand these building blocks.

Because of what Mingcong and Irina tell us about, we will be able to make a certain construction. Recall $\mathbb{S}_n = \mathrm{Aut}(\Phi)$. We can enhance it: we can create a group $\mathbb{G}_n \cong \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$.

Theorem 11.1 (Hopkins-Devinatz)**.** $L_{K(n)}S \simeq E_n^{h\mathbb{G}_n}$

Here $E_n^{h\mathbb{G}_n}$ is like taking $\mathbb{G}_n$ fixed points, but in a "good" homotopy-theoretic way. This is like the Golden Gate Bridge between number theory and homotopy theory. This tool gives a spectral sequence $H^*(\mathbb{G}_n, (E_n)_*) \implies \pi_* L_{K(n)}S$.

11.1. Elliptic curves and elliptic cohomology theories. Let E be a complex oriented ring spectrum. From this you get a formal group law F_E. Let C be an elliptic curve over E_*. It also has a formal group law, and if that coincides with F_E, say that E is an elliptic cohomology theory. For example, E_2 and $K(2)$ are elliptic cohomology theories.

Don yesterday told us about a Hopf algebroid (A, Γ) related to elliptic curves. It turns out that if you do things well enough, there's a map

$$\begin{array}{ccc} H^*(\mathbb{G}_2, (E_2)_*) & \Longrightarrow & \pi_* L_{K(2)}S \\ \downarrow & & \downarrow \\ H^*(A, \Gamma)[\Delta^{-1}]^{\wedge}_{??} & \Longrightarrow & \pi_* L_{K(2)} tmf \end{array}$$

Talk 12: Local chromatic homotopy theory (Leanne Merrill* and Jens Jakob Kjaer)

Topologists want to know about $\pi_* X$. We have a tool, called the Adams spectral sequence, which is an algebraic tool whose inputs are (co)homology and whose output is (something about) $\pi_* X$. You could use ordinary cohomology (what is traditionally called ASS), or some extraordinary cohomology theory. If you use $BP_* X$, this picks out the "p part" of $\pi_* X$. (So BP_* has a prime p that is always floating around.) The hope is that $BP_* X$ will be more computable, and you can use this to glue all the primes together. By gluing all that information, you have a better understanding of $\pi_* X$.

$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*X)$ is the origin of the Adams Novikov spectral sequence. There are several ways to compute this. The chromatic spectral sequence is a tool that computes $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*X)$, and it does so by breaking up X into "chromatic layers". These are determined by families of periodic self-maps. It turns out that these periodic self-maps are related to these $K(n)$'s. So as a topologist, I care about modular forms and elliptic curves because the theory around them will help me compute the ANSS E_2 term.

12.1. Bousfield localization. One way all of this gets tied together is through a tool called Bousfield localization.

Definition 12.1. Let X be a spectrum. Let E_* be a homology theory. We say X is E_*-local if "maps out of an E-acyclic thing to X are zero" – more precisely, for any W such that $E_*W = 0$, we have $[W, X] = 0$.

(If X is a space, not a spectrum, you have to make a slight modification of this, but it's not a big deal.)

Example 12.2. If X is any spectrum, and E is a ring spectrum, then I will show that $E \wedge X$ is E-local. Let W be such that $E_*W = 0$. I need to pick $f : W \to E \wedge X$, and I want to show that this is nullhomotopic. I claim the following diagram is commutative:

$$\begin{array}{ccccc} S^0 \wedge W = W & \xrightarrow{f} & E \wedge X & \xrightarrow{\mathbb{1}} & E \wedge X \\ \downarrow{\scriptstyle \eta\wedge\mathbb{1}} & & \downarrow{\scriptstyle \eta\wedge\mathbb{1}_{E\wedge X}} & \nearrow{\scriptstyle \mu\wedge\mathbb{1}} & \\ E \wedge W & \xrightarrow[\mathbb{1}_E\wedge f]{} & E \wedge E \wedge X & & \end{array}$$

$E \wedge W$ in the bottom left corner has $\pi_*(E \wedge W) = 0$ by assumption. So the top composition $\mathbb{1} \circ f$ is the same as the bottom composition, which factors through $*$. This shows that f is nullhomotopic.

Definition 12.3. An E_*-localization of a spectrum X is an E_*-local spectrum L_EX and a map $\lambda : X \to L_EX$ so that $E_*\lambda$ is an isomorphism.

Theorem 12.4 (Bousfield localization)**.** *For E and X, L_EX exists and is unique.*

Example 12.5. Set $E = K$ (ordinary complex K-theory). Set $X = S^0$. Then $\pi_{-2}(L_KS^0) = \mathbb{Q}/\mathbb{Z}$. In fact, it is *really* non-connective: $\pi_{-i}(L_KS^0) \neq 0$ for infinitely many i. Ravenel has a paper from 1984 in which he actually computes these.

Definition 12.6. E and F are Bousfield equivalent if, for any X, $E \wedge X \simeq *$ iff $E \wedge X \simeq *$. This is an equivalence relation; denote the Bousfield equivalence class by $\langle E \rangle$.

Exercise 12.7. $L_E = L_F \iff \langle E \rangle = \langle F \rangle$

There are two binary operations:

- $\langle E \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle$
- $\langle E \rangle \vee \langle F \rangle = \langle E \rangle \vee \langle F \rangle$

Define a partial ordering: $\langle E \rangle \geq \langle F \rangle$ if acyclicity w.r.t. E ($E \wedge X \simeq *$) implies acyclicity w.r.t. F ($F \wedge X \simeq *$). This means that "F has more acyclics than E", and hence is detecting *less* information.

Theorem 12.8. *If $\langle E \rangle \geq \langle F \rangle$, we have $L_FL_E = L_F$ and we have a map $L_E \to L_F$.*

12.2. "Classical" chromatic homotopy theory.

Definition/ Theorem 12.9. There exist homology theories $K(n)_*$ such that:

(1) $K(0)_*X = H_*(X;\mathbb{Q})$
(2) $K(1)$ is one of the summands of mod p complex K-theory
(3) $K(0)_*(*) = \mathbb{Q}$ and $K(n)_* = K(n)_*(*) = \mathbb{F}_p[v_n^{\pm}]$ where $|v_n| = 2(p^n - 1)$.
(4) (Künneth) $K(n)_*X = K(n)_*X \otimes_{K(n)_*} K(n)_*Y$
(5) Any graded module over $K(n)_*$ is free

These are "graded fields" in the category of cohomology theories.

$K(n)$ is a complex oriented cohomology theory, so it has a formal group law. So $[p](x) = x^{p^n}$ (the Honda formal group law).

Definition 12.10. A finite spectrum is called p-local if $H^*(X;\mathbb{Z}) \cong H_*(X;\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$.

Alternatively: it's the result of doing $L_{H\mathbb{Z}_{(p)}}$; or, it's a suspension spectrum of something built out of p-local spheres. For example, I'll tell you how to construct a 5-local sphere:

$$S^0 \xrightarrow{2} S^0 \xrightarrow{2\cdot 3} S^0 \xrightarrow{2\cdot 3\cdot 7} S^0 \xrightarrow{2\cdot 3\cdot 5\cdot 7\cdot 11} S^0 \to \dots$$

(where you do all the primes except 5) and form the mapping telescope to form $S_{(5)}$.

Theorem 12.11. *$K(n)_*$ has the following detection property. Let X be a finite p-local spectrum. If $K(n)_*X = 0$, then $K(n-1)_*X = 0$.*

Definition 12.12. The *type* of a finite p-local spectrum is the minimal n for which $K(n)_*X \neq 0$.

For example, S has type 0 and $S/p = \mathrm{cofib}(S \xrightarrow{p} S)$ has type 1.

"Bousfield equivalence classes detect type":

Theorem 12.13 (Class invariance theorem)**.** *Let X and Y be finite p-local spectra, with type n and m, respectively. Then $\langle X\rangle = \langle Y\rangle$ iff $m = n$, and $\langle X\rangle < \langle Y\rangle$ iff $m > n$.*

The proof of this is basically the thick subcategory theorem.

12.3. The chromatic tower.

Recall: $E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}]$. This was really nice – it was Landweber exact, and some quotienting process gave us the $K(n)$'s.

Definition 12.14. Write $L_nX := L_{E(n)}X$.

We want to "glue" information together about L_n and L_{n-1}. The chromatic approximates X using "chromatic layers" L_nX.

I want to show that there are maps $L_n \to L_{n-1}$.

Fact 12.15. $\langle E(n)\rangle = \bigvee_{i=0}^{n} \langle K(i)\rangle$

So $L_{E(n)} = L_{K(0)\vee\cdots\vee K(n)}$. If $K(n)_*X$ vanishes, so does $K(n-i)$. So if something is acyclic for $E(n)$, it is acyclic for $E(n-1)$, and it is another fact about the poset of Bousfield localization classes that this means there is a map $L_{E(n)} \to L_{E(n-1)}$ (a.k.a. $L_n \to L_{n-1}$).

The *chromatic tower* is

$$\cdots \to L_nX \to L_{n-1}X \to \cdots \to L_1X \to L_0X.$$

Theorem 12.16 (Chromatic convergence)**.** *$X = \varprojlim L_nX$.*

In practice, let's say X is type 0 (e.g. $S^0_{(p)}$). Then $p^{-1}S^0_{(p)}$ is the mapping telescope $S^0_{(p)} \xrightarrow{p} S^0_{(p)} \xrightarrow{p} S^0_{(p)} \to \ldots$, which is $\cong L_0(S^0_{(p)})$.

Example 12.17. The mod-p Moore spectrum is $M(p) = \mathrm{cofib}(S^0 \xrightarrow{p} S^0)$. It turns out (by the nilpotence theorem) that there exists a map $\Sigma^{?}M(p) \to M(p)$, which we call v_1 because it is multiplication by v_1 in BP_* homology. So we can form the telescope $M(p) \xrightarrow{v_1} M(p) \xrightarrow{v_1} \ldots$ (really, there should be suspensions there), which is $v_1^{-1}M(p)$. It turns out that this is $L_1M(p)$. Sadly, this pattern doesn't hold in general.

Definition 12.18. The chromatic filtration is given by $F_nX = \ker(\pi_*X \to \pi_*L_nX)$. This is a decreasing filtration.

Comment: $L_{n+1}L_nX = L_nX$.

The *chromatic fracture square* is a pullback square

$$\begin{array}{ccc} L_nX & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X \end{array}$$

The maps are induced by the Bousfield localization classes. So, what can we say about $L_{K(n)}X$? If X is type n, there is an Adams Novikov spectral sequence

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*L_{K(n)}X) \implies \pi_*L_{K(n)}X.$$

By **Magic** (a.k.a. "Morava change of rings"), the LHS is $\cong H^{()}\mathbb{G}_n; (E_n)_*X)$. The reason for this program is that this group cohomology is easier to compute.

Talk 13: Lubin-Tate cohomology (Mingcong Zeng* and Irina Bobkova*)

Let $\mathcal{C} = \text{Top}$.

Definition 13.1. An operad C in Top is a collection of spaces $\{C(n)\}_{n\geq 0}$ equipped with:

- a right action of the symmetric group Σ_n on $C(n)$
- a distinguished element $1 \in C(1)$
- maps

$$C(k) \times C(i_1) \times \ldots \times C(i_k) \to C(i_1 + \cdots + i_k)$$

Example 13.2 (Endomorphism operad of $X \in \text{Top}$)**.** Define the operad called $\text{End}(X)$ by $\text{End}(X)(n) = \text{Map}(X^n, X)$. The structure maps $\text{Map}(X^k, X) \times \text{Map}(X^{i_1}, X) \times \ldots \times \text{Map}(X^{i_k}, X) \to \text{Map}(X^{i_1+\cdots+i_k}, X)$ are given by taking $(g, f_{i_1}, \ldots, f_{i_k})$ and plugging the f_{i_j}'s into g.

Example 13.3 (Associative operad)**.** Define the operad As by $As(n) = \Sigma_n$.

Example 13.4 (A_∞ operad)**.** A is an A_∞ operad if $\pi_0 A(n) = \Sigma_n$, each component of $A(n)$ is contractible, and the action of Σ_n on $AN(n)$ is free.

Example 13.5 (Commutative operad)**.** The operad Com is defined by $Com(n) = *$.

Example 13.6 (E_∞ operad)**.** E is an E_∞ operad if each $E(n)$ is contractible and the action of Σ_n on E_n is free.

For every operad C, you can associate a functor $C : \text{Top} \to \text{Top}$ $C(X) = \coprod_{n\geq 0} C(n) \times_{\Sigma_n} X^n$ which is a monad (a monoid object in the category of endofunctors).

Definition 13.7. A topological space X is a C-algebra if it is equipped with a map $\lambda : C(X) \to X$ and diagrams encoding unit and associativity. Equivalently, X is equipped with a collection $C(n) \times_{\Sigma_n} X^n \to X$ (along with those diagrams).

The idea is that C acts on X.

Remark 13.8. Spectra is a category over spaces: there is a "module structure" Spaces $\times$ Spectra $\to$ Spectra. If C is an operad in spaces and X is a spectrum, X can be a C-algebra.

Example 13.9. Here are some examples of C-algebras:

(1) The free C-algebra out of X

$$C(X) = \coprod_{n\geq 0} C(n) \times_{\Sigma_n} X^n.$$

(2) X is an algebra over Com if there is exactly one map $X \times \ldots \times X \to X$, i.e. if X is a commutative ring object.

(3) X is an algebra over an E_∞ operad C if there is a map $C(n) \times_{\Sigma_n} X^n \to X$, so there is a contractible space of maps $X^n \to X$.

Topologists don't like strict commutativity.

Topology	Algebra
E_∞ ring spectrum	commutative rings
A_∞ ring spectra	associative rings
spectra	abelian groups

We would really like to define $E_n^{h\mathbb{G}_n}$ where $\mathbb{G}_m = \mathrm{Aut}(\Gamma_n) \rtimes \mathrm{Gal}$ and we would like it to be an E_∞-ring spectrum. We can prove that E_n is E_∞ and A_∞, and $\mathbb{G}_n$ acts on E_n by E_∞ and A_∞ maps. The A_∞ part is technical (but Mingcong will talk about it), and the E_∞ part is really technical.

For the rest of the talk, C is an A_∞ operad. Here's the plan for proving that E_n is A_∞:

(1) Assuming X is a C-algebra, we show
$$\mathrm{Hom}_{C\text{-alg}}(E_n, E_n) \simeq FG((\mathbb{F}_{p^n}, \Gamma), (\mathbb{F}_{p^n}, \Gamma))$$
where $\mathbb{G}_n$ acts.

(2) Check that E_n is a C-algebra.

Let $k = \mathbb{F}_{p^n}$, Γ_n the Honda formal group law of height n. We see that $\mathrm{Aut}(\Gamma_n)$ acts on $(E_n)_*$ on an algebra level. The automorphisms act on the spectrum up to homotopy. But we what we really want is to talk about fixed points, and for that we need a better action – we need it to happen in a category where we can take limits. So we're going to lift E_n to a better category by using operads.

We need:

(1) a theory to show why E_n is A_∞ or E_∞

(2) a theory to show $\mathrm{Map}_{A_\infty \text{ or } E_\infty}(E_n, E_n)$

We will start with (2) and look at (1). Suppose C is an A_∞-operad. Assume both E and F are C-algebras. We want to look at the mapping space $C\text{-alg}(F, E)$. We want to know $\pi_*(\text{this})$. Classically, we can do some manipulation on F and E if we only care about π_*. To study $\mathrm{Map}(X, Y)$, we can replace X by a simplicial complex and make use of the filtration on the simplicial complex. Recall F is a C-algebra; that means you have a map $CF \to F$, and moreover you have a diagram $F \leftarrow CF \leftleftarrows CCF$ from associativity, and you can continue this to the right to make a simplicial spectrum $C^{\bullet+1}F := CF \leftleftarrows CCF \Lleftarrow \dots$ (a simplicial object in the category of spectra). This object is a good approximation to F: if C and F are "good", $|C^{\bullet+1}F| \xrightarrow{\simeq} F$ so $C\text{-alg}(F, E) \xrightarrow{\simeq} C\text{-alg}(|C^{\bullet+1}F|, E)$

(Comment: A_∞ corresponds to having a sheaf of algebras on the stack, and E_∞ corresponds to having a sheaf of commutative algebras.)

Without geometric realization, C-alg$(C^{\bullet+1}F, E)$ is a cosimplicial object. There is a standard tool to attack such objects, called the Bousfield-Kan spectral sequence, and there is an obstruction theory associated to it.

Let $X_\bullet$ be a cosimplicial space. I can define the totalization in a dual fashion to geometric realization:

$$\operatorname{Tot} X_\bullet = \varprojlim X_n^{\Delta^{op}}.$$

Theorem 13.10.

(a) If $X_\bullet$ has a base point f (the spaces are pointed and all the maps preserve the base point), then there is a spectral sequence

$$E_2^{s,t} = \pi^s \pi_t(X_\bullet, f) \implies \pi_{t-s} \operatorname{Tot} X_\bullet.$$

(b) If $[f]$ is in the equalizer of $\pi_* X_0 \rightrightarrows \pi_* X_1$, plus some technical condition, then $\pi^s\pi_t(X_\bullet, [f])$ is well-defined. There are obstructions in $\pi^s\pi_{s-1}(X_\bullet, [f])$ to lifting $[f]$ to $\operatorname{Tot}_\bullet X$.

Here π^s is cohomology of the associated chain complex to the simplicial abelian group $\pi_t(\dots)$.

Theorem 13.11. *The spectral sequence/ obstruction has*

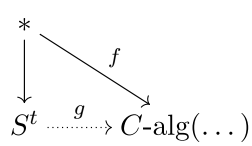
$$E_2^{0,0} \cong \operatorname{Hom}_{E_*\text{-}alg}(E_*F, E_*)$$
$$E_2^{s,t} \cong \operatorname{Der}^s_{E_*}(E_*F, E_{*+t})$$

I need to explain what this Der means. Start with a comutative ring R. Let A be an R-algebra and M an R-module. Then

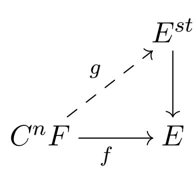
$$\operatorname{Der}^s_R(A.M) := H^s(\operatorname{Der}_R(P_\bullet, M))$$

where $P_\bullet \to A$ is a cofibrant replacement in simplicial associated R-algebras.

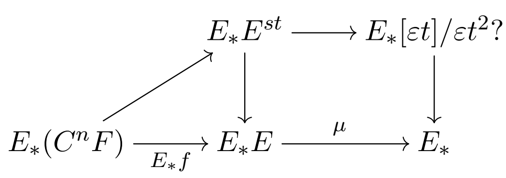
We're interested in $\pi_t(C\text{-alg}(C^nF, E), f)$. You want to find a dotted arrow



Apply E_* homology to



to get



$$\pi_t(C\text{-alg}(C^nF, E), f) \to \operatorname{Hom}_{E_*\text{-alg}}(E_*(C^nF), E_*[\varepsilon t]/\varepsilon t^2) \cong \operatorname{Der}_E(E_*(C^nF), E_{*+t})$$

Let $E = E(k_1, \Gamma_1)$ and $F = E(k_2, \Gamma_2)$.

Theorem 13.12.

$$\mathrm{Hom}_{E_*\text{-}alg}(E_*F, E_*) \cong FG((k_1, \Gamma_1), (k_2, \Gamma_2)) = \{(j, \varphi) \,:\, j : k_2 \to k_1, \varphi : \Gamma_1 \xrightarrow{\sim} j^*\Gamma_2\}$$

and $\mathrm{Der}^s_{E_*(E_*F, E_{*+t})} = 0$. *We have* $\pi_0(C\text{-}alg(F, E)) \cong FG(\Gamma_1, \Gamma_2)$ *and* π_i *for* $i > 0$ *are trivial.*

Now we can build the A_∞-structure. A map in $\mathrm{Oper}(C, \mathrm{End}(E))$ will make E a C-algebra. we want to know if the set of such maps is empty. Resolve C into a simplicial operad and use the Bousfield-Kan spectral sequence or the obstruction theory. It turns out that we get a similar spectral sequence

$$\begin{aligned} E_2^{0,0} &= \mathrm{Oper}(\pi_0 C, \pi_0 \mathrm{End}(E)) \\ E_2^{0,t} &= [\Sigma^t E, E] \\ E_2^{s,t} &= \mathrm{Der}^{s+t}_{E_*}(E_*E, E_{*+t}) \end{aligned}$$

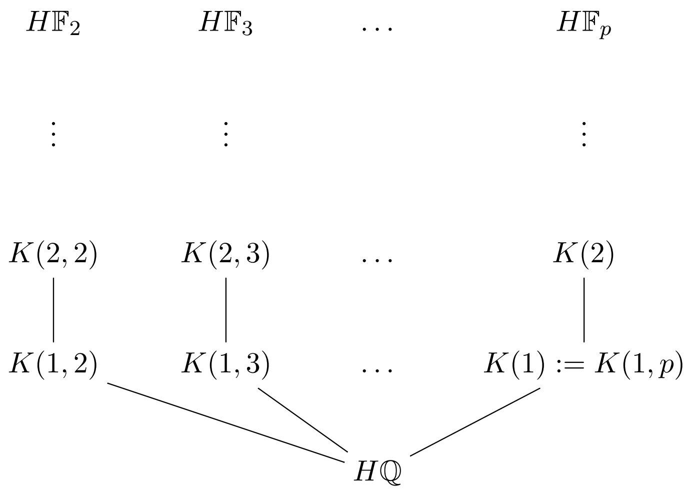
See $E_2^{s,s-1} = 0$ when E is Lubin-Tate. So there is no obstruction to constructing a single point in the space, so the space is nonempty.

Talk 14: Recap (Vesna Stojanoska)

Yesterday, we saw that localization plays an important role. Let me make a basic analogy. Suppose you were studying abelian groups, maybe starting with finitely generated abelian groups. Any such A can be written as $A_{\text{free}} \oplus \bigoplus_p A_p$ where A_p is the p-power torsion. If you want to understand an arbitrary abelian group, you can split up the work and study it "one prime at a time". A has a map to $\mathrm{Spec}\,\mathbb{Z}$, where the free part lives over the prime (0), and A_p lives over p. If you invert away from p and then mod out by it, you get $\mathbb{F}_p$. Doing the same thing for 0 produces $\mathbb{Q}$ instead.

A submodule is $\mathbb{Q}$-acyclic if, when you tensor with $\mathbb{Q}$, you get zero (this is the same as torsion). It is $\mathbb{Q}$-local if it is free. It is $\mathbb{F}_p$-acyclic if it consists of prime-to-p torsion, and $\mathbb{F}_p$-local if it consists of p-power torsion.

You can do the same thing for (finite) spectra: break them up into the $K(n)$-acyclic vs. $K(n)$-local part, for every n. Recall $K(0) = H\mathbb{Q}$, $K(1) = K(1,p) = K/p$, $K(\infty) = H\mathbb{F}_p$.



So you can think of finite spectra as decomposing into this picture. You can also think of the vertical dimension here as heights of associated formal groups.

Rationally, $\pi_* S \otimes \mathbb{Q}$ is really simple. We saw yesterday that the $K(1)$-local and $\mathbb{Q}$-local parts piece together into something fully understood in concrete terms (how divisible are some numbers by p). The $K(2)$-local sphere is more complicated, but it is in the process of being understood in the last 10-15 years. This relates to the mysterious patterns (lightning flashes) in the Adams spectral sequence.

(We have a localization map $Sp_p \overset{L_{K(n)}}{\to} Sp_{K(n)}$ from p-local spectra to $K(n)$-local spectra. These assemble in a more complicated way than in algebra: you need to use the fracture squares.)

To study all formal groups at the same time is a moduli problem. Moduli of elliptic curves is going to tell us about formal groups of height ≤ 2 so we can isolate the bottom of the picture. If we just want to study one prime, we can study "formal neighborhoods" of $K(n)$, which are governed by Morava E-theories E_n (which is acted on by $\mathbb{G}_n = \mathcal{O}_D^\times \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$). These formal neighborhoods are called Lubin-Tate spaces.

Once you've done the hard work of putting fancy multiplicative structures on these guys, you can do *actual algebras* with these, not just vague analogies with algebra. For example, the category of modules over a ring spectrum means something specific here.

The Gross-Hopkins period map: the Lubin-Tate space is like an open disc, and they construct an étale cover of projective space by an open disc. It only happens in p-adic geometry (usually, projective spaces don't have nontrivial covers).

Today we'll study moduli of things; we've chosen those moduli problems because they describe parts of the diagram above.

Talk 15: Moduli problems and stacks (Artur Jackson* and Yuri Sulyma*)

Slogan: groupoids + sites = stacks.

Let a group G act on a set X. We define the category $X//G$ so the objects are $x \in X$, and the morphisms are $\mathrm{Hom}(x, y) = \{g \in G \,:\, g \cdot x = yy\}$. We have $\pi_0(X//G) = X/G$ (here π_0 is the set of isomorphism classes), and $\pi_1(X//G, x)$ (automorphisms of the element in this category) is the stabilizer of x. For example, if $X \neq \emptyset$ is transitive, then this says that $\pi_0(X//G) = 0$. If X is a free G-set, then $\pi_1(X//G, x) = 0$ for all x. So this doesn't just remember that two things are isomorphic, but *why* they're isomorphic.

I claim $X = \varinjlim_{X//G} (G^{\mathrm{reg}})$ where G^{reg} means a regular representation. (Recall that a map $G^{\mathrm{reg}} \to X$ is the same as an element of X.) This groupoid has the property that the objects are X and the arrows are $G \times X$. Ravenel calls this a "split" groupoid (i.e. the morphisms is objects $\times$ something). For example, we had $\Gamma = A \otimes_{\mathbb{Z}}$ something.

Any time I have G acting on X (e.g. a group scheme acting on a scheme), I can form the groupoid $G \times X \rightrightarrows X$.

Example 15.1. We saw how to get the moduli space of elliptic curves as $\mathbb{H}/SL(2, \mathbb{Z})$. Instead, I could take the stacky quotient $H//SL(2, \mathbb{Z}) = (\mathbb{H} \times SL(2, \mathbb{Z}) \rightrightarrows \mathbb{H})$. This will be $\mathcal{M}_{\mathrm{ell}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C}$, the moduli stack of elliptic curves base-changed to $\mathbb{Z}$.

15.1. Pullback of categories.

Let X, Y, Z be categories. I want to define the pullback

$$\begin{array}{ccc} P = X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad f \quad} & Z \end{array}$$

For sets, I would say $P = \{x \in X, y \in Y \,:\, fx = gy\}$. But what does it mean for two things in a category to be equal? It means that there is an isomorphism between them. So for categories, $P = \{x \in X, y \in Y, \varphi : fx \xrightarrow{\cong} gy\}$. (The objects of P are "what they're supposed to be".)

For example, if $c, d \in C$ the following is a pullback

$$\begin{array}{ccc} \{\text{isos } c \cong d\} & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle d} \\ * & \xrightarrow{\quad c \quad} & C \end{array}$$

If $C = BG$ (one object with morphisms in bijection with elements of G), then the pullback is just G.

15.2. Sites.

Eventually, we want to talk about sheaves on categories, not just on spaces. It is easy to generalize the notion of a presheaf on a space X to a presheaf on a category C. A presheaf on X is a functor $\mathcal{O}(X)^{op} \to \text{Set}$, where $\mathcal{O}(X)$ is the category of open subsets (with inclusion). So a presheaf on C should just be a functor $C^{op} \to \text{Set}$.

Now we generalize the notion of open cover. If $\{U_i \subset X\}$ is an open cover, think of this as a family of morphisms $\{U_i \to X\}$ in Top. So for an object $c \in C$, we can consider families $\{U_i \to c\}$ of morphisms in C. When is such a family a cover of C?

Definition 15.2. A Grothendieck topology J on C assigns to each $c \in C$ a collection of covers $\{\{u_i \to c\}\}$.

In topology, we have the trivial cover $\{X\}$ of X. Analogously, a trivial cover of $c \in C$ is $\{f\}$, where $f : d \to c$ is an isomorphism. Call this axiom 1.

In topology: if $\{U_i\}$ covers X, then $\{U_i \cap V\}$ covers V for any $V \subset X$. Analogously, if $\{U_i \to c\}$ is a cover, then $\{U_i \times_c d \to d\}$ is a cover of d:

$$\begin{array}{ccc} U_i \times_c d & \longrightarrow & d \\ \downarrow & & \downarrow{\scriptstyle f} \\ U_i & \longrightarrow & c \end{array}$$

This is axiom 2 of covers.

In practice, you often show something is a cover by using the fact that "a cover of a cover is a cover": if $\{U_i \to c\}$ covers c and $\{V_{i,j} \to U_i\}$ covers U_i for all i, then $\{V_{i,j} \to c\}$ covers c.

If I have a cover, I can throw in more arrows and still get a cover. So when I say what the covers are, I mean that the covers contain those things.

Example 15.3.

- Let $C = \mathcal{O}(X)$, then $\{U_i \to U\}$ covers U if $\bigcup U_i = U$.
- Let $C = \text{Top}$. Then $\{U_i \to X\}$ covers X if each $U_i \hookrightarrow X$ is the inclusion of an open subspace, and $\bigcup U_i = X$. (This is a global version of the previous example.)
- Let C be the opposite of the category of finitely presented $\mathbb{Z}$-algebras. A cover of $\operatorname{Spec} R$ is $\{\operatorname{Spec} R[f_i^{-1}] \to \operatorname{Spec} R\}$ such that the f_i generate the unit ideal of R. This is the big Zariski site. (A small Zariski site would be the Zariski analogue of the first example.)
- If $C = \text{Top}$, I could also ask for covers $\{U_i \to X\}$ where $U_i \to X$ has open image where U_i is a covering space of its image. This is a topological analogue of the étale site.
- Say that $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is *flat* if $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$ for all x. It is *faithfully flat* if it is flat and surjective. Define a map between locally ringed space to be a cover if it is flat. This gives the flat topology. In algebraic geometry, there are two versions: fppf and fpqc. These are the same for affine things, so we can just say "flat topology".

The point of defining a site is to be able to say what it means for a presheaf to be a sheaf.

Definition 15.4. A presheaf $F : C^{op} \to \text{Set}$ is a *sheaf* if the following diagram

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(\underbrace{U_i \times_X U_j}_{U_{ij}})$$

is an equalizer for all $X \in C$, and for all covers $\{U_i \to X\}$ of X.

Definition 15.5. A *prestack* is a functor $M : C^{op} \to \text{Gpd}$. It is a *stack* if

$$M(X) \longrightarrow \prod_i M(U_i) \rightrightarrows \prod_{i,j} M(\underbrace{U_i \times_X U_j}_{U_{ij}}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \prod_{i,j,k} M(U_{ijk})$$

is an equalizer for all $X \in C$ and all covers $\{U_i \to X\}$.

This is part of a bar complex; if you want to talk about ∞-stacks, you'd need to write down the entire bar complex.

For example, let $M = \text{Vect}(-)$, where $\text{Vect}(X)$ is the category with objects = collections of vector bundles ξ_i on each U_i along with isomorphisms $\xi|_{U_i \cap U_j} \overset{\alpha_{ij}}{\to} \xi_j|_{U_i \cap U_j}$ satisfying $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$.

For example, let (B_0, B_1) be a Hopf algebroid. Then I have a prestack

$$\text{Hom}(X_1, B_0) \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \text{Hom}(X, B_1)$$

which I can stackify. This is what is meant by a stack associated to a Hopf algebroid.

Now specialize to C as the category of affine schemes with the flat topology. We have

$$\text{Zariski open inclusions} \subset \text{Étale morphisms} \subset \text{Flat morphisms.}$$

Étale means flat + unramified (this coincides with our earlier definition of unramified).

An example of something that isn't a Zariski open inclusion is the n-fold open cover of $\mathbb{G}_m$ by itself.

15.3. Descent.

Theorem 15.6 (Grothendieck, fpqc descent for quasicoherent sheaves)**.** *Let $U = \{f_i : U_i \to X\}$ be an fpqc cover of X. Let $\mathcal{F}_i$ be quasicoherent sheaves over every U_i. Descent data is isomorphisms $\sigma : \mathcal{F}_i|_{U_i \cap U_j} \cong \mathcal{F}_j|_{U_i \cap U_j}$. Then there exists a quasicoherent sheaf $\mathcal{F}$ on X such that $f_i^* \mathcal{F} \to \mathcal{F}_i$ is an isomorphism.*

Definition 15.7. Let $\text{QCoh}(-)$ be the prestack $X : \text{Aff}_{pf}^{op} \to \text{Gpd}$ sending T to the category whose objects are quasicoherent sheaves over T and morphisms are isomorphisms.

15.4. Morphisms. Let $\varphi : \mathcal{F} \to \mathcal{F}'$ be a morphism of stacks. A scheme S gives rise to a discrete stack $\underline{S}$ via the Yoneda embedding (it only has identity isomorphisms).

Definition 15.8. We say that φ is *schematic* (a.k.a. *representable*) if, for all maps $f : \underline{S} \to \mathcal{F}'$, the left vertical arrow in the pullback

$$\begin{array}{ccc} \mathcal{F} \times_\varphi \underline{S} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \varphi \\ \underline{S} & \xrightarrow{f} & \mathcal{F}' \end{array}$$

is a morphism of schemes.

(Sometimes this is called *relatively representable.*)

15.5. Algebraic stacks.

Definition 15.9. Let $T \in \mathrm{Aff}$. Let $a_1, a_2 \in \mathcal{F}(T)$. Then $\mathrm{Isom}_T(a_1, a_2)$ is a functor $\mathrm{Aff}^{op}/T \to \mathrm{Ens}$ (where Ens means "ensembles" *(possibly just Set modulo technicalities)*) sending $(\alpha : S \to T)$ to $\mathrm{Hom}(\alpha^*(a_1), \alpha^*(a_2))$.

For example, if E_1, E_2 are two curves over $T = \mathrm{Spec}\, R$ and $\alpha : S \to T$, I'm looking at the isomorphisms $\alpha^* E_1 \overset{\cong}{\to} \alpha^* E_2$ over S.

This theorem is not very hard.

Theorem 15.10. *Let $\mathcal{F}$ be a stack. TFAE:*

(1) The diagonal morphism $\mathcal{F} \overset{\Delta}{\to} \mathcal{F} \times \mathcal{F}$ is representable.

(2) All isoms are representable.

(3) Every map $\underline{S} \to \mathcal{F}$ is representable. Note that this representable is not the fancy stacky "representable" – it's a functor into sets!

This theorem is important for actually making definitions.

Let $x \in \{$étale, flat smooth, $\dots\}$. We say that a representable morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is x if f in the following pullback diagram is x.

$$\begin{array}{ccc} c, aF \times_G \underline{S} & \longrightarrow & \mathcal{F} \\ \downarrow f & & \downarrow \\ \underline{S} & \longrightarrow & G \end{array}$$

Definition 15.11. A stack X is *algebraic* if the hypotheses of Theorem 15.10 hold, and there is a smooth atlas $U = \bigcup U_i \to X$ where U is affine and $U \to X$ is smooth.

For example, $*//G$ is covered by the atlas which is just the point.

Given a Hopf algebroid (A, Γ), you can produce an algebraic stack $X_{A,\Gamma}$ with an atlas. (You can go backwards if you have an affine cover $\mathrm{Spec}\, A$ of X, and then you look at the pullback

$\operatorname{Spec} A \times_X \operatorname{Spec} A = \operatorname{Spec} \Gamma$.) If you put enough conditions on both sides, this is an equivalence. You need the stacks to be rigidified (a stack with a chosen affine cover), and the Hopf algebroids to be flat.

Talk 16: Stacky homotopy theory (Gabe Angelini-Knoll* and Eva Belmont*)

I can't liveTEX my own talk!

Talk 17: Moduli of elliptic curves (Allen Yuan* and Dimitar Kodjabachev*)

References: TMF book, compiled by Mike Hill and others.

Fix a prime p.

Theorem 17.1 (Landweber, Ravenel, Stong)**.** *Let C be an elliptic curve given by Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. This lives over $\widetilde{A} = \mathbb{Z}[a_1, \dots, a_6, \Delta^{-1}]$. This curve completes to give a formal curve $\widehat{C}$. There is also a formal group law, which corresponds to a map $L \to \widetilde{A}$.*

Then $\widehat{C}$ is Landweber exact.

Proof. Step 0: $v_0 = p$. OK.

Step 1: want to show that $v_1 \in \widetilde{A}/p\widetilde{A} = \mathbb{F}_p[a_1, \dots, a_6, \Delta^{-1}]$ is nonzero. As long as there exist non-supersingular curves, this is OK.

Step 2: Claim that $v_2 \in \widetilde{A}/(p, v_1)\widetilde{A}$ is a unit. If not, $\widetilde{A}/(p, v_1, v_2) \to K$ would give an elliptic curve of height 3, which doesn't exist. □

So we get a cohomology theory E_{Weier} satisfying

$$E_{\text{Weier}}(X) = \widetilde{A} \otimes_L MU_* X.$$

In some worlds, you might want to periodify this.

$$\begin{array}{ccc} \{\text{Elliptic coh. thry.}\} & \longrightarrow & \{\text{Cplx. orient. coh. thry.}\} \\ \downarrow & & \downarrow {\scriptstyle \operatorname{Spf} E^*(\mathbb{CP}^\infty)} \\ \{\text{Elliptic curve}\} & \xrightarrow{\widehat{C}} & \{\text{formal groups}\} \end{array}$$

Landweber exactness gives a partial section on the right vertical map.

Definition 17.2. An elliptic cohomology theory over a commutative ring R is:

(1) an elliptic curve C/R,

(2) a spectrum E (weakly even periodic),

(3) isomorphisms $\pi_0 E \cong R$ and $\mathrm{Spf}(E^*(\mathbb{C}\mathrm{P}^\infty)) \cong \widehat{C}$.

For the rest of the talk, I'll be taking apart the diagram above and taking it a little more seriously with stacks.

The first moduli we want to understand is $\mathcal{M}_{\mathrm{ell}}$, the moduli of elliptic curves. Over $\mathbb{C}$, we've seen that it looks like $\mathbb{H}/SL_2(\mathbb{Z}) = \mathcal{M}_{\mathrm{ell}} \times \mathrm{Spec}\,\mathbb{C}$. One way to picture this is by a fundamental domain in the upper half plane. Alternatively, there is a map $j : \mathcal{M}_{\mathrm{ell}} \times \mathrm{Spec}\,\mathbb{C} \to \mathbb{C}$ called the j-homomorphism that classifies elliptic curves up to isomorphism.

First, suppose we're working in nonzero characteristic. We want to study $[p] : C \to C$. The differential of this map at the origin is p. If p is invertible, the p-torsion is $C[p] = (\mathbb{Z}/p)^2$. In characteristic zero, the p-torsion could either look like $\mathbb{Z}/p$ with formal height 1, or $*$ with formal height 2. The first case is called the ordinary case, and the second case is called the supersingular case.

Most curves are ordinary, but there is a supersingular locus $\mathcal{M}_{\mathrm{ell}}^{ss}$.

Let $\widetilde{A} = \mathbb{Z}[a_1, \ldots, a_6, \Delta^{-1}]$, where $\mathrm{Spec}\,\widetilde{A}$ is the moduli of smooth Weierstrass equations, and $\widetilde{\Gamma} = \widetilde{A}[u^{\pm}, r, s, t]$ be the moduli of isomorphisms. We have three related moduli spaces:

$$\mathcal{M}_{\mathrm{ell}} \to \overline{\mathcal{M}_{\mathrm{ell}}} \to \overline{\mathcal{M}_{\mathrm{ell}}}^{+}$$

where

- $\mathcal{M}_{\mathrm{ell}} = \mathcal{M}_{(\widetilde{A},\widetilde{\Gamma})}$
- $\overline{\mathcal{M}_{\mathrm{ell}}}$ is the compactification, which also parametrizes curves with nodal singularities
- $\overline{\mathcal{M}_{\mathrm{ell}}}^{+} =: \mathcal{M}_{A,\Gamma}$ parametrizes curves with cusps and nodes

The map assigning a formal group to an elliptic curve (i.e. $C \mapsto \widehat{C}$) gives rise to a map of stacks $\mathcal{M}_{\mathrm{ell}} \xrightarrow{\Phi} \mathcal{M}_{\mathrm{fg}}$.

Theorem 17.3. Φ *is flat.*

PROOF. Take the pullback

$$\begin{array}{ccc} PB & \longrightarrow & \mathcal{M}_{\mathrm{ell}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,R & \longrightarrow & \mathcal{M}_{\mathrm{fg}} \end{array}$$

Add the faithfully flat cover $\operatorname{Spec} \widetilde{A} \to \mathcal{M}_{\text{ell}}$ and take another pullback:

$$\begin{array}{ccc}
PB' & \longrightarrow & \operatorname{Spec} \widetilde{A} \\
\downarrow & & \downarrow \\
PB & \longrightarrow & \mathcal{M}_{\text{ell}} \\
\downarrow & & \downarrow \\
\operatorname{Spec} R & \longrightarrow & \mathcal{M}_{\text{fg}}
\end{array}$$

We had a faithfully flat cover $\operatorname{Spec} \widetilde{A} \to \mathcal{M}_{\text{ell}}$, which shows that $PB' \to PB$ is faithfully flat. By Landweber, $\operatorname{Spec} \widetilde{A} \to \mathcal{M}_{\text{fg}}$ is flat, which shows that $PB' \to \operatorname{Spec} R$ is flat. $\square$

Corollary 17.4. *Given a flat map $f : \operatorname{Spec} R \to \mathcal{M}_{ell}$, you get an elliptic cohomology theory E_f:*

$$\begin{array}{ccc}
\operatorname{Spec} R & \xrightarrow{f} & \mathcal{M}_{ell} \\
& \searrow & \downarrow \\
& & \mathcal{M}_{fg}
\end{array}$$

In other words, there's a presheaf $\mathcal{O}^{hom}$ of homology theories on $\mathcal{M}_{ell}$.

Look at the geometric fibers of Φ. Fix an algebraically closed field k. We want to study the pullback

$$\begin{array}{ccc}
PB & \longrightarrow & \mathcal{M}_{\text{ell}} \\
\downarrow & & \downarrow \Phi \\
\operatorname{Spec} k & \longrightarrow & \mathcal{M}_{\text{fg}}
\end{array}$$

In characteristic p, let's focus on height 2. What this looks like is one point with the action of the Morava stabilizer group S_n. This can be called BS_n. Living over this is the stack of supersingular elliptic curves. We want to take the pullback

$$\begin{array}{ccc}
\text{“}\mathcal{M}_{\text{ell}}^{ss} \times S_n\text{”} & \longrightarrow & \mathcal{M}_{\text{ell}}^{ss} \\
\downarrow & & \downarrow \\
\operatorname{Spec} k & \longrightarrow & BS_n
\end{array}$$

In general, what we're doing is

$$\begin{array}{ccc}
\text{Geometric point} & \longrightarrow & \text{Stack or scheme} \\
& \searrow \quad \nearrow_{\text{flat}} & \\
& \text{Formal nbd.} &
\end{array}$$

E.g. if the stack is $\mathcal{M}_{fg}$, then this is a way to engineer something that is Landweber flat.

Geometric point	Formal neighborhood	Stack or scheme
$\operatorname{Spec} R/\mathfrak{m}$	$\operatorname{Spf} \widehat{R_{\mathfrak{m}}}$	$\operatorname{Spec} R$
$K(2)_* = \mathbb{F}_p[v_2^{\pm}]$	E_2	$\mathcal{M}_{\text{fg}}$
Elliptic curve$/k$	Elliptic curve$/R$	$\mathcal{M}_{\text{ell}}$

If K is a field with characteristic p and R is a nilpotent thickening, we want to study the square

$$\begin{array}{ccc} \{\text{Supersingular ell. curves}/R\} & \longrightarrow & \{\text{formal groups}/R\} \\ \downarrow & & \downarrow \\ \{\text{Ss. ell. curves}/k\} & \longrightarrow & \{\text{formal groups}/k\} \end{array}$$

Theorem 17.5 (Serre-Tate)**.** *The above square is a pullback.*

Let ω be the module of invariant differentials ω over $\mathcal{M}_{\text{fg}}$. You can pull this back over $\mathcal{M}_{\text{ell}} \xrightarrow{\Phi} \mathcal{M}_{\text{fg}}$ to get $\omega_{ell} = \Phi^*\omega$. Recall, a modular form of weight n is a global section of $\omega_{ell}^{\otimes n}$, i.e. $H^0(\mathcal{M}_{\text{ell}}, \omega^{\otimes n})$.

Along the map $\mathcal{M}_{\text{fg}}^s \to \mathcal{M}_{\text{fg}}$, a tangent vector on $\mathcal{M}_{\text{fg}}^s$ goes to a $\mathbb{G}_m$-torsor on $\mathcal{M}_{\text{fg}}$. From the viewpoint of topology, a $\mathbb{G}_m$-torsor is a "grading sheaf". It's like a Serre twist on projective space. Recall:

$$\begin{aligned} \operatorname{Ext}(MU_*, MU_*X) &= R\operatorname{Hom}(\mathcal{O}_{\mathcal{M}_{\text{fg}}^s}, \mathcal{F}_X) \\ &= H^*(\mathcal{M}_{\text{fg}}^s, \mathcal{F}_X) \\ H^*(\mathcal{M}_{\text{ell}}; \omega^{\otimes q}) &= H^*(\mathcal{M}_{\text{fg}}; \mathcal{F} \otimes \omega^{\otimes q}) \implies \pi_* TMF \end{aligned}$$

Recall stacky Landweber flatness says that $\Phi : \mathcal{M}_{\text{ell}} \to \mathcal{M}_{\text{fg}}$ is flat. There is a functor

$$\mathcal{O}^{\text{hom}} : (\operatorname{Aff}/\mathcal{M}_{\text{ell}})^{op} \to \{\text{elliptic cohomology theories}\}$$

taking $C/R \mapsto \operatorname{Ell}_C/R$. We want a universal elliptic cohomology theory.

Theorem 17.6 (Goerss-Hopkins-Miller)**.** *There exists a sheaf*

$$\mathcal{O}^{top} : (\operatorname{Aff}/\overline{\mathcal{M}_{ell}})^{op}_{\acute{e}t} \to \{E_\infty\text{-rings}\}$$

whose associated presheaf is $\mathcal{O}^{hom}$ the sheaf sending $\operatorname{Spec} R$ to the Landweber flat cohomology theory associated to the formal group classified by $\operatorname{Spec} R \to \overline{\mathcal{M}_{ell}} \to \mathcal{M}_{fg}$ (note étale $\implies$ flat which is why this is Landweber flat). This satisfies:

- $\mathcal{O}^{\text{top}}(f)$ is even and weakly periodic
- *if $f : \operatorname{Spec} R \to \overline{\mathcal{M}_{ell}}$ then $\pi_*(\mathcal{O}^{\text{top}}(f)) \cong R$*
- $\Gamma_{\mathcal{O}^{\text{top}}(f)} \cong \widehat{C}$ *I think this means that $\widehat{C}$ is the formal group associated to the spectrum $\mathcal{O}^{top}(f)$*

Construction of $\mathcal{O}^{\mathrm{top}}$. (1) Assemble $\mathcal{O}^{\mathrm{top}}$ from an arithmetic square

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{top}} & \longrightarrow & \prod_p i_{p,*}\mathcal{O}_p^{\mathrm{top}} \\ \downarrow & & \downarrow \\ i_{\mathbb{Q},*} \times \mathcal{O}_{\mathbb{Q}}^{\mathrm{top}} & \longrightarrow & (\prod_p i_{p,*}\mathcal{O}_p^{\mathrm{top}})_{\mathbb{Q}} \end{array}$$

where $\mathcal{O}_p^{\mathrm{top}}$ is a sheaf over $(\overline{\mathcal{M}_{\mathrm{ell}}})_p$ and $\mathcal{O}_{\mathbb{Q}}^{\mathrm{top}}$ is a sheaf over $(\overline{\mathcal{M}_{\mathrm{ell}}})_{\mathbb{Q}}$.

We have morphisms

$$\mathcal{M}_{\mathrm{ell}}^{\mathrm{ord}} \overset{i_{\mathrm{ord}}}{\to} (\overline{\mathcal{M}_{\mathrm{ell}}})_p \overset{i_{ss}}{\leftarrow} \mathcal{M}_{\mathrm{ell}}^{ss}.$$

(2) Constructing $\mathcal{O}_{\mathbb{Q}}^{\mathrm{top}}$ is easy. $\mathcal{O}_p^{\mathrm{top}}$ is subtle. It is assembled from a Hasse square

$$\begin{array}{ccc} \mathcal{O}_p^{\mathrm{top}} & \longrightarrow & i_{ss,*}\mathcal{O}_{K(2)}^{\mathrm{top}} \\ \downarrow & & \downarrow \\ i_{\mathrm{ord},*}\mathcal{O}_{K(1)}^{\mathrm{top}} & \overset{\beta}{\longrightarrow} & (i_{ss,*}\mathcal{O}_{K(2)}^{\mathrm{top}})_{K(1)} \end{array}$$

The idea is that this is supposed to come from a fracture square. □

Now we can believe in the existence of $\mathcal{O}^{\mathrm{top}}$.

Definition 17.7 ("The mother of all elliptic cohomology theories")**.** Take $\mathcal{M}_{\mathrm{ell}}$, do something mysterious involving $\Phi(\mathcal{O}^{\mathrm{top}}, -)$, and get TMF (topological modular forms).

There are variations:

- $Tmf = \Gamma(\mathcal{O}^{\mathrm{top}}, \overline{\mathcal{M}_{\mathrm{ell}}})$ is nonperiodic
- $tmf = \tau_{\geq 0} Tmf$ (connective version)

The intuition is that $\mathcal{O}^{\mathrm{top}}$ is a topological analogue of $\omega^{\otimes *}$. Define $MF_* = \Gamma(\omega^{\otimes *}\mathcal{M}_{\mathrm{ell}})$.

Question: what is $\pi_* TMF$?

Claim 17.8. *There is a spectral sequence with E_2-term*

$$E_2^{p,q} = H^q(\mathcal{M}_{ell}, \pi_p^{\dagger}(\mathcal{O}^{top})) \overset{strong}{\Longrightarrow} \pi_{p-q}(TMF).$$

(Here † means sheafification.) Using the sheaf of invariant differentials, you can rewrite this as

$$H^q(\mathcal{M}_{ell}, \omega^{\otimes p}) \implies \pi_{2p-q}(TMF).$$

To construct the spectral sequence, take $\mathcal{M}$ to be a site with coproducts, a sheaf $\mathcal{O} : \mathcal{M}^{op} \to Sp$, and $C = \{U_i \to U\}_{i \in I}$ a covering. Define $U_\bullet : \Delta^{op} \to \mathcal{M}$ taking $[n] \mapsto \coprod_{i_0,\dots,i_n} U_{i_0,\dots,i_n}$. Applying $\mathcal{O}$, $\mathcal{O}(U_\bullet) : \Delta \to Sp$, $X \in Sp^{\Delta}$, get $\mathrm{Tot}\, X =$ the equalizer of $\prod_{[n]} (X^n)^{\Delta(-,[n])} \rightrightarrows \prod_{\varphi:[m]\to[n]} (X^m)^{\Delta(-,[m])}$. Claims:

- We get a tower of spectra
$$\cdots \to \mathrm{Tot}^n(X) \to \mathrm{Tot}^{n-1}(X) \to \dots$$
where $\mathrm{Tot}^n(X) = \mathrm{Tot}(n\text{-coskeleton of } X)$.
- $\mathrm{Tot}(X) = \lim_i \mathrm{Tot}^i(X)$
- You can put a model structure on this model category. If F is fibrant in the Reedy model structure, this tower $\cdots \to \mathrm{Tot}^n(X) \to \dots$ is a tower of fibrations. Now take the homotopy fiber at each stage, take π_*, take the LES in homotopy, wrap these around to get an exact couple and hence a spectral sequence.

Rationally, $\pi_*(TMF) = MF_* = \mathbb{Z}[c_4, c_6, \Delta^{-1}]/(c_4^3 - c_6^2 - 1728\Delta)$ where MF_* is the classical ring of modular forms. There is something similar for $\pi_* Tmf = mf_*$, where mf_* is the ring of meromorphic integral modular forms.

Talk 18: Gross-Hopkins period map (Sean Howe* and Paul VanKoughnett)

Notation:

- $k = \mathbb{F}_{p^n}$
- $W = W(k)$ (Witt vectors)
- $K = \mathrm{Frac}(W)$.
- G_0/k is the Honda formal group law of height n (so $[p]x = x^{p^n}$)
- LT is the Lubin-Tate space over $\mathrm{Spf}\, W$. (Think of this as non-canonically a unit ball.) $LT(R) = \{(G, i) \,:\, G \text{ a formal group}/R, i : G_k \xrightarrow{\cong} G_0\}$ where G_k is the special fiber.
- LT_K is the rigid analytic fiber on LT. Think of it as an open unit disc with "more functions".
- $\mathbb{G}_n = \mathrm{Aut}(G_0) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. This acts on the formal group by changing the trivialization of the special fiber.

Theorem 18.1 (Gross-Hopkins)**.** *There is an étale surjective map $\pi_{GH} : LT_K \to \mathbb{P}_K^{n-1}$.*

Functions on LT_K are elements of $K[[t_1, \dots, t_{n-1}]]$ that converge for all $|t_i| < 1$, $t_i \in \overline{K}$. *I think functions on LT aren't allowed to have denominators – i.e. coefficients in the Witt ring.*

$\mathbb{G}_n$ acts by i, and Gal acts because I base-changed my Honda formal group. The $\mathbb{G}_n$ action is an explicit linear action by matrices on $\mathbb{P}^n$(?).

The important part is that this gives a tool for studying equivariant sheaves on Lubin-Tate space LT (or at least on the generic fiber).

Topologist question:

- Why should $\mathbb{P}^1$ have connected étale covers? This doesn't happen in the archimedean topology; it is a phenomenon in rigid analytic geometry.
- Why do we have to pass to LT_K?

- Why is this a period map?
- What does $\pi_{GH}(X)$ remember about X? (I.e. what are the fibers?)

How does this show up in topology? We want to compute stable homotopy groups of a finite CW complex X. Chromatic homotopy theory breaks it into chunks: isolate some of the periodicity. There's also a way to glue the chunks back together. The chunks are Bousfield localizations $L_{K(n)}X$. This is still a triangulated symmetric monoidal category with unit given by localizing S. If you ask for symmetries of all spectra, you get a group $\mathbb{Z}$ generated by suspension. (By symmetries, I mean automorphisms preserving the triangulation and the symmetric monoidal structure.)

The symmetries we're looking for are all given by tensoring with an invertible object. In general, the symmetries in the $K(n)$-local category are given by tensoring with an invertible object in the $K(n)$-local category. That is, the symmetries are the Picard group Pic (isomorphism classes of invertible objects). From this, you can ask about Pic of $\mathbb{G}_n$-equivariant line bundles on LT.

18.1. Archimedean analogue. Let $E/\mathbb{C}$ be an elliptic curve. I can look at de Rham cohomology $H^1_{dR}(E(\mathbb{C}))$ (this has $\mathbb{C}$-coefficients). Inside, I get a 1-dimensional sub-vector space ω_E spanned by holomorphic differential forms (closed and harmonic);

$$0 \to \omega_E \to H^1_{dR}(E(\mathbb{C})) \to \operatorname{Lie} E^\vee \to 0$$

There is a classical isomorphism $H^1(E(\mathbb{C}), \mathbb{C}) \cong H^1_{dR}(E(\mathbb{C}))$ gotten by integrating topological forms across cycles.

This works in families, like $S/\mathbb{C}$ or E/S.

$$0 \to \omega_{E/S} \to H^1_{dR}(E) \to \operatorname{Lie} E/S \to 0$$

$\omega_{E/S}$ comes from the Hodge filtration and H^1 is a vector bundle on S. I can push forward to get $R^i_{f_*}\mathbb{C} \otimes \mathcal{O}_S$. I have a canonical way to identify nearby fibers in this local system. (The singular cohomology only depends on the topological structure of E, and locally this is a fiber bundle in the topological category.)

H^1_{dR} is a filtered vector bundle with integrable (can locally find a basis of sections, called flat sections) connection (a way to differentiate sections along a vector field). The flat sections are $R^1F_*\mathbb{C}$.

Suppose S is simply connected. Then I can choose a basis of this local system – I can choose a flat basis. This gives an isomorphism; in particular, after applying the comparison theorem (the previous isomorphism with H^1_{dR}), I get $H^1_{dR}(E) \cong \mathcal{O}_S^n$. The vector bundle still has a filtration, and this transfers it to $\mathcal{O}_S^n$. (Actually, $n = 2$.) This map depended on my choice of flat basis; a different choice corresponds to an action by $GL_2(\mathbb{C})$.

S could be something that parametrizes elliptic curves (plus a basis for their integral Betti cohomology), say the upper half plane. Integral Betti cohomology is a flat basis; if I use that to construct this map, I get the standard identification of this with $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. I have an action of $GL_2(\mathbb{C})$ just by changing the trivialization of Betti cohomology. The connection is visibly equivariant w.r.t. that action – you're not changing the curve; you're just changing the identification of the fiber. GL_2 acts also on the universal elliptic curve over this moduli space. Because of this, your action of GL_2 on the moduli space is equivariant on the linear action on $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$.

Suppose $[a_1, a_2] \in \mathbb{P}^1$ are coordinates for the Hodge filtration $W_S \subset H^1_{dR}(E_S)$ w.r.t. a flat basis.

The goal is to construct a filtered vector bundle on LT_K with integrable connection. Let's simply and ask for this to be $\mathrm{Aut}(G_0)$-equivariant. What does "integrable connection" mean in the non-archimedean world? We want to be able to give a basis of flat sections.

Take the map that sent a point to the filtration expressed in this flat basis. Why should this map be equivariant for the linear action on $\mathbb{P}^{n-1}$? That's because the connection is equivariant; the actions preserves flat sections, which gives the linear action on the flat sections.

We can do all of this except the *basis of flat sections* on the formal scheme – that is the only time you have to pass to the rigid analytic generic fiber.

First we talk about how to produce this filtered vector bundle. I'm going to produce it on Lubin-Tate space (without passing to the generic fiber). Start with a formal group G over a complete local Noetherian ring over $\mathbb{Z}_p$. I have to produce something like de Rham cohomology in this setting. I'm going to do something naïve: I have a de Rham complex $R[[x]] \xrightarrow{d} R[[x]]dx$ for my formal group (formal differentiation). Now take H^1 of this. This isn't enough: I haven't used the formal group structure, and also this turns out to be generally quite large. Let's take translation-invariant classes, and call that H^1_{TI}.

Inside of here are translation-invariant differentials, which gives a closed form.

$$0 \to \omega_G \to H^1_{TI} \to \mathrm{Lie}\, G^\vee \to 0$$

H^1_{TI} is n-dimensional, and ω_G is a 1-dimensional filtration inside. This gives me my vector bundle with filtration. H^1_{TI} is the Dieudonné module (to be defined later, for a formal group over a field of characteristic p).

Aside: the Dieudonné module $\mathbb{D}$ for a formal group G_0/k (where k is of characteristic p) is $H^1_{TI}(G)$ for any lift of G_0 to $W(R)$. When looking at the Witt vectors, you're only taking points in a disc of radius $\frac{1}{p}$. You can't take an arbitrary lift to $\mathcal{O}_{\overline{K}}$ because you have no canonical way to identify these.

I want to say this is a vector bundle with connection. If you have a connection on a vector bundle in this setting, you can't extend sections beyond a ball of radius $p^{\frac{1}{p}}$. Fix two points in formal Lubin-Tate space (over an extension now – some $\mathcal{O}_{\overline{K}}$-point). There are disjoint balls around them. The connection will give a basis of flat sections over these disjoint balls. You can cover your space by open balls where you can give flat sections, but I can't give them over the whole thing. The reason is I have to solve some differential equation. Fix a basis of sections over the whole thing. Write down $\nabla = 0$. Because the connection is integrable, I can formally solve this in power series, but I need to divide by $n!$'s, and when doing that you have a $\frac{1}{p}$ in there (which is really big, in non-archimedean topology). So the $\frac{1}{p}$'s are only cancelled out when you take something sufficiently close to your point.

So giving a connection gives a way to identify fibers that are close enough, and the converse also holds. You want to think of it as a connection when you're doing rigid analytic geometry. You can't put the balls together, so you can't get flat sections over the whole thing. If you take flat sections, you need to prove the coefficients aren't growing too fast. You have a map $t \mapsto t^p$ on the coordinate for your ball (to $\mathbb{P}^1$). You get some Frobenius-equivariant structure. If you have a basis of flat sections of $\mathbb{P}^1$, you can pull them back to get rigid analytic sections. Somehow claim that you can extend these, and iterate this to show that you can extend to the whole unit ball.

Why is this map étale? In archimedean settings, things are completely determined by the Hodge structure. This is why the period map is an isomorphism. If I work on one of these small neighborhoods, then the same thing is true here. You get surjectivity from the equivariant action.

The fibers are almost isogeny classes.

Talk 19: Recap (Jared Weinstein)

Let k be a field of characteristic p. Let $\mathcal{O}_K = W(k)$ and $K = \mathcal{O}_K[\frac{1}{p}]$. We have a functor from (some) formal schemes over $\operatorname{Spf} \mathcal{O}_K$ to rigid spaces, sending $X \mapsto X_K$. X_K is a functor from K-affinoid algebras to sets. An important class of K-affinoid algebras is valued fields L/K. For example, $(\operatorname{Spf} A)_K(L) = \operatorname{Hom}_{\mathcal{O}_K\text{-linear}}^{\text{cts}}(A, L)$.

If $X = \operatorname{Spf} \mathcal{O}_K[[u]]$, then $X_K(L) = \operatorname{Hom}_{\text{cts}}$. This is determined by where u goes. Say $u \mapsto u_0$; this has to be a place that makes all integral power series in u_0 converge. Powers of u_0 have to tend towards 0, so the condition on u_0 is $|u_0| < 1$. So, this shows that $X_K(L) = \{u_0 \in L : |u_0| < 1\}$.

Suppose $G = \mathbb{G}_m = \operatorname{Spf} \mathbb{Z}_p[[x]]$. There is a logarithm $\log : G_{\mathbb{Q}_p} \to \mathbb{A}^1_{\mathbb{Q}_p}$. Evaluated on a field L, this is $\mathfrak{m}_{\mathcal{O}_L} = G_{\mathbb{Q}_p}(L) \to \mathbb{Q}^1_{\mathbb{Q}_p}(L) = L$, and $x \in \mathfrak{m}_{\mathcal{O}_L}$ gets sent to $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Since there are denominators, you have to pass to the generic fiber.

The exponential doesn't converge everywhere, but if you multiply something in $\mathbb{A}^1_{\mathbb{Q}_p}$ by a large enough power of p, it lands somewhere where exp converges. $\ker \log$ is the roots of unity, so the fibers of this map are μ_{p^∞}. This is an ∞-to-1 étale surjective map. This shows that the affine line is not simply connected as a rigid analytic space.

Is it surjective on L-points? No, but you can find an étale extension of L, such that that has a preimage. It's surjective on the level of étale sheaves.

Let's move to $\mathcal{M}_{\text{fg}}$. Suppose G_0 is a formal group over $k = \overline{k}$. Declare it's dimension 1, and the height is n (e.g. the Honda formal group). I have LT, the deformation space for G_0. There is a noncanonical isomorphism $\operatorname{Spf} \mathcal{O}_K[[u_1, \ldots, u_{n-1}]]$. As a functor, $LT(\mathcal{O}_L)$ is the set of pairs (G, i) up to isomorphism, where G is a formal group over $\mathcal{O}_L$, and $i : G \otimes_{\mathcal{O}_L} k \xrightarrow{\cong} G_0$.

Let $\Gamma = \operatorname{Aut} G_0$. It acts on the set of pairs (G, i). Therefore it acts on $\operatorname{Spf} \mathcal{O}_K[[u_1, \ldots, u_{n-1}]]$, and hence on the ring. This is rather mysterious.

If L/K is a field extension, $LT_K(L)$ is the set of homomorphisms out of $\mathcal{O}_K[[u_1, \ldots, u_{n-1}]]$ that send the u_i's somewhere where the power series can converge. So $LT_K(L) = \{(u_1, \ldots, u_{n-1}) : u_i \in \mathfrak{m}_L\} = LT(\mathcal{O}_L)$.

What is the Gross-Hopkins map? There is a map, kind of like the logarithm, but that only appears on the generic fiber. It takes $LT_K \stackrel{\pi_{GH}}{\to} \mathbb{P}^{n-1}_K$, so on L, it's $LT(\mathcal{O}_L) \to \mathbb{P}^{n-1}(L)$.

$H^1_{dR}(G) = \operatorname{coker}(d : \mathcal{O}_L[[x]] \to \mathcal{O}_L[[u]]dx)$. This is not going to be surjective: undoing it involves denominators. Inside of this, there is a space $D(G)$ of translation-invariant differentials (or, translation-invariant modulo exact differentials). This $D(G)$ has a special name: it's called the Dieudonné module of G. There are many constructions of the Dieudonné module. When $L = K$, the morphism $F \mapsto D(G)$ from formal groups over $\mathcal{O}_K$ to $\mathcal{O}_K = W(k)$-modules with some extra structure factors through formal groups over k.

The true reason this works is that $\frac{p^n}{n!}$ is in $\mathbb{Z}_{(p)}$.

Now I can define the Gross-Hopkins period map. I have an isomorphism $i : G_0 \stackrel{\cong}{\to} G \otimes k$. That means that there is an isomorphism $D(G_0) \cong D(G)$. Inside $D(G)$ (dimension h = height of G_0) is the line ω consisting of holomorphic 1-forms on G. The Gross-Hopkins period map takes $LT_K \to \mathbb{P}(D(G_0))_K$. This is linear w.r.t. the action of Γ; it acts on LT_K by definition and on the projective space because Dieudonné modules are functorial.

Talk 20: p-divisible groups 1 (Tony Feng* and Alexander Bertoloni Meli*)

As number theorists, we're interested in studying objects like an elliptic curve E over K. This is a 1-dimensional proper group variety. We're also interested in the higher dimensional analogues (abelian varieties A/K). If K is a number field, then there is a construction that gives a linear algebraic object that contains a lot of information about these objects. Take the geometric points of the p^n torsion and take the inverse limit: $T_pA := \varprojlim_n A[p^n](\overline{K})$.

Theorem 20.1. $\operatorname{Hom}_K(A, B) \otimes \mathbb{Z}_p \cong \operatorname{Hom}_{G_K}(T_pA, T_pB)$ *where* G_K *is the absolute Galois group of* K.

Work over characteristic p.

	ordinary elliptic curves	supersingular elliptic curves
# p-torsion points	p	1
formal group	height 1	height 2

Maybe we can patch up these missing p-torsion points with the formal group. Now look at $\varinjlim A[p^n]$ (so not just geometric points – also get nilpotence).

Theorem 20.2 (Serre-Tate)**.** $\mathrm{Def}(G(A)) \cong \mathrm{Def}(A)$

Here Def means lifts to a nilpotent thickening.

Theorem 20.3. *Let R be a ring. A p-divisible group of height n is a system (G_v, i_v) indexed by $v \in \mathbb{N}_{\geq 0}$, where G_v is a finite flat group scheme over R of order p^{nv}, and i_v is a map in an exact sequence $G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1}$. (This exact sequence is part of the data.)*

Example 20.4. Let G be a finite group. Take the R-algebra $R[e_g]$ for $e_g \in G$. This has $|G|$ copies of R, where the idempotent for each copy is e_g and $e_g e_h = 0$. The comultiplication is given by $e_g \mapsto \sum_{\sigma\gamma = g} e_\sigma \otimes e_\gamma$. This is called the constant group scheme $\underline{G}$. Note that there are natural inclusions $\underline{\mathbb{Z}/p^n\mathbb{Z}} \overset{i_n}{\hookrightarrow} \underline{\mathbb{Z}/p^{n+1}\mathbb{Z}}$. Then $(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, i_n)$ is a p-divisible group.

Example 20.5. $\ker([p^n] : G_m \to G_m) = \mathrm{Spec}\, R[x]/x^{p^n} - 1$. Then there is a p-divisible group $\mu_{p^\infty} = (\mu_{p^n}, i_n)$ where i_n is the natural inclusion.

Example 20.6. If $[p^n] : A \to A$ then $(A[p^n], i_n)$ is a p-divisible group of height $2 \dim A$.

Suppose $G = \mathrm{Spec}\, A$ comes with operations $N : A \to A \otimes A$ and $m : A \otimes A \to A$. Look at $\mathrm{Spec}\, \mathrm{Hom}_{R\text{-mod}}(A, R) =: G^*$. Claim that the operation $(-)^*$ is nice: $(G^*)^* = G$.

Take a p-divisible group (G_v, i_v). I claim that $G_{v+1} \xrightarrow{[p^v]} G_{v+1} \xrightarrow{[p]} G_{v+1}$ is exact, and so the second map factors through G_v:

$$\begin{array}{ccccccc} G_{v+1} & \xrightarrow{[p^v]} & G_{v+1} & \xrightarrow{j_v} & G_v & \longrightarrow & 0 \\ & & & \searrow^{[p]} & \downarrow & & \\ & & & & G_{v+1} & & \end{array}$$

and so, taking duals, there is an exact sequence $0 \to G_v^* \xrightarrow{j_v^*} G_{v+1}^* \xrightarrow{[p^v]} G_{v+1}^*$. Checking that the dual of multiplication by p^v is multiplication by p^v, the upshot is that there is a p-divisible group (G_v^*, j_v^*).

Example 20.7.

- $(\mathbb{Q}_p/\mathbb{Z}_p)^* = N_{p^\infty}$
- $G(A)^* \cong G(A^*)$
- Elliptic curves are self-dual

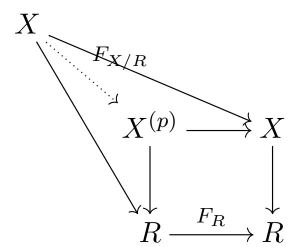
I'm going to say more about the structure theory of p-divisible groups, mostly bootstrapped off the theory of finite type group schemes.

Let R be a ring with $p = 0$, and $X = \mathrm{Spec}\, A$ over R. I want to define a "Frobenius map". The starting point is that $F_A : A \to A$ sending $a \mapsto a^p$ is a ring homomorphism (because

$p = 0$). However, it's not an R-algebra homomorphism. So it won't respect the structure map to R, so we need to modify this. Instead, define some twist of the R-algebra structure on A.

Define $A^{(p)} := A \otimes_{R,F_R} R$ (I'm viewing R as an algebra over itself, not by the identity, but by this Frobenius map). There is a map $F_{A/R} : A^{(p)} \to A$ sending $a \otimes_\lambda \mapsto a^p \lambda$ for $a \in A, \lambda \in R$.

Geometrically, we have a map $X \to R$, and $\operatorname{Spec} A^{(p)}$ is just the base-change



Definition 20.8. $F_{X/R} = \operatorname{Spec} F_{A/R}$

This is sometimes called the "relative Frobenius".

If X is a group scheme G, then $G^{(p)}$ is a group scheme as well. I can take all the diagrams defining the group structure on X, and base-change them.

Example 20.9. Suppose $R = \mathbb{F}_q$ and $A = \mathbb{F}_q[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$. Think of $\operatorname{Spec} A$ as $\{(x_1, \ldots, x_n) : f_i(x) = 0\}$; then $F(x_1, \ldots, x_n) = (x_1^p, \ldots, x_n^p)$ satisfy $f_i^{(p)} = 0$.

Dualize $F_{G^*/R} : G^* \to G^{*(p)} = G^{(p)^*}$ to get a map $V : G^{(p)} \to G$ which is called the Verschiebung.

Fact 20.10. *The composition* $G \xrightarrow{F} G^{(p)} \xrightarrow{V} G$ *is* $[p]$*. Similarly,* $G^{(p)} \xrightarrow{V} G \xrightarrow{F} G^{(p)}$ *is also multiplication by* p*.*

If $G = (G_v, i_v)$ is a p-divisible group, I get $F : G \to G^{(p)}$ and $V : G^{(p)} \to G$ corresponding to the Frobenius and Verschiebung. These also satisfy $FV = p = VF$ just because it's true at each layer.

Let R be a henselian local ring (the slogan for this is "field or complete DVR"). There is an exact sequence $0 \to G^0 \to G \to G^{\text{ét}} \to 0$, where G^0 is connected and $G^{\text{ét}}$ is étale. You have to show that the connected component is a subgroup, which is not obvious and not true in general.

Let $G = \operatorname{Spec} A$ over a ring R. Assume A is a finite flat R-module.

Example 20.11. If $R = \mathbb{F}_p$, consider the scheme μ_p of p^{th} roots of unity. This is $\operatorname{Spec} \mathbb{F}_p[x]/(x^p - 1) = \mathbb{F}_p[x]/(x-1)^p$. This has only one geometric point, but it's a non-reduced point – it has a nontrivial tangent space.

Example 20.12. $\alpha_p := \ker(F : \mathbb{G}_{a,\mathbb{F}_p} \to \mathbb{G}_{a,\mathbb{F}_p}) = \operatorname{Spec} k[x]/x^p$

A connected group scheme is a group scheme that is connected as a scheme. An étale group scheme $G = \operatorname{Spec} A$ is a group scheme such that A is étale over R. (Equivalently, $\Omega^1_{A/R} = 0$.)

Slogan: think of a finite étale map as a covering map.

Example 20.13. The constant group schemes are étale.

Over a field of characteristic 0, all finite flat group schemes are étale. This is the class of group schemes that correspond to your traditional geometric intuition.

Proposition 20.14. *G is étale iff $F : G \to G^{(p)}$ is a monomorphism. This is equivalent to it being an isomorphism.*

That F exists implies we're working over a ring with $p = 0$.

PROOF SKETCH. $d(x^p) = px^{p-1} = 0$, so $dF = 0$ on Ω^1. If it's a monomorphism, then dF is injective, so Ω^1 is already zero (remember this implies étale-ness). The other direction is easy. □

Remember the SES $0 \to G^0 \to G \to G^{\text{ét}} \to 0$. There is an equivalence between étale group schemes and group schemes with a π_1-action, which helps us understand $G^{\text{ét}}$. The connected part G^0 is more mysterious, but if $G = (G_v, i_v)$ is a p-divisible group, then you get $G^0 = (G_v^0)$. Also $G^{\text{ét}} = (G_v^{\text{ét}})$.

We can understand G^0 in terms of formal groups. A formal Lie group is $\Gamma = \operatorname{Spf} R[[x_1, \ldots, x_d]]$ along with a power series $F(x_1, \ldots, x_d, y_1, \ldots, y_d)$ satisfying the axioms of formal group laws. This is different from a group object in the category of formal schemes, which doesn't have to have this form.

Given such a Γ, I can form a p-divisible group $\Gamma(p) = (\Gamma[p^v], i_v)$.

Theorem 20.15 (Tate)**.** *This induces an equivalence*

$$\{\textit{divisible formal Lie groups}\} \longleftrightarrow \{\textit{connected } p\textit{-divisible groups}\}$$

when R is complete, Noetherian, and local, with residue field of characteristic p.

Definition 20.16. Define $\dim G$ to be the dimension of the formal Lie group Γ associated to G^0; this dimension is just the number of power series variables (d).

Remark 20.17. If X is an abelian variety and $G = (X[p^v], i_v)$, then $G^0 = \widehat{X}$.

Example 20.18.

p-div. group	height	dimension
μ_{p^∞}	1	1
$\mathbb{Q}_p/\mathbb{Z}_p$	1	0
$X[p^\infty]$	$2d$	d

Proposition 20.19. *Let G be a p-divisible group of height h and dimension d, such that $\dim G^* = d'$. Then $h = d + d'$.*

Proof. We know that $G \xrightarrow{F} G^{(p)} \xrightarrow{V} G$ is multiplication by p. That gives a SES $0 \to \ker F \to \ker[p] \to \ker V \to 0$. So we will calculate the dimension of each of these pieces. By the definition of height, $|\ker[p]| = p^h$. Recall we have $0 \to G^0 \to G \to G^{\text{ét}} \to 0$. The Frobenius is injective on étale group schemes, so $\ker F = \ker(F|_{G^0})$. But G^0 comes from $\operatorname{Spf} R[[x_1, \dots, x_d]]$ and I know that Frobenius on this just takes $x_i \mapsto x_i^p$. So $\ker F = \operatorname{Spec} R[[x_1, \dots, x_d]]/(x_1^p, \dots, x_d^p)$, which implies $|\ker F| = p^d$. Dually, $|\ker V| = p^{d'}$. □

Talk 21: p-divisible groups and Dieudonné modules (Serin Hong* and Zijian Yao*)

Title: $\{p\text{-divisible groups}\} \xrightarrow{\mathbb{D}} \{\text{“Diedonné modules”}\}$.

References: Tate, “p-divisible groups” and “Classes d'isogénie de variet]'es abeliannes”; Mann, “Theory of formal groups over finite characteristic”; Demazane, “Lectures on p-divisible groups”.

21.1. Motivation. The generic fiber is related to Tate modules. The idea is that the Tate module determines arithmetic properties. If A, B are abelian varieties,

$$\operatorname{Hom}_K(A, B) \cong \operatorname{Hom}_G(T_p(A), T_p(B))$$

where $T_p(-)$ is the Tate module. This was conjectured by Tate and proven by Faltings. It says that if you want to understand abelian varieties, all you have to know is data about the linear algebraic object T_pA. Faltings' proof uses p-divisible groups extensively.

There is also a correspondence between deformations of abelian varieties and deformations of p-divisible groups. The latter is slightly easier; we can use about the functor $\mathbb{D}$, where Dieudonné modules are essentially linear algebra, which is much easier than the geometry of p-divisible groups.

Topologists care about $\mathcal{M}_{\text{fg}_p}$; geometric points correspond to the height of formal groups, and there is a chain of inclusions of neighborhoods. There is a map $\mathcal{M}_{\text{ell}_p} \to \mathcal{M}_{\text{fg}_p}$. $\mathcal{M}_{\text{ell}}$ has a supersingular locus that maps to a point, and the other locus that maps to another point. This is flat but not étale. Serre's theorem says that $\mathcal{M}_{\text{ell}_p} \to \mathcal{M}_{\text{fg}_p}$ factors through another stack $\mathcal{M}_{p\text{-div}}$ with a formally étale map from $\mathcal{M}_{\text{ell}_p}$.

The idea is that Dieudonné modules are like $W(k)$-modules with Frobenius and Verschiebung maps. We will make this precise.

The map $\mathbb{D}$ factors through p-divisible groups over R/p. We will actually be talking about the diagonal map "$\mathbb{D}$":

$$\begin{array}{ccc} \{p\text{-divisible groups}/R = W(k)\} & \xrightarrow{\quad\mathbb{D}\quad} & \{\text{linear algebra}\} \\ & \searrow \quad\quad \nearrow_{\text{“}\mathbb{D}\text{”}} & \\ & \{p\text{-divisible groups}/(R/k)\} & \end{array}$$

21.2. p-divisible groups over a perfect field k of characteristic p. A p-typical group is a system $\{G_v, i_v : G_v \to G_{v+1}\}$ satisfying several conditions. Each of these is a finite flat group scheme of p-power order over the field. If G is a finite flat group scheme over k, we have the connected étale sequence $0 \to G^0 \to G \to G^{\text{ét}} \to 0$.

Lemma 21.1. *If k is perfect, then this sequence splits: $G = G^0 \times G^{\acute{e}t}$.*

By considering the Cartier dual, $G = G^{0,0} \times G^{0,\text{ét}} \times G^{\text{ét},0} \times G^{\text{ét},\text{ét}}$. ($\mu_p$ and $\mathbb{Z}/p\mathbb{Z}$ are Cartier duals.) These factors are $\alpha_p, \mu_p, \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/\ell\mathbb{Z}$, respectively, where $(\ell, p) = 1$. Recall $\mu_p = \operatorname{Spec} k[x]/x^p - 1$. The relative Frobenius is the dotted map in

$$\begin{array}{ccc} \mu_p & & \\ & \ddots\searrow \quad \searrow & \\ & \mu_p^{(p)} \longrightarrow & \mu_p \\ & \downarrow & \downarrow \\ & \mathbb{F}_p \longrightarrow & \mathbb{F}_p \end{array}$$

Here is what F, V do to the various pieces:

	α_p	μ_p	$\mathbb{Z}/p\mathbb{Z}$	$\mathbb{Z}/\ell\mathbb{Z}$
F	0	0	$\cong$	$\cong$
V	N	$\cong$	N	$\cong$

If I restrict attention to p-power group schemes, I can ignore the last column. The upshot is that F and V remember structure of G.

Theorem 21.2. *There is a functor*

$$\mathbb{D} : \left\{ \begin{array}{c} \textit{finite flat groups} \\ \textit{of } p\textit{-power order over } K \end{array} \right\}^{op} \longrightarrow \left\{ \begin{array}{c} W(k)\textit{-modules with action of} \\ F \textit{ and } V \textit{ s.t. (*)} \end{array} \right\}$$

where the conditions (*) are:

- $Fx = \sigma(x)F$

- $Vx = \sigma^{-1}(x)V$
- $FV = VF = p$

where σ is a Frobenius lift. Then $\mathbb{D}$ is an equivalence of categories.

Properties of $\mathbb{D}$:

- It is stable under perfect extensions
- G has order p^h iff the length of $\mathbb{D}(G)$ is h
- $\mu_p \mapsto K$ where F acts by p and V acts by 1
- $\mathbb{Z}/p \mapsto K$, where F acts by 1 and V acts by p
- $\alpha/p \mapsto K$, where F acts by p and V acts by p

This behaves well w.r.t. limits.

Passing to the limit, you get an equivalence of categories

$$\mathbb{D} : \{p\text{-divisibles}/k\} \longrightarrow \{\text{Dieudonné modules free } W(k)\text{-modules of finite rank}\}$$

This sends $\mu_{p^\infty} \mapsto \mathbb{Z}_p$ with F acting by p and V acting as the identity.

For example, an elliptic curve E/k could be supersingular or ordinary. In both cases, $E(p^\infty) \mapsto \mathbb{Z}_p^2$. In the supersingular case, the Frobenius action is given by $\begin{pmatrix} p^{\frac{1}{2}} & 0 \\ 0 & p^{\frac{1}{2}} \end{pmatrix}$ and in the ordinary case, it acts by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

21.3. Classification of Dieudonné modules up to isogeny.

Definition 21.3. A map $f : G_1 \to G_2$ of p-divisible groups is an *isogeny* if it is surjective with finite kernel.

Because there is a notion of isogeny on p-divisible groups, via $\mathbb{D}$ there is also a notion of isogeny on Dieudonné modules.

Theorem 21.4. *Let $f : G_1 \to G_2$. TFAE:*

(1) f is an isogeny

(2) $\mathbb{D}(f) : \mathbb{D}(G_2) \to \mathbb{D}(G_1)$ is injective

(3) After tensoring with $\mathbb{Q}_p$, $\mathbb{D}(f) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p : \mathbb{D}(G_2) \otimes \mathbb{Q}_p \to \mathbb{D}(G_1) \otimes \mathbb{Q}_p$ is an isomorphism. That is, $\mathbb{D}(G_2)[\frac{1}{p}] \cong \mathbb{D}(G_1)[\frac{1}{p}]$.

Definition 21.5. A morphism of Dieudonné modules $g : M \to N$ is an isogeny if $g \otimes \mathbb{Q}_p$ is an isogeny.

Remark 21.6. If $f : G_1 \to G_2$ is an isogeny, $\mathrm{rank}(G_1) = \mathrm{rank}(G_2)$ and $\dim(G_1) = \dim(G_2)$.

Let $A := W(k)[\frac{1}{p}][F]$. Then $\mathbb{D}(G) \otimes \mathbb{Q}_p$ becomes a finitely generated A-modules.

Theorem 21.7 (Manin). *The category of finitely generated A-modules is semisimple. The simple objects are*

$$E_{d/n} = W(k)[\frac{1}{p}][F]/(F^h - p^d) \qquad \text{where } (d,h) = 1.$$

21.4. Newton polygons. Define the slope of E_λ to be λ.

Given a collection of slopes $\lambda_1 \le \cdots \le \lambda_n$, this gives an A-module $E_{\lambda_1} \oplus E_{\lambda_2} \oplus \ldots \oplus E_{\lambda_n}$, which gives the isogeny class of some Dieudonné module.

Define its Newton polygon to be $(0,0) \to (h_1, d_1) \to (h_1 + h_1, d_1 + d_2) \to \cdots \to (h_1 + \cdots + h_n, d_1 + \cdots + d_n)$. So it's a bunch of line segments of slopes $\lambda_1, \ldots, \lambda_n$. If $k \neq \overline{k}$, pass to $\overline{k}$.

Remark 21.8. If G is a p-divisible group such that $\mathbb{D}(G) \otimes \mathbb{Q}_p = E_{d/n}$, then $h = \mathrm{ht}(G)$ and $d = \dim(G)$.

There are maps

$$\{\text{abelian varieties}\} \longrightarrow \{p\text{-divisible groups}\} \to \{\text{Newton polygons}\}$$

sending $X \mapsto X[p^\infty] \mapsto$ its Newton polygon.

Remark 21.9. The Newton polygon has the following properties:

(1) the break points are integer points

(2) the slope is a rational number

Theorem 21.10. *A Newton polygon arises from an abelian variety if:*

(1) $0 \le \lambda \le 1$ *for all slopes* λ

(2) It is symmetric in the following sense: the slopes λ *and* $1 - \lambda$ *appear with the same multiplicity.*

Recall that $\lambda = d/h$, where $d = \dim(G) + \dim(G^\vee)$. The symmetric part is also easy, because the polarization gives a map $X \to X^*$ which induces a map on p-divisible groups $X[p^\infty] \to X^*[p^\infty]$. This sends $E_\lambda \mapsto E_{\lambda^*}$.

Definition 21.11. Let ν_1 and ν_2 be two Newton polygons. Say $\nu_1 \le \nu_2$ if ν_1 lies above ν_2.

By the theorem, there is a unique maximal Newton polygon, and a unique minimal Newton polygon. The maximal one is $(0,0) \to (g,0) \to (2g,g)$, and the minimal one is the single line $(0,0) \to (2g,g)$. The maximal one is called the "ordinary" Newton polygon, and the minimal one is called the "supersingular" Newton polygon.

Definition 21.12. An abelian variety is ordinary [supersingular] if $X[p^\infty]$ has the ordinary [supersingular] Newton polygon.

Example 21.13. Go back to the example of supersingular or ordinary elliptic curves. The coefficients in the matrix defining F determine the Newton polygon slopes.

Remark 21.14. Let $k = \mathbb{F}_{p^n}$, and G a p-divisible group over k. The Newton polygon of G is the Newton polygon of $\mathrm{char}(F)$.

Talk 22: Building cohomology theoreis from p-divisible groups

(Charmaine Sia* and Callan McGill*)

Throughout the talk, I will be using even periodic theories instead of complex oriented.

Suppose we have a flat map $X \to \mathcal{M}_{\mathrm{fg}}$ and choose some topology (étale topology).

Question 22.1. Can we construct a sheaf of E_∞ and/or A_∞ rings such that for any affine flat map $\Gamma : \operatorname{Spec} R \to X \to \mathcal{M}_{\mathrm{fg}}$ such that $\mathcal{O}^{\mathrm{top}}(R) = E(R,\Gamma)$, we have

- $E(R,\Gamma)_0 = R$
- $\operatorname{Spf} E(R,\Gamma)^0(\mathbb{CP}^\infty) = \Gamma$
- $E(R,\Gamma)_{2n} = \omega^{\otimes n}$

For example, $X = LT_\Gamma \to \mathcal{M}_{\mathrm{fg}}$.

If you could upgrade the structure sheaf from $(X, \mathcal{O}_X)$ to $(X, \mathcal{O}^{\mathrm{top}})$, you would get a descent spectral sequence

$$H^s(X, \omega^{\otimes t}) \implies \pi_{t-2s}(\Gamma(\mathcal{O}^{\mathrm{top}})).$$

If you could do this for the identity map $\mathcal{M}_{\mathrm{fg}} \to \mathcal{M}_{\mathrm{fg}}$, you would get the Adams Novikov spectral sequence $H^s(\mathcal{M}_{\mathrm{fg}}, \omega^{\otimes t}) \implies \pi_*(S^0)$ (but this doesn't quite work).

Theorem 22.2 (Lurie's machine)**.** *Let $(A, \mathfrak{m})$ be a local ring, where $A/\mathfrak{m} = k$ is a characteristic p perfect ring. Let X/A be a stack such that*

- *X is locally noetherian,*
- *X is separated,*
- *X is Deligne-Mumford (think of this as a quotient of something by a group),*
- *the map $f : X \to \mathcal{M}_{fg}$ factors as*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathcal{M}_p(n) \\ & {\scriptstyle f}\searrow & \downarrow \\ & & \mathcal{M}_{fg} \end{array}$$

such that $\varphi : X \to \mathcal{M}_p(n)$ to be formally étale (for any point $x \in X$ with p-divisible group $\varphi(x)$, we want $\mathrm{Def}_x \cong \mathrm{Def}_{\varphi(x)}$)

then there is a canonical and functorial way to upgrade $(X, \mathcal{O}_X)$ *to* $(X, \mathcal{O}^{top})$*.*

Example 22.3. If $X = \operatorname{Spf} \mathbb{Z}_p = LT_1$ (height 1 Lubin-Tate space), then the machine associates to this $\widehat{KU_p}$.

$\mathcal{M}_p(n)$ is the moduli stack of height n, dimension 1 p-divisible groups. It has n geometric points $G_{h,n} := \Gamma_h \times (\mathbb{Q}_p/\mathbb{Z}_p)^{n-h}$. We have $\operatorname{Aut}(G_{h,n}) = \mathbb{G}_n \times GL_{n-h}(\mathbb{Z}_p)$.

The deformation space $\operatorname{Def}_{G_{h,n}}$ is noncanonically isomorphic to $\operatorname{Spf} \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]]$. There is a SES $0 \to H_{\text{for}} \to H \to (\mathbb{Q}_p/\mathbb{Z}_p)^{n-h} \to 0$ which corresponds to an element of $\operatorname{Ext}^1(H_{\text{for}}, (\mathbb{Q}_p/\mathbb{Z}_p)^{n-h})$ (this has $(n-1) - (h-1)$ coordinates).

$\mathcal{M}_p(n)$ is not representable: the pullback in this diagram is not affine.

$$\begin{array}{ccc} *//GL_{n-h}(\mathbb{Z}_p) & \longrightarrow & \mathcal{M}_p(n) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \mathcal{M}_{\text{fg}} \end{array}$$

Serre-Tate theory says that deforming abelian varieties is the same as deforming p-divisible groups. Serre-Tate theory + Lurie gives

$$\begin{array}{ccc} & \nearrow & \mathcal{M}_p(2) \\ & & \downarrow \\ \mathcal{M}_{\text{ell}} & \longrightarrow & \mathcal{M}_{\text{fg}} \end{array}$$

The *tmf* story is for heights ≤ 2. We want to construct a version for heights $n \geq 3$. We want to apply Lurie's theorem. One of the inputs you need is moduli problems with 1-dimensional p-divisible groups of height n.

Attempt 1: elliptic curves. The problem is that they have height ≤ 2.

Attempt 2: We want some kind of generalization of elliptic curves, so we try abelian varieties of dimension 2. The problem is that $\dim A[p^\infty] \geq 2$. We want to cut this down to a p-divisible group of dimension 1.

Attempt 3: Try abelian varieties that have a 1-dimensional summand of $A[p^\infty]$. In order to use the Serre-Tate theorem, we need some condition that guarantees that the 1-dimensional summand controls the entire p-divisible group.

There is a solution from number theory: Shimura varieties associated to the unitary group $U(1, n-1)$. A Shimura variety is essentially a higher-dimensional analogue of a moduli curve. We are classifying abelian varieties of dimension n with complex multiplication $\mathcal{O}_F \hookrightarrow \operatorname{End}(A)$ (where F is quadratic imaginary extension of $\mathbb{Q}$), plus additional data in the form of a compatible polarization and a level structure. The upshot of specifying all this information is the following: if p splits in F as $u\overline{u}$, then $A[p^\infty] \cong A[p^\infty]_1 \oplus A[p^\infty]_{n-1}$, where the first summand has dimension 1 and height n (so the formal group has height $\leq n$) and the second summand has dimension $n-1$ and height n.

Denote this moduli by Sh. Then $\dim \mathrm{Sh} = n-1$.

Example 22.4. For $n=1$, $U(1,0)$ corresponds to the Shimura set, the moduli of complex multiplication elliptic curves.

For $n=2$, $U(1,1)$ corresponds to a Shimura curve, a generalization of $\mathcal{M}_{\mathrm{ell}}$.

For $n=3$, $U(1,2)$ corresponds to a Picard surface, which is related to plane quartics.

For general n, $U(1,n-1)$ correspond to Shimura varieties of Harris-Taylor type. These were used to prove local Langlands for GL_n.

By Serre-Tate and Lurie, you get a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞-ring specra over $\mathscr{S}h = \mathrm{Sh}_p^1$ (a formal stack). This sheaf spits out the corresponding R and formal group.

Definition 22.5. *TAF* is defined to be the global sections $\mathcal{O}^{\mathrm{top}}(\mathscr{S}h)$. You can think of this as being some sort of derived p-adic automorphic forms.

You get a descent spectral sequence

$$E_2^{s,t} = H^s(\mathscr{S}h, \omega^{\otimes t}) \implies \pi_{2t-s} TAF$$

where ω is the line bundle of invertible 1-forms over $\mathbb{G}_{\mathrm{for}}$.

The idea is that *TAF* and $\mathscr{S}h$ capture information about chromatic levels $\leq n$.

Fact: *TAF* is $E(n)$-local. Let $\mathrm{Sh}^{[n]}$ be the substack of Sh where $(A[p^\infty])_{\mathrm{for}}$ has height n.

Theorem 22.6 (Harris-Taylor)**.** $\mathrm{Sh}^{[n]}$ *is 0-dimensional.*

Theorem 22.7 (Behrens, Lawson)**.**

$$TAF_{\overline{\mathbb{F}}_p, K(n)} \cong \prod_{x \in \mathrm{Sh}^{[n]}(\overline{\mathbb{F}}_p)} E_n^{h\,\mathrm{Aut}(x)}$$

Example 22.8. When $n=1$, $TAF_{\overline{\mathbb{F}}_p}$ and *TAF* are $E(1)$-local, and (along with everything this machinery returns) p-complete. So it's already $K(1)$-local, and we don't need to localize again.

$$TAF_{\overline{\mathbb{F}}_p} \cong \prod_{x \in \mathrm{Sh}(\overline{\mathbb{F}}_p)} E_1^{h\,\mathrm{Aut}(x)} \cong \prod_{\mathrm{Cl}(F)} KU_p^{h\mathcal{O}_F^\times}$$

Here $\mathrm{Cl}(F)$ is the class group.

Example 22.9. When $n = 2$,

$$\coprod_{\mathrm{Cl}(F)} \mathcal{M}_{ell,\mathbb{Z}_p} \to \mathscr{S}h$$

is a Galois cover of degree $|\mathcal{O}_F^\times/\{\pm 1\}|$.

$$TAF \simeq \prod_{\mathrm{Cl}(F)} TMF_p^{h(\mathcal{O}_F^\times/\{\pm 1\})}$$

F	$\mathrm{Cl}(F)$	$\lvert\mathcal{O}_F^\times/\{\pm 1\}\rvert$
$\mathbb{Q}[i]$	1	2
$\mathbb{Q}[\omega_3]$	1	3
else		1

$$TMF_{\overline{\mathbb{F}}_p, K(2)} \cong \prod_x E_2^{h\,\mathrm{Aut}(x)}$$

where the indexing set is isomorphism classes of supersingular elliptic curves.

Talk 23: The relation between formal and rigid geometry

(Brian Hwang* and Tobias Barthel*)

Let K be a complete and non-archimedean field. Assume that it is discretely valued, e.g. $\mathbb{Q}_p$ (but this isn't necessary).

Let $R = \{z \in K \,:\, |z| \leq 1\} \supset \mathfrak{m} = \{|z| < 1\}$. Let $k = R/\mathfrak{m}$.

Goal:

(1) What are rigid analytic spaces?

(2) What is the way to go from formal schemes over R to rigid spaces over K? This is a process called "passing to the generic fiber of a formal scheme". This is not literally true, but there is an easy enough way of interpreting this.

The primary problem is that the natural analytic topology induced by the non-archimedean valuation results in a space that is totally disconnected: every point is an open and closed set. This causes a lot of problems, e.g. for integration or anything else that requires continuity. The solution is to redefined what "connected" means, i.e. redefine open sets. We use $\mathbb{C}$-analytic geometry as a guide.

Tate first came up with a notion of a wobbly analytic space, and then considered *rigid* analytic spaces, and the name stuck.

Definition 23.1. The Tate algebra is

$$T_n(K) = K\langle Z_1, \ldots, Z_n\rangle = \{\sum_{k\in\mathbb{N}^n} a_{\mathbf{k}} T^{\mathbf{k}} \,:\, \lim_{k_1+\cdots+k_n\to\infty} |a_{\mathbf{k}}| = 0\}$$

$$= \text{“holomorphic” functions on } \{(z_1, \ldots, z_n) \in K^n : |z_i| \le 1\}$$

Let $I \subset T_n(K)$ be an ideal, and $A = T_n/I$. Define $\mathrm{Sp}(A) = (\mathrm{MaxSpec}(A), \text{“}\mathcal{O}\text{”})$, where you need a different Grothendieck topology, and lots of other details to define this properly. This is called a K-affinoid space.

A *rigid analytic space* is a pair $(X, \mathcal{O})$, where X is a (G-top) space, and $\mathcal{O}$ is a sheaf of K-algebras such that there exists an “admissible” open cover $\{X_i\}$ of X such that $(X_i, \mathcal{O}|_X)$ are affinoids. The idea is that you can make a rigid analytic space by gluing affinoids together.

An admissible open cover is something that admits a finite refinement by affinoids. Obviously you want the unit ball to be connected. An example of a non-admissible cover is something that's just the boundary of the unit circle together with the open unit ball. An admissible open cover would be the union of an annulus and a smaller disc. The second one can be refined into a smaller cover by affinoids, but the first one can't.

Facts 23.2.

- If A is K-affinoid and has no idempotents, then $\mathrm{Sp}(A)$ is connected.
- You can “analytify” varieties. There are formal GAGA theorems.

The slogan is that the “generic fibers” of formal schemes are rigid analytic. For example, let $P = R[Z_1, \ldots, Z_n]$ and let $\mathfrak{m} \subset R$ denote the maximal ideal. Taking the formal completion (endowing with the $\mathfrak{m}$-adic topology) is:

$$\widehat{P_{\mathfrak{m}}} = \varprojlim_m P/\mathfrak{m}^m P = \mathcal{T}_n.$$

In more accessible language,

$$\begin{aligned}\mathcal{T}_n &= \{\sum_{\mathbf{k} \in \mathbb{N}^m} a_{\mathbf{k}} \mathbf{Z}^k \ : \ a_{\mathbf{k}} \in R, \ \forall m > 0, \ \#\{\mathbf{k} \ : \ a_{\mathbf{k}} \notin \mathfrak{m}^m R\} < \infty\} \\ &= \text{elements of } T_n \text{ with } R\text{-coefficients.}\end{aligned}$$

The associated formal scheme is:

$$\mathrm{Spf}(\mathcal{T}_n) = (R/\mathfrak{m}[Z_1, \ldots, Z_n], \mathcal{T}_n)$$

which we call the “formal n-ball”.

What do we mean by “take the generic fiber”? Since $K \otimes_R \mathcal{T}_n = T_n$, the generic fiber of $\mathrm{Spf}(\mathcal{T}_n)$ is

$$\mathrm{Sp}(K \otimes_R \mathcal{T}_n) = \mathrm{Sp}(T_n).$$

Theorem 23.3 (Raynaud, ~1970)**.** *There is an equivalence of categories*

$$\{\textit{qcqs rigid spaces}/R\} \longleftrightarrow \{\textit{qc “adm.” formal schemes } R\} \left[\frac{1}{\textit{adm. formal blowups}}\right].$$

My aim is to make the Gross-Hopkins map more accessible and intuitive.

Let $\mathbb{G}_0$ be a height h dimension 1 formal group over k, a perfect field of characteristic p. Let $W = W(k)$, $K = W(k)[\frac{1}{p}]$, $\mathbb{G}$ a deformation of $\mathbb{G}_0$ to W.

You can look at these as sheaves of groups on various things. It makes sense to look at extensions

$$0 \to V \to E \to \mathbb{G} \to 0$$

where V is a vector bundle. This theorem isn't very hard:

Theorem 23.4. *There exists an extension*

$$0 \to V\mathbb{G} \to E\mathbb{G} \to \mathbb{G} \to 0$$

that is universal, i.e. any extension comes with a map from this one. This is called the universal vector extension*.*

Proof. You can show that $V\mathbb{G} \cong \mathrm{Ext}^1(\mathbb{G}, \mathbb{G}_a)^\vee$ (where $\mathbb{G}_a$ is the additive group). $V\mathbb{G}$ is also isomorphic to the bundle of invariant 1-forms $\omega_{\mathbb{G}^\vee}$. This has dimension $h-1$, and $\mathbb{G}$ has dimension 1, so $E\mathbb{G}$ has dimension h. There is a SES

$$0 \to \omega_{\mathbb{G}^\vee} \to \mathrm{Lie}(E\mathbb{G}) \cong M(\mathbb{G}_0) \to \mathrm{Lie}(\mathbb{G}) \to 0$$

There's an extension $(p) \to W(k) \to k$, and you need $\frac{p^n}{n!}$ to live in the kernel (p). If $p = 2$, this is not true, but there's some other way to make this work out(?). All the problems go away when you invert p.

Dualize:

$$0 \to \omega_{\mathbb{G}} \to \mathrm{Lie}(E\mathbb{G})^\vee \cong H^1_{TI}(\mathbb{G}) \to \mathrm{Lie}(\mathbb{G}^\vee) \to 0.$$

Since we're over $W \subset K$, any of these 1-forms has an integral over K. So we can identify $\omega_{\mathbb{G}} = \{f \in XK[[x]] \,:\, df \in W[[X]]\}$. (We've normalized by saying that the constant term is 0.) Define $\partial f(X,Y) = f(X +_{\mathbb{G}} Y) - f(X) - f(Y)$. This is the condition for df to be a translation-invariant 1-form.

$\mathrm{Lie}(\mathbb{G}^\vee)$ consists of $\{\partial f \,:\, \dots\}$ and

$$H^1_{TI}(\mathbb{G}) \cong \{f \in XK[[X]] \,:\, df = 0, \partial f \in W[[X]]\}/\{f \in XW[[X]]\}$$

You get an exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_{\mathbb{G}^\vee} & \longrightarrow & E\mathbb{G} & \longrightarrow & \mathbb{G} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \log} & & \\
0 & \longrightarrow & \omega_{\mathbb{G}^\vee}[\frac{1}{p}] & \longrightarrow & \underbrace{\mathrm{Lie}(E\mathbb{G})}_{=(M\mathbb{G}_0)[\frac{1}{p}]\cong K^h}\ [\frac{1}{p}] & \longrightarrow & \mathrm{Lie}(\mathbb{G})[\frac{1}{p}] & \longrightarrow & 0
\end{array}$$

The image of the log map is a hyperplane, a $(h-1)$-dimensional vector space. Next you need to use rigid analytic geometry.

If you replace $\mathbb{G}$ with something d-dimensional you can do something similar. □

Let me say what this has to do with level structures. There is a moduli space LT which parametrizes deformations of $\mathbb{G}_0$. Recall $LT(R)$ is the set of deformations of $\mathbb{G}_0$ to R. We can cover this by moduli of deformations with level structure: $LT_n(R)$ is the set of deformations of $\mathbb{G}_0$ to R with $\alpha : (\mathbb{Z}/p^n)^h \xrightarrow{\sim} \mathbb{G}(R \otimes K)[p^n]$. If we take the limit of all these things, you get

$LT_\infty(R)$, the set of deformations of $\mathbb{G}_0$ to R together with a map $\mathbb{Q}_p^h \cong (\lim \mathbb{G}[p^n]) \otimes \mathbb{Q}$. (It's not $\mathbb{Q}_p/\mathbb{Z}_p$ because I'm taking a limit, not a colimit.) I also get a map $\mathbb{Z}_p^h \to \lim \mathbb{G}[p^n]$.

We can use this stack to describe the Gross-Hopkins period map more precisely. Essentially it's giving coordinates on this hyperplane $M\mathbb{G}_0[\frac{1}{p}]$. There is a map from the basis $\mathbb{Z}_p^h \to \lim \mathbb{G}[p^n]$, and there's a map from that to $E\mathbb{G}$.

Why should topologists care about perfectoid spaces?

The observation is that moduli spaces "with infinite level structure" at p are perfectoid. Examples include LT_∞ and modular curves $X_{\Gamma(p^\infty)}$ (take the inverse limit over the p^n structures). These all have a natural perfectoid structure.

Here's a 1-minute cartoon of what a perfectoid space is. You have a space X over $K = \mathbb{Q}_p(p^{\frac{1}{p^\infty}})$ and you're relating this to a tilted thing $X^\flat$ over $K^\flat = \mathbb{F}_p((t))[t^{\frac{1}{p^\infty}}]$. Idea: the way you describe $\mathbb{Z}_p$ is like $\mathbb{F}_p$, except you have carrying $\mathbb{Z}_p$ but not in $\mathbb{F}_p$. Then $H^*(X_{\text{ét}}, \mathcal{F})$ is "sort of isomorphic to" $H^*(X^\flat, \mathcal{F}^\flat)$. ("Sort of isomorphic" has something to do with the kernel and cokernel being killed by powers of p(?).)

Consequences: the ∞-level is simple. Also, perfectoid spaces satisfy strong cohomological vanishing statements. Using the cartoon, if X is affinoid perfectoid, then $H^*(X, \mathbb{F}_p) = 0$ for $i > 0$.

Proof: pass to the tilt. In characteristic p, we have a Frobenius, and so we have a sequence $0 \to \mathbb{F}_p \to \mathcal{O}_{X^\flat} \to \mathcal{O}_{X^\flat} \to 0$, and $H^i(X^\flat, \mathcal{O}_{X^\flat}) = 0$ for $i > 0$. Use the LES to get this vanishing almost immediately.

Here's a result that seems to be begging for a homotopical application:

Theorem 23.5 (Scholze). *There exists a $GL_2(\mathbb{Q}_p)$-equivariant map, called the Hodge-Tate period map:*

$$\pi_{HT} : X_{p,\infty} \to \mathbb{P}^1.$$

This is very close to what we've done with the Gross-Hopkins map. We have an analogue of the Hodge filtration on varieties over p-adic fields. If A is an elliptic curve over $C = \widehat{\mathbb{Q}_p}$, we have the sequence

$$0 \to (\operatorname{Lie} A)(1) \to T_p A \otimes_{\mathbb{Z}_p} C \to (\operatorname{Lie} A^*)^* \to 0.$$

I think the (1) *is a twisting.* We have a characterization of the image of this map using the following fact:

Fact 23.6 (Scholze-Weinstein). *$(\operatorname{Lie} A)(1) \to T_p A \otimes_{\mathbb{Z}_p} C$ is $\mathbb{Q}_p$-rational iff A is ordinary. $\mathbb{P}^1(C)$ is in bijection with the p-primary(?) p-divisible groups over $\mathcal{O}_C$ plus a trivialization of the Tate module.*

Corollary 23.7. *The Hodge-Tate period map has restrictions*

$$\pi_{HT}|_{X^{ord}} \to \mathbb{P}^1(\mathbb{Q}_p)$$
$$\pi_{HT}|_{X^{ss}} \cong \bigsqcup Dr_\infty \to \mathbb{P}^1 - \mathbb{P}^1(\mathbb{Q}_p) = \Omega^2$$

The story generalizes to all Hodge-type Shimura varieties.

Talk 24: Further directions (Organizers)

Mike Hill: Question: why are we doing this? Why would I, as a topologist, spend a lot of time focusing on the number theory here?

Given a finite CW complex X, I can associate a sheaf $\mathcal{X}$ over $\mathcal{M}_{\mathrm{fg}}$, which is described by the fact that MU_*X is a MU_*MU-comodule. This gives a functor from finite spectra to sheaves on $\mathcal{M}_{\mathrm{fg}}$. How good an approximation is this functor? I never send a finite spectrum to the zero sheaf. Let's call this condition "injective". But what happens on maps? Is it full or faithful? This is where this starts to break down. But it is also where you start to see the ANSS coming in. The set of maps $[X, Y]$ corresponds to $\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$, but a better thing to do is work in the derived category, so you worry about $\mathrm{Ext}(\mathcal{X}, \mathcal{Y})$. The ANSS is precisely the spectral sequence that takes that algebraic data and tries to recover $[X, Y]$.

Rationally, $\mathcal{M}_{\mathrm{fg}}$ is kind of boring – everything is additive – but rational spectra are also pretty boring – it's the category of graded abelian groups. There is a height stratification of $\mathcal{M}_{\mathrm{fg}}$ that can be pulled back to a stratification (the chromatic filtration) of spectra.

You also try to build sheaves of spectra on $\mathcal{M}_{\mathrm{fg}}$ or replacements like $\mathcal{M}_{\mathrm{ell}}$ to get better approximations of the topology.

The slogan is "keep calm and compute on".

Tyler Lawson: The origin of all this is an invariant $e : \pi_*S \to \mathbb{Q}/\mathbb{Z}$. Elements here detect the 1-line of the ANSS. Bernoulli numbers show up in this which was unclear. Later work was about the f-invariant, which tells information in terms of modular forms which detects the 2-line of the ANSS.

There have been other attempts to "algebraize" things, but this one has stuck because it gives a vision of where the subject should be going.