18.917: The Gamma function and $\mathbb{F}_1$

(lecture notes)

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Disclaimer

These are my notes from Prof. Barwick's course on gamma functions and $\mathbb{F}_1$, given at MIT in spring 2017. I have made them public in the hope that they might be useful to others, but these are not official notes in any way. In particular, mistakes are my fault; if you find any, please report them to:

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$\Psi \subset \Lambda$ generated by ψ_n's; lifting Ψ-algebra structures to Λ-algebra structures; ψ_p should be a lift of Frobenius; examples and non-examples of lifting

Kan-extending $\mathbb{W}$ to get a monad on $\mathrm{Shv}(\mathbb{Z})$; definition of $\mathrm{Shv}(\mathbb{F}_1)$ as algebras over this monad; geometric vs. étale morphisms of topoi; introduction to Beck/ Quillen modules and connection to derivations; using Beck/ Quillen modules to define modules over a Λ-algebra; quasi-coherent sheaves over $X \in \mathrm{Shv}(\mathbb{F}_1)$

Modules over an $\mathbb{F}_1$-algebra A = left modules over the twisted monoid ring $A^\Phi \mathbb{N}^\times$; definition of $\mathbb{F}_{1^n}$; $\mathbb{F}_{1^n}$-points on both affine lines

$\mathbb{W}_S(R)$ for truncation set S; restriction, Frobenius, and Verschiebung on these; use analogy with induction/ restriction of groups to define a pullback diagram of $\mathbb{W}_{\langle n \rangle}$'s; Witt vectors as $\mathrm{Ring} \to \mathrm{Fun}(F_C, \mathrm{Ab})$; effective Burnside category $A^{\mathrm{eff}}(F_C)$; definition of $\mathcal{O}_{\mathbb{C}_N}$, $F_{\mathbb{C}_N}$, $A^{\mathrm{eff}}(F_{\mathbb{C}_N})$

Rough plan involving biconstructible sheaves and Mackey functors; $\mathrm{Mack}(A^{\mathrm{eff}}(F_\mathbb{C}), \mathrm{Ab})$; Day convolution; recollement and Artin gluing; recollement for $\mathrm{Mack}(F_\mathbb{C}, \mathrm{Spectra})$

Cyclotomic objects in spectra; definition of Witt complex; comparison between structure in cyclotomic objects and Witt complexes; de Rham-Witt complex

General goal: get an $\mathbb{F}_1$-version of L-functions for varieties over $\mathbb{F}_q$ which involves getting an $\mathbb{F}_1$-analogue of de Rham cohomology; claim this is related to THH; results by Connes-Consani and Hesselholt relating L-functions to HC and TP; really abstract definition of THH and Dennis trace map using Mot^{nc}_{add}; TR and Tate cohomology in this setting

Descriptions of TR for smooth schemes over perfect field of characteristic k; crystalline cohomology and the de Rham complex in this setting; conjugate spectral sequence for crystalline cohomology; Hodge spectral sequence for computing TP

Regularized determinants; Berthelof's formula for $\zeta(X, s)$ in terms of crystalline cohomology; relating this to TP

Review of Hodge theory; Γ-function associated to a Hodge structure; Hodge structure on $H^w(X(K_v)^{an}, \mathbb{C})$ and resulting L-function $L_v(H^w(X); s)$ (LHS of Connes-Consani); Deligne cohomology; relationship between Deligne cohomology and HC

Summary; HC^{an}; proof of Riemann hypothesis

LECTURE 1: FEBRUARY 14

The topic of this course is the Γ function. You use this as an extra factor when you write down ζ functions and L-functions. The question is why is that the thing to put there? The answer is rather more complicated than I expected. This is inspired by a rather mysterious program by Lars Hesselholt, and by the end of the semester I'll be able to connect this to algebraic topology.

The course begins with the Mellin transform. Let

$$\langle \alpha, \beta \rangle = \{s \in \mathbb{C} : \mathrm{Re}(s) \in [\alpha, \beta]\}$$
$$\rangle \alpha, \beta \langle = \{s \in \mathbb{C} : \mathrm{Re}(s) \in]\alpha, \beta[\}$$

Definition 1.1. Suppose $f : \mathbb{R}_{>0} \to \mathbb{C}$ is integrable and $f(t) = O(t^{-\alpha})$ as $t \to 0$ and $f(t) = O(t^{-\beta})$ as $t \to \infty$. Then the Mellin transform of f is

$$M\{f\}(s) = \int_{\mathbb{R}_{>0}} t^s f(t) d\log t.$$

(This is a Fourier transform w.r.t. the usual multiplicative Haar measure $d\log t$.) This converges on the open strip $\rangle \alpha, \beta \langle$, and it defines a holomorphic function there.

Let's begin with a nonexample: polynomials don't admit Mellin transforms (the integral doesn't converge).

Example 1.2. For $a > 0$ we have

$$M\{\chi_{[a,+\infty[}\}(s) = -\frac{a^s}{s}$$

and the strip of definition is $\rangle - \infty, -a \langle$. Similarly, $M\{\chi_{]0,1[}\}(s) = \frac{1}{s}$.

Example 1.3. If $f(x) = (1+x)^{-1}$ then $M\{f\}(s) = \pi \csc(\pi s)$. This is a fun exercise.

If $f(x) = (1-x)^{-1}$ then $M\{f\}(s) = \pi \cot(\pi s)$. In both of these, the strip of definition is $\langle 0, 1 \rangle$.

Example 1.4. If $f(x) = \tan^{-1}(x)$ then $M\{f\}(s) = -\frac{\pi}{2} s^{-1} \sec(\frac{\pi}{2} s)$.

I want you to see that you get sophisticated functions out of simple functions.

Example 1.5. If $f(x) = \log \left| \frac{1+x}{1-x} \right|$ then $M\{f\}(s) = \pi s^{-1} \tan(\frac{\pi}{2} s)$.

It is clear from the definition that the Mellin transform is linear. In the following, I'm going to leave the strip of definition out, but a good exercise is to put it in.

6

$g(x)$	$M\{g\}(s)$
$f(ax)$ (for $a > 0$)	$a^{-s}M\{f\}(s)$
$x^z f(x)$ (for $z \in \mathbb{C}$)	$M\{f\}(z + s)$
$f(x^a)$ (for $a > 0$)	$a^{-1}M\{f\}(a^{-1}s)$
$f(x^a)$ (for $a < 0$)	$-a^{-1}M\{f\}(a^{-1}s)$
$f'(x)$	$(1 - s)M\{f\}(s - 1)$
$\log x \cdot f(x)$	$\frac{d}{ds}M\{f\}(s)$

Definition 1.6 (Mellin inversion formula). If φ is holomorphic on $\rangle\alpha, \beta\langle$, then

$$M^{-1}\{\varphi\}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \varphi(s)x^{-s}ds \quad \text{for } x > 0, c \in \rangle\alpha, \beta\langle.$$

Theorem 1.7.
$$f = M^{-1}\{M\{f\}\}$$

Question 1.8. Is there a decent analytic continuation of the Mellin transform? If so, what is it?

Proposition 1.9. *Suppose* $f : \mathbb{R}_{>0} \to \mathbb{C}$
- *is rapidly decreasing at* $+\infty$ *(i.e. a Schwartz function)*
- *admits an asymptotic expansion* $f(x) \approx \sum_{n\in\mathbb{N}} a_n x^{\alpha_n}$ *as* $x \to 0$ *where* $\lim_{n\to\infty} \operatorname{Re}(a_n) = +\infty$ *(note this doesn't mean the sum converges – it probably doesn't).*

Then $M\{f\}$ *is meromorphic on* $\mathbb{C}$ *with simple poles at* $s = -\alpha_n$ *with residue* $\operatorname{res}_{-\alpha_n} M\{f\} = a_n$.

    PROOF. We'll prove this when $\alpha_n = n$ to reduce subscripts. Write

$$M\{f\}(s) = \int_{\mathbb{R}_{>0}} t^s f(t)d\log t = \int_0^1 t^s f(t)d\log t + \int_1^\infty t^s f(t)d\log t.$$

By the "rapidly decreasing" hypothesis, the second piece is an entire function, so we focus on the first piece.

$$\int_0^1 \ldots = \int_0^1 t^s\Big(f(t) - \sum_{m=0}^{N-1} a_m t^m\Big)d\log t + \sum_{m=0}^{N-1} \frac{a_m}{m + s}.$$

The first part converges for $\operatorname{Re}(s) > -N$, and the second piece introduces the poles. Since N is arbitrary we conclude. $\qquad\square$

Definition 1.10. If $f(x) = \exp(-x)$. Then Euler's Γ function is

$$\Gamma(s) := M\{f\}(s) = \int_{\mathbb{R}_{>0}} t^s \exp(-t)d\log t.$$

Thanks to the proposition, we see that it admits an analytic continuation to $\mathbb{C}$, which is meromorphic with simple poles at $0, -1, -2, -3, \ldots$, and $\mathrm{res}_{-n}(\Gamma) = \frac{(-1)^n}{n!}$.

Some facts:

- $\Gamma(1) = 1$
- $\Gamma(1 + s) = s\Gamma(s)$
- $\Gamma(1 + n) = n!$ (using the first two facts)

I want a better functional equation. You could use the Weierstrass product formula and use facts about sec. I'm going to do it in a way that gives an excuse to talk about the beta function.

Definition 1.11.
$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)}$$

This isn't the normal way it's defined.

Exercise 1.12. If $u, v \in {]0, \infty[}$, then show that $B(u, v) = M\{f_v\}(u)$ where $f_v = \chi_{]0,1[}(x)(1 - x)^{v-1}$.

Proposition 1.13. *For $s \in \mathbb{C} \setminus \mathbb{Z}$, we have*
$$B(s, 1 - s) = \Gamma(s)\Gamma(1 - s) = \pi \csc(\pi s).$$

PROOF. Reduce to the strip ${\rangle 0, 1 \langle}$ and see that if $f(x) = \frac{1}{1+x}$, we have $M\{f\}(s) = B(s, 1 - s)$. $\square$

Example 1.14.
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$
$$\Gamma(m + \tfrac{1}{2}) = \frac{(2m - 1)!!}{2^m}\sqrt{\pi}$$

These are the values that appear when you compute the volume of the n-ball.

I'm going to define a partial Γ function:
$$\Gamma_m(s) = \frac{m^s m!}{s(1 + s)(2 + s)\ldots(m + s)} = \frac{m^s}{s(1 + \frac{s}{1})(1 + \frac{s}{2})\ldots(1 + \frac{s}{m})}.$$

Theorem 1.15 (Euler product).
$$\Gamma(s) = \lim_{m \to \infty} \Gamma_m(s).$$

PROOF. Note that both sides satisfy the functional equation. So it's enough to check for $\mathrm{Re}(s) > 0$. So now write

$$\exp(-t) = \lim_{m \to \infty} (1 - \tfrac{t}{m})^m$$

Since stuff converges absolutely here,

$$\Gamma(s) = \lim_{m \to \infty} \int_0^m (1 - \tfrac{t}{m})^m t^s d\log t$$

$$= \lim_{m \to \infty} \frac{m!}{s(s+1)\dots(s+m-1)m^m} \int_0^m t^{s+m-1} dt = \lim_{m \to \infty} \Gamma_m(s)$$

(by integration by parts). $\qquad\qquad\square$

We can make this look nicer:

$$\Gamma(s) = \frac{1}{s} \prod_{n \in \mathbb{N}} \frac{(1 + \tfrac{1}{n})^s}{1 + \tfrac{s}{n}}.$$

Notation 1.16. For $m \in \mathbb{N}_0$, let

$$h_m = \sum_{k=1}^m \frac{1}{k}.$$

Definition 1.17. There is a constant γ such that

$$h_N = \gamma + \log N + O(\tfrac{1}{N})$$

for all N. This is the Euler-Mascheroni constant.

Theorem 1.18 (Weierstraß).

$$\Gamma(s) = \frac{\exp(-\gamma s)}{s} \prod_{n \in \mathbb{N}} \frac{\exp\left(\tfrac{s}{n}\right)}{1 + \tfrac{s}{n}}$$

PROOF.

$$m^s = \exp(s \log m)$$

$$= \exp(s \log m - s h_m) \cdot \exp(s h_m)$$

$$\Gamma_m(s) = \tfrac{1}{s} \exp(s \log m - s h_m) \prod_{n=1}^m \frac{\exp\left(\tfrac{s}{n}\right)}{1 + \tfrac{s}{n}}.$$

Now let $m \to \infty$. $\qquad\qquad\square$

Exercise 1.19 (Gauss).

$$\prod_{k=0}^{m-1} \Gamma(s + \tfrac{k}{m}) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2} - ms} \Gamma(ms).$$

(Just use the Weierstraß formula.)

We're going to use Tate's thesis to write down a new form of the Gamma function. That will give us functional equations for L-functions. Every time, Γ-factors will turn up. There will be some choice in this; we want to make this more canonical. We'll look at regularized determinants and regularized products to get one (err... three) choice(s) that is more canonical. By learning more about $\mathbb{F}_1$, you can use it to find a relevant cohomology that forces a particular normalization for the Γ function, which is not the one that usually appears in the literature.

Along the way, I'll prove the Ramanujan master theorem.

Definition 1.20.
$$\Psi(s) = \frac{d}{ds}\log\Gamma(s)$$

Proposition 1.21.
$$\Psi(s) = \lim_{m\to\infty}\left(\log m - \sum_{k=0}^{m}\frac{1}{k+s}\right)$$
$$= -\gamma + \sum_{n\in\mathbb{N}}\frac{s-1}{n(s-1+n)}$$
$$\Psi(1+m) = h_m - \gamma$$

There are two functional equations (immediately obtained from the functional equations for the gamma function):
$$\Psi(1+s) = \tfrac{1}{s} + \Psi(s)$$
$$\Psi(s) - \Psi(1-s) = -\pi\cot(\pi s).$$

Exercise 1.22. Let $f(t) = \frac{t}{e^t-1}$. Compute $M\{f\}(s)$ in terms of $\Gamma(s)$.

LECTURE 2: FEBRUARY 16

Last time, we analytically continued the Γ-function.

Here is an extremely cheap way to compute the volume of the n-ball $v_n = Vol(B^n)$.
$$\pi^{n/2} = \left(\int_{\mathbb{R}}\exp(-x^2)dx\right)^n$$
$$= \int_{\mathbb{R}^n}\exp(-x_1^2 - x_2^2 - \cdots - x_n^2)dx_1\ldots dx_n$$
Now use the fact that $\int_X f d\mu = \int_0^{+\infty}\mu\{x\in X : f(x) > t\}dt$ (here μ means "measure").
$$= \int_0^1 v_n(-\log t)^{n/2}dt$$
$$= v_n\int_0^\infty s^{n/2}\exp(-s)ds$$
$$= v_n\Gamma(1+n/2)$$

Theorem 2.1 (Ramanujan's Master Theorem)**.** *For $f(t) = \sum_{m \in \mathbb{N}_0}(-t)^m\varphi(t)$ then $M\{f\}(s) = \pi\csc(\pi s)\varphi(-s)$. If $\lambda(s) = \varphi(s)\Gamma(1+s)$ then $\Gamma(s)\lambda(-s) = M\{g\}(s)$ where $g(t) = \sum_{m \in \mathbb{N}_0}\frac{(-t)^m}{m!}\lambda(m)$.*

Ramanujan's original proof was purely formal – he didn't talk about the convergence issues. Hardy wrote the "proof for mortals" that addresses convergence.

Definition 2.2 (Hardy class)**.** If $A, P, \delta \in \mathbb{R}$ are constants such that $A < \pi$ and $\delta \in {]}0, 1]$ then define $H(A, P, \delta)$ to be the set of holomorphic functions φ on $\rangle - \delta, \infty\langle$ such that $\varphi(s) = O(\exp(-P\operatorname{Re}(s) + A|\operatorname{Im}(s)|))$.

Theorem 2.3. *If $\varphi \in H(A, P, \delta)$ then $f(t) = \sum_{m \in \mathbb{N}}(-t)^m\varphi(m)$ converges for $t \in {]}0, \exp(P)]$, admits an analytic extension to $\rangle 0, \infty\langle$, and for any $s \in \rangle 0, \delta\langle$,*

$$M\{f\}(s) = \pi\csc(\pi s)\varphi(-s).$$

    PROOF. Use the Cauchy residue theorem

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\pi\csc(\pi s)\varphi(-s)t^{-s}ds$$

for any $c \in {]}0, \delta[$. Now we're done by Mellin Inversion. $\square$

Example 2.4. Take

$$f(t) = \frac{1}{\exp(t) - 1} = \sum_{m \in \mathbb{N}_0}\frac{B_m}{m!}t^{m-1}$$

where B_m is a Bernoulli number. This is a nice function with rapid decay at ∞, the power series converges in a disc around the origin, so the Mellin transform is meromorphic on $\mathbb{C}$ with simple poles at $s = 1 - m$ for $m \in \mathbb{N}_0$. The residues are

$$\operatorname{res}_{1-m}M\{f\} = \frac{B_m}{m!}.$$

If $t > 0$ then $\exp(t) > 1$, so we can also write

$$f(t) = \sum_{m \in \mathbb{N}}\exp(-mt).$$

Using the rules from last time,

$$M\{f\}(s) = \sum_{m \in \mathbb{N}}\Gamma(s)m^{-s} = \Gamma(s)\zeta(s)$$

where $\zeta(s) = \sum_{m \in \mathbb{N}}m^{-s}$. So ζ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. For $m \in \mathbb{N}_0$,

$$\zeta(-m) = (-1)^m\frac{B_{m+1}}{m+1}.$$

Definition 2.5 (Hurwitz zeta function)**.**

$$\zeta(s, q) = \sum_{m \in \mathbb{N}}(m + q)^{-s}$$

If

$$f(t) = \frac{\exp(-qt)}{1 - \exp(-t)} - \frac{1}{t}$$

(the $\frac{1}{t}$ is just for normalization), then $M\{f\}(s) = \Gamma(s)\zeta(s, q)$. We have the power series expansion

$$\frac{t \exp(qt)}{\exp(t) - 1} = \sum_{m \in \mathbb{N}_0} \frac{B_m(q)}{m!} t^m.$$

So $\zeta(1 - m, q) = -\frac{B_m(q)}{m}$. (Recall $B_m(1) = B_m$ but $B_m(0) = \pm B_m$.)

This is a special case of a Dirichlet series.

Definition 2.6. A *Dirichlet series* is a series

$$L(s) = \sum_{m \in \mathbb{N}} a_m m^{-s}$$

for some coefficients a_m. A *generalized Dirichlet series* is

$$L(s) = \sum_{m \in \mathbb{N}} a_m \lambda_m^{-s}$$

where $\lambda_1 < \lambda_2 < \ldots$ and $\lambda_j \to \infty$ faster than j^r for some $r > 0$.

Theorem 2.7. *A generalized Dirichlet series $L(s)$ admits an abscissa of convergence $\sigma_c \in \mathbb{R} \cup \{\pm\infty\}$ such that if $s \in \rangle\sigma_c, \infty\langle$ then $L(s)$ converges and if $s \in \rangle - \infty, \sigma_c\langle$ then $L(s)$ does not converge.*

It's really hard to figure out what happens on the line $z = \sigma_c$.

Example 2.8. If $L(s)$ is a generalized Dirichlet series, then

$$f(t) = \sum_{m \in \mathbb{N}} a_m \exp(\lambda_m t) \quad t > 0.$$

If f admits an asymptotic expansion

$$f(t) \approx \sum b_m t^m$$

as $t \to 0$ then we can take the Mellin transform

$$M\{f\}(s) = \Gamma(s)L(s).$$

So $L(s)$ admits a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$, and $L(-m) = (-1)^m m! b_m$.

Definition 2.9. A *Dirichlet character of modulus k* is a homomorphism $\chi^* : (\mathbb{Z}/k)^\times \to \mathbb{C}^\times$. Extending this to zero gets a character $\chi : \mathbb{Z}/k \to \mathbb{C}$ with

$$\mathbb{Z}/k \xrightarrow{\chi} \mathbb{C}$$
$$\uparrow \qquad \nearrow$$
$$\mathbb{Z}$$

But I'll probably just call both of these χ.

Definition 2.10 (Dirichlet L-series)**.** Define

$$L(s, \chi) = \sum_{m \in \mathbb{N}} \chi(m) m^{-s}.$$

It is an L-function: you can come up with an asymptotic expansion and use the recipe in the example to produce a meromorphic continuation.

Write $f(t, \chi) = \sum_{m \in \mathbb{N}} \chi(m) e^{-mt}$. Then $M\{f(-, \chi)\}(s) = \Gamma(s) L(s, \chi)$. We have

$$L(1 - m, \chi) = -\frac{B_{m\chi}}{m}$$

where $B_{m\chi}$ is defined by

$$\sum_{m \in \mathbb{N}} \chi(m) \frac{t e^{mt}}{e^{mt} - 1} = \sum_{m \in \mathbb{N}_0} \frac{B_{m\chi}}{m!} t^m.$$

In general, these things are fairly hard to compute.

Subexample 2.11. χ_4 has modulus 4 and is defined by $+1, 0, -1, 0, \ldots$. Genuinely unhelpful form:

$$L(s, \chi_4) = \sum_{m \in \mathbb{N}} (-1)^{m+1} (2m - 1)^{-s}.$$

Think of the previous expression for $f(t, \chi)$ as a geometric series, and get in this case $f(s, \chi_4) = \frac{1}{2} \operatorname{sech} t = \text{const} \cdot \frac{-e^{-t}}{1 - e^{-2t}}$. This is helpful because we know things about the power series expansion:

$$\frac{-e^{-t}}{1 - e^{-2t}} = \frac{1}{2} \sum_{m \in \mathbb{N}_0} \frac{E_m}{m!} t^m$$

where E_m are Euler numbers.

Now we know

$$L(1 - m, \chi_4) = \frac{E_m}{2}.$$

(Since we have the analytic continuation we can call $L(s, \chi)$ an L-function as opposed to just an L-series.) $B_{m\chi}$ is related to torsion in algebraic K-theory of the integers (although part of that story is conjectural).

Exercise 2.12. Write $L(s) = \sum_{m \in \mathbb{N}} a_m m^{-s}$ and $A(t) = \sum_{m \leq t} a_m$. Also write $f(t) = A(\frac{1}{t})$. Show that if $s \in \langle \max\{0, \sigma_c\}, +\infty \rangle$, then $M\{f\}(s) = \frac{L(s)}{s}$.

LECTURE 3: FEBRUARY 23

I want to talk about a proof that Riemann gave for the functional equation for the ζ function, and generalize it to Dirichlet characters.

Definition 3.1. Jacobi's theta function is

$$\theta(z) := \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z}.$$

As written, this converges on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$; it defines a holomorphic function there. This is not the right kind of function to do a Mellin transform to, but we can fix it up and renormalize so we get the kind of convergence at ∞ that we need. Set

$$f(x) = \tfrac{1}{2}(\theta(ix) - 1)$$

and define

$$Z(s) = M\{f\} \left(\tfrac{s}{2}\right).$$

You can begin by trying to understand one summand

$$g_n(x) = e^{-\pi n^2 x}.$$

Then

$$M\{g_n\}(s) = \pi^{-s}\Gamma(s)n^{-2s}.$$

But $f(x) = \sum_{n \in \mathbb{N}} g_n$, and so

Proposition 3.2. $Z(s) = \pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s).$

Key point 3.3. $\varphi(t) = \exp(-\pi t^2)$ is its own Fourier transform.

Theorem 3.4 (Poisson summation)**.** *For any Schwartz function ψ, we have*

$$\sum_{m \in \mathbb{Z}} \psi(m) = \sum_{m \in \mathbb{Z}} \widehat{\psi}(m).$$

If $x > 0$, write

$$\gamma_x(t) := \varphi(\sqrt{x}t) = \exp(-\pi x t^2)$$

so

$$\gamma_x(n) = g_n(x)$$

$$\widehat{\gamma}_x(y) = \frac{1}{\sqrt{x}}\gamma_x\left(\tfrac{y}{x}\right)$$

(using the Key Point).

Corollary 3.5.

$$\theta\left(-\tfrac{1}{z}\right) = \left(\tfrac{z}{i}\right)^{1/2}\theta(z)$$

For f, we get

$$f(x) = x^{1/2}f\left(\tfrac{1}{x}\right) + \tfrac{1}{2}x^{-1/2} - \tfrac{1}{2}$$

and

$$M\{f\}(s) = M\{f\}(\tfrac{1}{2} - s).$$

(What's with that $(-)^{1/2}$? I literally mean $\left(\tfrac{z}{i}\right)^{1/2} = e^{1/2\log\left(\tfrac{z}{i}\right)}$ using the principal branch of the logarithm – remember this is all going on in the upper half plane.) The last two terms in $f(x)$ are just there to make it converge, but the integral behaves according to the first term.

14

Corollary 3.6 (Functional equation)**.**
$$Z(s) = Z(1 - s)$$

Let χ be a nontrivial primitive Dirichlet character of modulus k. (It's the modulus that's primitive, not the character.)

Definition 3.7. The *exponent* of χ is $\varepsilon = \varepsilon(\chi) \in \{0, 1\}$ such that
$$\chi(-1) = (-1)^\varepsilon \chi(1).$$

Define
$$\theta(\chi, z) = \sum_{m \in \mathbb{Z}} \chi(m) m^\varepsilon \exp(i\pi \tfrac{m^2}{k} z).$$

As before, I want to contemplate
$$f(\chi, x) = \tfrac{1}{2} \theta(\chi, ix).$$

If you want a general formula that works for everything, you might want $\frac{1}{2}(\theta(\chi, ix) - \chi(0))$, but $\chi(0) = 0$ since the character is nontrivial.

Exercise 3.8. The Mellin transform is
$$M\{f(\chi, -)\}\left(\tfrac{s+\varepsilon}{2}\right) = 2 \left(\tfrac{k}{\pi}\right)^{\frac{s+\varepsilon}{2}} L(\chi, s)\Gamma\left(\tfrac{s+\varepsilon}{2}\right).$$

Now we want to produce a functional equation. We want to express $\theta(\chi, -\frac{1}{z})$ in terms of something involving θ-functions. Actually, I'll show
$$\theta(\chi, -\tfrac{1}{z}) = \text{coefficient} \cdot \theta(\overline{\chi}, z).$$

Definition 3.9 (Gauss sum)**.**
$$\tau(\chi) = \sum_{m=0}^{k-1} \chi(m) \exp(i2\pi \tfrac{m}{k})$$

$$|\tau(\chi)| = \sum_{m=0}^{k-1} \sum_{n=0}^{k-1} \chi(n) \exp(i2\pi mn/k) \exp(-i2\pi m/k) = k$$
$$= \sum_{n=0}^{k-1} \chi(n) \sum_{m=0}^{k-1} \exp(i2\pi m(n-1)/k)$$

The contributions from $n \neq 0$ cancel out because you're adding up roots of unity so this is just the contribution from $n = 0$. I claim
$$\theta(\chi, -\tfrac{1}{z}) = \frac{\tau(\chi)}{i^\varepsilon \sqrt{k}} \left(\tfrac{z}{i}\right)^{\varepsilon + 1/2} \theta(\overline{\chi}, z).$$

Corollary 3.10. *If*

$$\Lambda(\chi, s) = \left(\frac{k}{\pi}\right)^{s/2} L(\chi, s)\Gamma\left(\tfrac{s+\varepsilon}{2}\right)$$

then

$$\Lambda(\chi, s) = \frac{\tau(\chi)}{i^\varepsilon \sqrt{k}}\Lambda(\overline{\chi}, 1-s).$$

I won't prove this because I'll give a massive generalization when we do Tate's thesis.

This is another way to prove that this admits an analytic continuation.

I want to begin the process of generalizing this picture.

Let (S, σ) be a finite $\mathbb{Z}/2$-set (where σ stands for the nontrivial automorphism). We can create some nice vector spaces

$$\mathbb{C}^S = \mathrm{Map}(S, \mathbb{C}).$$

This has a $\mathbb{Z}/2$-action: send $z \mapsto \overline{z \circ \sigma}$. Define

$$\mathbb{R}^S := (\mathbb{C}^S)^{\mathbb{Z}/2} \cong \mathrm{Map}(S, \mathbb{R})$$

(i.e. $\mathbb{Z}/2$-fixed points of $\mathbb{C}^S$). In the case where $S = \mathrm{Hom}(K, \mathbb{C})$, $\mathbb{R}^S \cong K \otimes_\mathbb{Q} \mathbb{R}$. This is the C_2-fixed points of the isomorphism $\mathbb{C}^S \cong K \otimes_\mathbb{Q} \mathbb{C}$.

If K is a number field, a typical setting is

$$S = \mathrm{Hom}(K, \mathbb{C}).$$

Then $\mathbb{C}^S \cong K \otimes_\mathbb{Q} \mathbb{C}$ and $\mathbb{R}^S \cong K \otimes_\mathbb{Q} \mathbb{R}$. I'm going to try to define an analogue of the θ-function. We need an analogue of $\mathbb{Z}$. It is convenient to consider not just the square lattice in $\mathbb{R}^S$, but any lattice.

If $W \subset \mathbb{R}^S$ is a $\mathbb{Z}$-structure (free abelian group in $\mathbb{R}^S$ such that $W \otimes_\mathbb{Z} \mathbb{R} = \mathbb{R}^S$ – a.k.a. complete lattice), then we can form

$$\theta_W(z) = \sum_{w \in W} \exp(i\pi(wz, w))$$

where (wz, w) is the Hermitian form on $\mathbb{C}^S$ defined by

$$(z, w) = \sum_{s \in S} z_s \cdot \overline{w_s} = \mathrm{tr}(z\overline{w}).$$

(Warning: some textbooks use $\overline{w}$ to mean the complex conjugation composed with σ, not just the complex conjugation.) This is invariant under the $\mathbb{Z}/2$-action, so gives rise to an inner product on $\mathbb{R}^S$.

Fact 3.11. $\theta_W(z)$ *converges absolutely and uniformly on a compactification of*

$$\mathbb{H}^S = \{z \in \mathbb{C}^S : z = z \circ \sigma \text{ and } \tfrac{1}{2i}(z - \overline{z \circ \sigma}) > 0\}.$$

(That second condition is supposed to be the analogue of "$\mathrm{Im}(z) > 0$".)

All of Fourier analysis on LCA groups goes here. I'm really thinking of Schwartz functions $\varphi : \mathbb{R}^S \to \mathbb{C}$. Here's the key example:

$$\varphi(x) = \exp(-\pi(x,x))$$

is its own Fourier transform.

Theorem 3.12. *If $W \subset \mathbb{R}^S$ is a $\mathbb{Z}$-structure, and*

$$W^\vee = \{v \in \mathbb{R}^S : (w,v) \in \mathbb{Z} \; \forall w \in W\}$$

(the dual lattice) then for any Schwartz function $\mathbb{R}^S \to \mathbb{C}$

$$\sum_{w \in W} \psi(w) = \frac{1}{\operatorname{covol} W} \sum_{v \in W^\vee} \widehat{\psi}(w).$$

If $W = \mathbb{Z}\{w_1, \ldots, w_n\}$ then $\operatorname{covol}(W) = |\det(w_1, \ldots, w_n)|$ (it's the volume of the fundamental domain). This is the analogue of the Poisson summation formula.

Corollary 3.13.

$$\theta_W\left(-\tfrac{1}{z}\right) = \frac{\sqrt{N(z/i)}}{\operatorname{covol}(W)} \theta_{W^\vee}(z)$$

where N is the norm:

$$N(w) = \prod_{s \in S} w(s).$$

(Again, this square root involves the principal branch of the logarithm.) We're going to do this next time; the story is the same in each case. This will allow us to access situations for bigger number fields than just $\mathbb{Q}$.

LECTURE 4: FEBRUARY 28

We had a C_2-set (S, σ) (i.e. σ is the nontrivial automorphism). You're supposed to imagine $S = \operatorname{Hom}(K, \mathbb{C})$ with σ as the Galois action. We built $\mathbb{C}^S$ which is what you think it is; there's a C_2 action on this which uses both σ and complex conjugation: $z \mapsto \overline{z \circ \sigma}$. We discovered $\mathbb{R}^S$ (which in our special case coincides with the Minkowski space) is the C_2-fixed points for this action.

Last time we were looking at the analogue of the upper half plane. You think I would define $\mathbb{H}^S = \{z \in \mathbb{C}^S : \operatorname{Im}(z) > 0\}$. But instead, we also added this extra condition that $z = z \circ \sigma$. Why? If you think of the example, it's the difference between the set of embeddings of K into $\mathbb{C}$ and the set of places.

Definition 4.1. For a $\mathbb{Z}$-structure $W \subset \mathbb{R}^S$,

$$\theta_W(z) = \sum_{w \in W} \exp(i\pi(wz, w))$$

where $(-, -)$ is the Hermitian inner product given by $(x, y) = \sum_{s \in S} x(s) \cdot \overline{y(s)}$. (Here $z \in \mathbb{H}^S$ and wz is the product formed pointwise (so $\mathbb{C}^S$ is an algebra).)

This converges absolutely and uniformly on the compactification of $\mathbb{H}^S$.

Definition 4.2. The dual of the lattice is
$$W^\vee = \{v \in \mathbb{R}^S \; : \; \forall w \in W, (w, v) \in \mathbb{Z}\}.$$

This is important for the Poisson summation formula: if φ is a Schwartz function,
$$\sum_{w \in W} \varphi(w) = \frac{1}{\mathrm{covol}(W)} \sum_{v \in W^\vee} \widehat{\varphi}(v).$$

Corollary 4.3 (Functional equation for generalized θ-function).
$$\theta_W(-\tfrac{1}{z}) = \frac{\sqrt{N\left(\frac{z}{i}\right)}}{\mathrm{covol}(W)} \theta_{W^\vee}(z)$$
where $N(z) = \prod_{s \in S} z_s$.

The idea is to sum up (integrate) a bunch of θ-functions, take the Mellin transform, and then you get a completed form of the ζ-function; the functional equation for the ζ-function comes from this functional equation for the θ-function.

We also need Gamma-functions, so you need to take the Mellin transform, i.e. you need to integrate along a ray. We need to define
$$(\mathbb{R}^S)_{>0} = \{x \in \mathbb{R}^S \; : \; x = x \circ \sigma \text{ and } x > 0\}.$$
The idea is that this is the same condition for $\mathbb{H}^S$, where you're ignoring the distinction you normally have between conjugate embeddings. This is equal to the product of a bunch of $\mathbb{R}_{>0}$'s, but I want to think of it as the specific product indexed as $\prod_{p \in S/C_2} \mathbb{R}_{>0}$. Pick the isomorphism $(\mathbb{R}^S)_{>0} \to \prod_{p \in S/C_2} \mathbb{R}_{>0}$ sending $x \mapsto (\prod_{s \in p} x_s)_{p \in S/C_2}$. This specifies a normalization of the Haar measure: $d \log x$ gets sent to $\pi^*(\prod d \log)$.

Note that $\mathbb{R}^S$ isn't literally the real numbers embedded in $\mathbb{C}$, but the condition $x = x \circ \sigma$ forces x to be real so $x > 0$ makes sense as a condition. (In textbooks this is sometimes called **R** and **C** instead of $\mathbb{R}^S$ and $\mathbb{C}^S$.)

Definition 4.4. For $\underline{z} \in \mathbb{C}^S$,
$$\Gamma_S(\underline{z}) = \int_{(\mathbb{R}^S)_{>0}} N(\exp(-x)x^{\underline{z}}) d \log x.$$
Here $x^{\underline{z}}$ is done pointwise.

Here's an easy proposition:

Proposition 4.5.
$$\Gamma_S(\underline{z}) = \prod_{p \in S/C_2} \Gamma_p(\underline{z}_p) \; where \; \Gamma_p(\underline{z}_p) = \begin{cases} \Gamma(\underline{z}_p) & p \; real \\ 2^{1-\mathrm{tr}(z_p)}\Gamma(\mathrm{tr}(z_p)) & p \; complex. \end{cases}$$

Possible wrong factor of 2 somewhere? We're breaking up S as $S = \bigsqcup_{p \in S/C_2} p$, where p is a coset that has either one or two elements. If $\#p = 1$, say p is real. If $\#p = 2$, say p is complex. (You should think of this as the set of places of the number field.)

This is supposed to be an origin story for the Gamma-factors at infinity.

Definition 4.6. Let $\underline{z} \in \mathbb{C}^S$ (call this a generalized complex number). Then define

$$L_S(\underline{z}) = N(\pi^{-z/2})\Gamma_S\left(\tfrac{z}{2}\right) = \prod_{p \in S/C_2} L_p(\underline{z}_p)$$

where

$$L_p(\underline{z}_p) = \begin{cases} \pi^{-z_p/2}\Gamma\left(\frac{z_p}{2}\right) & p \text{ real} \\ 2(2\pi)^{-\operatorname{tr}(\underline{z}_p)/2}\Gamma\left(\frac{\operatorname{tr}(\underline{z}_p)}{2)}\right) & p \text{ complex.} \end{cases}$$

Let $s \in \mathbb{C}$. Break up S as $S = r_1(C_2/C_2) \sqcup r_2(C_2/e)$. Let $n = \#S = r_1 + 2r_2$. For $z \in \mathbb{C}$, $\Gamma_S(z) := \Gamma_S(\operatorname{const}_z)$. Now

$$\Gamma_S(s) = 2^{(1-2s)r_2}\Gamma(s)^{r_1}\Gamma(2s)^{r_2}$$

$$L_S(s) = \pi^{-ns/2}\Gamma_S(s/2)$$

$$L_{\mathbb{R}}(S) := L_{C_2/C_2}(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})$$

$$L_{\mathbb{C}}(s) := L_{C_2/e}(s) = 2(2\pi)^{-s}\Gamma(s)$$

$$L_S = L_{\mathbb{R}}^{r_1} L_{\mathbb{C}}^{r_2}$$

Here are some identities (analogues of formulas that relate the Gamma function to factorials):

$$L_{\mathbb{R}}(1) = 1$$

$$L_{\mathbb{C}}(1)\frac{1}{\pi}$$

$$L_{\mathbb{R}}(2+s) = \frac{s}{2\pi}L_{\mathbb{R}}(s)$$

$$L_{\mathbb{C}}(1+s) = \frac{s}{2\pi}L_{\mathbb{C}}(s)$$

$$L_{\mathbb{R}}(1-s)L_{\mathbb{R}}(1+s) = \sec(\tfrac{\pi}{2}s)$$

$$L_{\mathbb{R}}(s)L_{\mathbb{R}}(1+s) = L_{\mathbb{C}}(s)$$

$$L_{\mathbb{C}}(1-s)L_{\mathbb{C}}(s) = 2\csc(\pi s)$$

Here is a general functional equation for L_S:

$$L_S(s) = \cos(\tfrac{\pi}{2}s)^{r_1+r_2}\sin(\tfrac{\pi}{2}s)^{r_2}L_{\mathbb{C}}(s)^n L_S(1-s).$$

All of this is obvious from recent stuff and stuff from last class.

Let K be a number field.

$$\zeta_K(s) = \sum_{0 \neq a \trianglelefteq \mathcal{O}_K} N(a)^{-s} = \prod_{0 \neq p \in \operatorname{Spec} \mathcal{O}_K} (1 - N(p)^{-s})^{-1}.$$

This converges on the strip $\rangle 1, \infty \langle$. (Here $\trianglelefteq$ means ideal.)

For each $\Phi \in Cl(K)$, define

$$\zeta_K(\Phi, s) = \sum_{\substack{a \triangleleft \mathcal{O}_K \\ a \in \Phi}} N(a)^{-s}$$

$$Z(\Phi, s) = |d_K|^{s/2} L_X(s) \zeta(\Phi, s)$$

where $X = \mathrm{Hom}(K, \mathbb{C})$.

First we need to cut out a hypersurface in $(\mathbb{R}^S)_{>0}$. Define

$$\mathbb{S}^S := \{x \in (\mathbb{R}^S)_{>0} \; : \; N(x) = 1\}.$$

Note $\mathcal{O}_K^\times / \mu_K \subset \mathbb{S}^S$, and after taking log it's a lattice in the trace-0 hyperplane. I have the decomposition

$$(\mathbb{R}^S)_{>0} \cong \mathbb{S}^S \times \mathbb{R}_{>0}.$$

On the left the measure is $d \log x$, the measure on $\mathbb{S}^S$ is $d^\times x$, and the measure on $\mathbb{R}_{>0}$ is $d \log x$. I can consider the log

$$
\begin{array}{ccc}
(\mathbb{R}^S)_{>0} & \xrightarrow{\;\;\log\;\;} & \{x \in \mathbb{R}^S \; : \; x = s \circ \sigma\} \\
\uparrow & & \uparrow \\
\mathbb{S}^S & \longrightarrow & \{s \in \mathbb{R}^S \; : \; x = x \circ \sigma \text{ and } \mathrm{Tr}(x) = 0\} \\
\uparrow & & \uparrow \\
\mathcal{O}_K^\times / \mu_K \cong |\mathcal{O}_K^\times| & \longrightarrow & \log |\mathcal{O}_K^\times|
\end{array}
$$

By the Dirichlet unit theorem, $|\mathcal{O}_K^\times|$ is a maximal $\mathbb{Z}$-structure *(lattice)* in $\{s \in \mathbb{R}^S \; : \; x = x \circ \sigma \text{ and } \mathrm{Tr}(x) = 0\}$. (See Milne's notes in algebraic number theory.)

We also have $a \triangleleft \mathcal{O}_K$ is a $\mathbb{Z}$-structure in $\mathbb{R}^\times$ with $\mathrm{covol}(a) = \sqrt{d_a}$ where $d_a = N(a)^2 |d_K|$. You want to integrate over $\mathbb{S}^S$ a theta-function, define a function using that, perform a Mellin transform, and get this completed ζ-function. That's almost true, but if you try to write down the integral you realize you're counting too many times: a representative a of a class φ, look at the action of $\mathcal{O}_K^\times$ and you're counting that stuff multiple times. So you have to look at a fundamental domain of (twice) $\log |\mathcal{O}_K^\times|$, and you need to integrate over that instead. Then we'll make the θ-functions more complicated, and define Hecke L-functions.

LECTURE 5: MARCH 2

Recall we had $\mathcal{O}_K^\times / \mu_K \cong |\mathcal{O}_K^\times| \subset \mathbb{S}^S \subset (\mathbb{R}^S)_{>0}$, where $S = \mathrm{Hom}(K, \mathbb{C})$.

$$
\begin{array}{ccc}
(\mathbb{R}^S)_{>0} & \xrightarrow{\;\;\log\;\;} & \{x \in \mathbb{R}^S \; : \; x = s \circ \sigma\} \\
\uparrow & & \uparrow \\
\mathbb{S}^S & \longrightarrow & \{s \in \mathbb{R}^S \; : \; x = x \circ \sigma \text{ and } \mathrm{Tr}(x) = 0\} \\
\uparrow & & \uparrow \\
\mathcal{O}_K^\times / \mu_K \cong |\mathcal{O}_K^\times| & \longrightarrow & \log |\mathcal{O}_K^\times|
\end{array}
$$

Define F as the inverse image in $\mathbb{S}^S$ of any fundamental domain of $2\log|\mathcal{O}_K^\times|$.

Exercise 5.1. $\mathrm{vol}(F) = 2^{r_1+r_2-1}R_K$, where R_K is the Dirichlet regulator, which is defined by the equality $\mathrm{covol}(\log|\mathcal{O}_K^\times|) = \sqrt{r_1+r_2}R_K$.

What measure? We had a Haar measure $d\log x$ on $\mathbb{R}_{>0}$ and a product measure on $(\mathbb{R}^S)_{>0}$; since $(\mathbb{R}^S)_{>0}$ decomposes as $\mathbb{S}^S \times \mathbb{R}_{>0}$, this induces a measure on $\mathbb{S}^S$.

Recall $\Phi \in Cl(K)$ was an ideal class, and $a \in \Phi$ was an integral ideal.

Theorem 5.2. *Write*
$$f_F(a,t) = \frac{1}{\#\mu_K}\int_F \theta_a(ix(\tfrac{t}{d_a})^{1/n})d^\times x - \frac{\mathrm{vol}(F)}{\#\mu_K}.$$
Then
$$Z(\Phi,s) = M\{f_F(a,-)\}.$$

    PROOF. Let Y denote the quotient of the action of $\mathcal{O}_K^\times$ on a and form
$$g(x) = \sum_{y\in Y}\exp(-\pi(y\left(\tfrac{x}{d_a}\right)^{1/n},x)).$$
Recall $d_a = N(a)^2|d_K|$ where d_K is the discriminant. This exercise is just a definition chase.

Exercise 5.3. Check that
$$|d_K|^s\pi^{-ns}\Gamma_X(s)\zeta(\Phi,2s) = \int_{(\mathbb{R}^S)_{>0}}g(x)N(x)^s d\log x.$$

Rewriting the exercise content,
$$Z(\Phi,2s) = \int_{\mathbb{R}_{>0}}\underbrace{\left\{\int_{\mathbb{S}^S}\sum_{y\in Y}\exp(-\pi(yx\left(\tfrac{t}{d_a}\right)^{1/n},x))d^\times x\right\}}_{A(t)}t^s d\log t.$$

$$A(t) = \sum_{\eta\in|\mathcal{O}_K^\times|}\int_{\eta^2 F}\sum_{y\in Y}\exp(-\pi(yx\left(\tfrac{t}{d_a}\right)^{1/n},x))d^\times x$$
$$= \frac{1}{\#\mu_K}\sum_{\eta\in\mathcal{O}_K^\times}\int_{\eta^2 F}\sum_{y\in Y}\exp(-\pi(yx\left(\tfrac{t}{d_a}\right)^{1/n},x))d^\times x - \frac{1}{\#\mu_K}\int_F(\theta_a(ix\left(\tfrac{t}{d_a}\right)^{1/n})-1)d^\times x$$

$\square$

Corollary 5.4. $Z(\Phi,s)$ *admits a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$, and*
$$\mathrm{res}_{s=0}Z(\Phi,s) = -\frac{2^{r_1+r_2}}{\#\mu_K}R_K$$

$$\operatorname{res}_{s=1} Z(\Phi, s) = \frac{2^{r_1 + r_2}}{\#\mu_K} R_K$$

It satisfies

$$Z(\Phi, s) = Z(\Phi^{-1} \otimes \omega, 1 - s)$$

where ω is the codifferent ideal:

$$\omega := \{x \in K \ : \ \operatorname{tr}_{K/\mathbb{Q}}(x\mathcal{O}_K) \subset \mathbb{Z}\}.$$

Powers of ω are like a twist, *and $\omega \otimes \omega = \mathcal{O}_K$ in* Pic(?).

PROOF. In order to use the Poisson summation formula, we have the understand the dual lattice $a^\vee$. It's almost $b := a^{-1} \otimes \omega = \{x \in \mathbb{R}^S \ : \ xa \subset \omega\}$. What's true instead is that

$$\overline{a^\vee} = \{x \in \mathbb{R}^s \ : \ \operatorname{tr}(xa) \subset \mathbb{Z}\}$$
$$= \{x \in \mathbb{R}^S \ : \ \forall r \in a, \operatorname{tr}_{K/\mathbb{Q}}(xr\mathcal{O}_K) \subset \mathbb{Z}\}$$
$$= \{x \in \mathbb{R}^S \ : \ xa \subset \omega\} = b$$

Remember that $\mathbb{R}^S$ isn't necessarily conjugation-invariant. $\qquad\qquad\square$

Here's another definition-chase exercise:

Exercise 5.5. Check:

- $d_b = \frac{1}{d_a}$

- $\theta_{\overline{W}} = \theta_W$ for a $\mathbb{Z}$-structure $W \subset \mathbb{R}^S$ (here $\overline{W}$ means take complex conjugate pointwise; one also has $\overline{W} = \sigma^* W$)

$$f_F(a, \tfrac{1}{t}) = \frac{1}{\#\mu_K} \int_F \theta_a(ix(td_a)^{-1/n}) d^\times x - \frac{\operatorname{vol}(F)}{\#\mu_K}$$

Using the functional equation for θ

$$= \frac{1}{\#\mu_K} \frac{(td_a)^{1/2}}{\operatorname{covol}(a)} \int_{F^{-1}} \theta_b(ix(td_a)^{1/n}) d^\times x - \frac{\operatorname{vol}(F)}{\#\mu_K}$$

Last time, we saw $\operatorname{covol}(a) = \sqrt{d_a}$.

$$= t^{1/2} f_{F^{-1}}(b, t) + \frac{\operatorname{vol}(F)}{\#\mu_K} t^{1/2}$$

By the usual Mellin stuff,

$$Z(\Phi, s) = M\{f_F(a, -)\}(s) = M\{f_{F^{-1}}(b, -)\}(1 - s) = Z(\Phi^{-1} \otimes \omega, 1 - s)$$

I ignored the analytic continuation part, but it follows from the same Mellin stuff as before. Let

$$Z_K(s) := |d_K|^{s/2} L_S(s) \zeta_K(s).$$

Corollary 5.6. $Z_K(s)$ *admits a meromorphic continuation to* $\mathbb{C}$ *with simple poles at* $s = 0$ *and* $s = 1$ *with residues*

$$\operatorname{res}_{s=0} Z_K(s) = -\frac{2^{r_1+r_2}}{\#\mu_K} R_K \cdot (\#Cl_K)$$

$$\operatorname{res}_{s=1} Z_K(s) = \frac{2^{r_1+r_2}}{\#\mu_K} R_K \cdot (\#Cl_K)$$

Moreover,

$$Z_K(s) = Z_K(1-s).$$

$L_S(s)$ came from a machine. It came from creating Minkowski spaces, but there's no a priori reason why $L_S(s)$ should be "the correct factor at ∞". We've seen it's true, but we'll eventually get a better reason.

There has been a basic series of steps, which we'll imitate once we get to the Tate world:

(1) find a Fourier self-dual function

(2) sum it up over a lattice to get a θ-function

(3) "simple" integral transform to get the functions we've been calling f_F

(4) Mellin transform (multiplicative Fourier transform) gives the completed ζ, L, whatever function (i.e. the one described in terms of Dirichlet series)

We've done it three times – for the ζ-function, for L-functions of Dirichlet characters, and for the Dedekind ζ-functions.

LECTURE 6: MARCH 7

Here are some locally compact abelian groups:

- finite (or discrete) abelian groups (e.g. $\mathbb{Z}$)
- $\mathbb{T} := \mathbb{R}/\mathbb{Z}$
- $\mathbb{T}^n$
- $\mathbb{R}$, $\mathbb{C}$
- Any finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, with the obvious topology (sometimes called vector groups)
- *not* $\mathbb{Q}$ (with the subspace topology)
- Given a family $\{A_\alpha\}_{\alpha \in \Lambda}$ where A_α is LCA where all but finitely many are compact, then $\prod_{\alpha \in \Lambda} A_\alpha$ is LCA. (The product and sum coincide, so we write $\oplus$ for this.)
- Limits of LCA groups where all but finitely many are compact, with continuous homomorphisms.
   - $\widehat{\mathbb{Z}} = \lim_{m \in \Phi^{op}} \mathbb{Z}/m$ where Φ is the poset $\mathbb{N}$ ordered by divisibility
   - $\mathbb{Z}_p = \lim_{n \in \mathbb{N}_0} \mathbb{Z}/p^n$
   - $\widehat{\mathbb{Z}}^\times$, $\mathbb{Z}_p^\times$
   - the solenoid $\widehat{\mathbb{S}}^1 := \lim_{m \in \Phi^{op}} \mathbb{R}/m\mathbb{Z}$

- $\mathbb{S}_p^1 = \lim_{n \in \mathbb{N}_0} \mathbb{R}/p^n\mathbb{Z} \cong (\mathbb{R} \times \mathbb{Z}_p)/\mathbb{Z} \cong (\mathbb{R} \times \mathbb{Q}_p)/\mathbb{Z}[\frac{1}{p}]$ (with $\mathbb{Z}$, $\mathbb{Z}[\frac{1}{p}]$ as the diagonal copy)

- any quotient of an LCA group is LCA

- any closed subgroup of an LCA group is LCA

- filtered colimits of open continuous homomorphisms
  - $\mathbb{Q}/\mathbb{Z} = \text{colim}_{m \in \Phi} \mathbb{Z}/m$ (this has the discrete topology)

  - $\mathbb{Q}_p = \text{colim}_{n \in \mathbb{N}_0} p^{-n}\mathbb{Z}_p$

  - Fact: $\mathbb{Q}_p/\mathbb{Z}_p \cong \text{colim}_{n \in \mathbb{N}_0} \mathbb{Z}/p^n$ (this is sometimes called the Prüfer group); it is sometimes called $\mathbb{Z}/p^\infty$ or $\mathbb{Z}_{p^\infty}$. More helpful notation is $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

Fact: torsion LCA groups are discrete.

Exercise 6.1. Describe the topology on an infinite compact Hausdorff ring. (They are all homeomorphic.)

"Idèle" appeared first; it is (in French) short for "ideal element". Then "adèle" was defined; it is short for "additive idèle".

Definition 6.2. The rational finite idèles are:
$$\mathbb{A}_{\mathbb{Q},\text{fin}} := \widehat{\mathbb{Z}} \otimes \mathbb{Q} \cong \text{colim}_{m \in \Phi} m^{-1}\widehat{\mathbb{Z}}.$$
The rational adèles are:
$$\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_{\mathbb{Q},\text{fin}}.$$

If you glom all the p's together in $\mathbb{S}_p^1$, you get $\widehat{\mathbb{S}}^1 \cong \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$. (These are topological isomorphisms.)

Given a family $\alpha_{\alpha \in \Lambda}$, you have a map $\bigsqcup_{\alpha \in \Lambda} A_\alpha \to \prod_{\alpha \in \Lambda} A_\alpha$. Warning: in terms of the topology, this is not the inclusion of a subspace.

Definition 6.3. Say we have a family $\{A_\alpha\}_{\alpha \in \Lambda}$ a family of LCA groups, and a finite subset $J_\infty \subset \Lambda$, and for all $\beta \in \Lambda \backslash J_\infty$, suppose $K_\beta \subset A_\beta$ is open and compact. In this setting, we can define the restricted product

$$^{\{K_\beta\}}\coprod_{\alpha \in \Lambda} A_\alpha = \text{colim}_{\substack{S \text{ finite} \\ J_\infty \subset S \subset \Lambda}} \left(\prod_{\alpha \in S} A_\alpha \times \prod_{\beta \in \Lambda \backslash S} K_\beta \right)$$

This is a filtered colimit of open inclusions.

We have

$$^{\{K_\beta\}}\coprod_{\alpha \in \Lambda} A_\alpha = \bigoplus_{\alpha \in J_\infty} A_\alpha \oplus \left(\prod_{\beta \in \Lambda \backslash J_\infty} A_\beta \times_{\prod_\beta A_b/K_\beta} \bigsqcup_\beta A_\beta/K_\beta \right) \hookrightarrow \prod_{\alpha \in \Lambda} A_\alpha.$$

Warning: this map is a continuous homomorphism and a set inclusion, but not a subspace homomorphism.

24

Now we can define the adèles in general. We had

$$\mathbb{A}_{\mathbb{Q},\mathrm{fin}} := {}^{\{\mathbb{Z}_p\}}\coprod_{p \in \Pi} \mathbb{Q}_p$$

where Π is the set of primes, and

$$\mathbb{A}_{\mathbb{Q}} \cong {}^{\{\mathcal{O}_{\mathbb{Q}v}\}}\coprod_{\text{places } v} \mathbb{Q}_v.$$

This makes sense for any number field, so we can write

$$\mathbb{A}_K \cong {}^{\{\mathcal{O}_{K_v}\}}\coprod_{\text{places } v} K_v.$$

Then J_∞ in the definition of restricted product is the set of infinite places. There's also

$$\mathbb{A}_{K,\mathrm{fin}} \cong {}^{\{\mathcal{O}_{K_v}\}}\coprod_{\text{finite places } v} K_v.$$

We also have a K-solenoid

$$\widehat{\mathbb{S}}^1_K := \mathbb{A}_K/K.$$

Define the *idèles* are

$$\mathbb{I}_K \cong {}^{\{\mathcal{O}_{K_v}^\times\}}\coprod_{\text{places } v} K_v^\times.$$

Warning: the topology on the idèles is not the subspace topology from $\mathbb{I}_K \to \mathbb{A}_K$. It is the canonical topology on the invertible elements: it *can* be written as a subspace $\coprod_K \subset \mathbb{A}_K \times \mathbb{A}_K$ via the map $x \mapsto (x, x^{-1})$. (The problem is that formation of the inverses might not be continuous.) We have

$$\mathbb{I}_{\mathbb{Q}} \cong \mathbb{R}_{>0} \oplus \mathbb{Q}^\times \oplus \widehat{\mathbb{Q}}^\times.$$

Let $\mathbb{I}^1$ denote the norm-1 idèles. There is an exact sequence $\mathbb{I}_K^1 \to \mathbb{I}_K \to \mathbb{R}_{>0}$.

Definition 6.4. Define $\widehat{A} := \mathrm{Hom}(A, \mathbb{T})$. (Sorry, this conflicts with $\widehat{\mathbb{Z}}$ etc.)

Theorem 6.5 (Pontryagin)**.** *The functor $A \mapsto \widehat{A}$ is an equivalence $LCA^{op} \xrightarrow{\sim} LCA$ and it is its own inverse.*

A	$\widehat{A}$
$\mathbb{Z}$	$\mathbb{T}$
V	$V^\vee$
$\mathbb{Z}/p$	$\mathbb{Z}/p$
finite	finite
discrete	compact
$\widehat{\mathbb{Z}}$	$\mathbb{Q}/\mathbb{Z}$
$\mathbb{Z}_p$	$\mathbb{Q}_p/\mathbb{Q}_p$
profinite	ind-finite (i.e. torsion)
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Q}_p$	$\mathbb{Q}_p$
$\{K_\beta\} \prod_{\alpha \in \Lambda} A_\alpha$	$\{K_\beta^\perp\} \prod_{\alpha \in \Lambda} \widehat{A}_\alpha$
$\mathbb{A}_{\mathbb{Q},\mathrm{fin}}$	$\mathbb{A}_{\mathbb{Q},\mathrm{fin}}$
$\mathbb{A}_{\mathbb{Q}}$	$\mathbb{A}_{\mathbb{Q}}$
$\mathbb{A}_K$	$\mathbb{A}_K$
K	$\mathbb{A}_K/K = \widehat{\mathbb{S}}_K^1$
$K^\times$	$\mathbb{I}_K^1/K^\times$
$\mathbb{I}_K^1$	$\mathbb{I}_K^1$
$\mathbb{I}_K$	$\mathbb{I}_K$
$\mathcal{O}_{K_v}$	$K_v/\mathcal{O}_{K_v} = \mathbb{S}_{K_v}^1$
$\mathcal{O}_{K_v}^\times$	??
$\mathrm{Gal}(F^{ab}/F)$, F local	$\widehat{\widehat{F^\times}}$ (one is completion, one is dual) (???)
$\mathrm{Gal}(F^{ab}/F)$	$\mathbb{I}_F^0 \backslash \widehat{\mathbb{I}_F/F^\times}$ (???)
$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$	$\mathbb{Q}^\times$

All the proofs I've found of $\widehat{\mathbb{Z}}$ being dual to $\mathbb{Q}/\mathbb{Z}$ are really complicated. But you can just "take the corresponding colimit to the limit". $\mathbb{Q}_p$ is canonically its own dual (there is a preferred choice of pairing), $\mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{T}$ sending $(x,y) \mapsto \{xy\}_p \in \mathbb{Q}_p/\mathbb{Z}_p$. This relates to $\mathbb{R}$ being its own dual, as $\mathbb{R}$ is the completion of $\mathbb{Q}$ at the infinite prime.

Suppose I have a closed subgroup $B \le A$. Then $\widehat{B} = \widehat{A}/B^\perp$ (where $B^\perp$ is the set of characters (i.e. things in $\widehat{A}$) that annihilate B).

The *idèle class group* is $\mathbb{I}_K/K^\times$.

LECTURE 7: MARCH 9

Today we will give a maximally offputting proof of the Pontryagin duality theorem, i.e. one that does not use the classification theorem.

We begin with the category of discrete abelian groups. Then we'll look at S^1, regarded as a discrete abelian group. Then we're going to do a formal construction that has the duality properties we want. Then we embed the category of LCA groups into this.

Let Ab be the category of abelian groups (with no topology). Note $\mathbb{T}$ is an object here, but we'll write $\mathbb{T}^\delta$ to emphasize there is no topology (i.e. the discrete topology). This is additive, symmetric monoidal, has internal Homs, etc. There's a general procedure, given such a category, that produces a category that has duality with respect to a certain object (here, $\mathbb{T}$). The resulting category will be called $\mathbb{D}(\text{Ab}, \mathbb{T}^\delta)$; its objects are triples (A, A', η) where $A, A' \in \text{Ab}$ and η is a pairing $A \otimes A' \to \mathbb{T}^\delta$ that is nondegenerate. (For every $a \in A$ there exists $a' \in A'$ such that $\eta(a, a') \neq 1$, and the other way around as well.) A morphism $(A, A', \eta) \to (B, B', \theta)$ is a pair $(A \overset{\varphi}{\to} B, B' \overset{\psi}{\to} A')$ such that $\eta(a, \psi(b')) = \theta(\varphi(a), b')$.

Properties:

(1) It is easy to see that this is an additive category, and self-opposite. The obvious functor $\mathbb{D}(\text{Ab}, \mathbb{T}^\delta)^{op} \to \mathbb{D}(\text{Ab}, \mathbb{T}^\delta)$ will be an extension of the duality between LCA and LCA^{op}.

(2) $\mathbb{D}(\text{Ab}, \mathbb{T}^\delta)$ is presentably symmetric monoidal:
$$(A, A') \otimes (B, B') = \big(A \otimes B, \underline{\text{Hom}}(A, B') \times_{\underline{\text{Hom}}(A \otimes B, \mathbb{T}^\delta)} \underline{\text{Hom}}(B, A')\big)$$
and this has a right adjoint that you can write down (but I won't). So it has internal Homs.

(3) There is a natural dual object $D := (\mathbb{T}^\delta, \mathbb{Z}, \text{obvious pairing})$. It is dual in the sense that the natural map
$$A \to \underline{\text{Hom}}(\underline{\text{Hom}}(A, D), D)$$
is an isomorphism in this category.

(4) The unit is $(\mathbb{Z}, \mathbb{T}^\delta, \text{obvious pairing})$. Call this "1". Then I have a duality functor
$$\underline{\text{Hom}}(-, D) : \mathbb{D}^{op} \to \mathbb{D}$$
(where $\mathbb{D}$ is short for $\mathbb{D}(\text{Ab}, \mathbb{T}^\delta)$). Also we have $D = \underline{\text{Hom}}(1, D)$ (this is a stupid observation).

Now I'm going to show this category is equivalent to a category of topological abelian groups with a certain property, and that that contains the category of LCA groups. The dual will go to the dual, and since the dual of an LCA group is LCA, we have our theorem.

First let's throw out some obviously terrible topological abelian groups.

Definition 7.1. Say that a topological abelian group is *admissible* if it can be exhibited as a topological subgroup (not necessarily closed) of a product (not necessarily finite) $\prod_{\alpha \in \Lambda} A_\alpha$ where the A_α's are all LCA.

This gives me access to infinite limits and colimits of LCA groups.

Definition 7.2. If A is a topological abelian group, then a topology τ on the underlying set $|A|$ is *A-characteristic* if

- $(|A|, \tau)$ is admissible, and
- $\text{Hom}^{cts}(A, \mathbb{T}) = \text{Hom}^{cts}((|A|, \tau), \mathbb{T})$ (where Hom^{cts} is the set of continuous homomorphisms).

Proposition 7.3. *For any admissible topological abelian group A, there is a coarsest A-characteristic topology on $|A|$ and a finest A-characteristic topology on $|A|$.*

(The poset of all topologies on A, and look only at the ones that agree with the original one about what the characters are, there is a maximal and a minimal object.)

PROOF. The coarsest one is formal – take this as an exercise. (This is the same argument for compactly generated spaces.) Call this one $\tau_{+\infty}$.

Let's build the finest one; this is really not formal. Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be the set of all A-characteristic topologies on $|A|$. Take $\prod_{\alpha \in \Lambda}(|A|, \tau_\alpha)$. These are all no coarser than the coarsest one, so we can map each one to $\tau_{+\infty}$, and so we can get a map

$$\prod_{\alpha \in \Lambda}(|A|, \tau_\alpha) \to (|A|, \tau_{+\infty})^\Lambda.$$

Now define $\tau_{-\infty}$ by the pullback

$$
\begin{array}{ccc}
(|A|, \tau_{-\infty}) & \longrightarrow & \prod_{\alpha \in \Lambda}(|A|, \tau_\alpha) \\
\downarrow & & \downarrow \\
(|A|, \tau_{+\infty}) & \xrightarrow{\;\Delta\;} & (|A|, \tau_{+\infty})^\Lambda
\end{array}
$$

It's obviously finest, but you do have to check that it's A-characteristic, i.e. $\text{Hom}^{cts}((|A|, \tau_{-\infty}), \mathbb{T}) = \text{Hom}^{cts}(A, \mathbb{T})$. (That's an exercise, which is critically going to use the admissibility criterion.) □

Definition 7.4. Say that an admissible topological abelian group A is $\mathbb{T}$-*cogenerated* if its topology is $\tau_{-\infty}$. Say that A is $\mathbb{T}$-*generated* if its topology is $\tau_{+\infty}$.

We have
$$\mathbb{T}\text{-cogenerated} \subset \text{Admissible} \supset \mathbb{T}\text{-generated}.$$
We've produced adjoint functors to these inclusions, namely the one taking an admissible topological abelian group A to $(|A|, \tau_{+\infty})$ (this is a left adjoint) or $(|A|, \tau_{-\infty})$ (this is a right adjoint). What this is really doing is inverting a class of weak equivalences: we're inverting the maps $A \to B$ such that $\text{Hom}^{cts}(B, \mathbb{T}) \to \text{Hom}^{cts}(A, \mathbb{T})$ is an isomorphism.

Note that the categories of $\mathbb{T}$-cogenerated and $\mathbb{T}$-generated objects are equivalent.

Exercise 7.5. LCA groups are already $\mathbb{T}$-cogenerated. (This is easy, and trivial if you use the classification of LCA groups, but the whole point is we're trying to do without that.)

Now we'll show that the category of $\mathbb{T}$-cogenerated things (or, equivalently, $\mathbb{T}$-generated things), is equivalent to the category $\mathbb{D}$ from above, and moreover this equivalence sends the dual to the dual. Then we'll be done.

Theorem 7.6. $\mathbb{T}$-*cogen* $\simeq \mathbb{D}$ *under a functor sending* $\mathbb{T} \mapsto D$, *and moreover we have*

$$
\begin{array}{ccc}
\mathbb{T}\text{-}cogen & \xrightarrow{\ \simeq\ } & \mathbb{D} \\
{\scriptstyle \underline{\mathrm{Hom}}(-,\mathbb{T})}\Big\downarrow & & \Big\downarrow{\scriptstyle \underline{\mathrm{Hom}}(-,D)} \\
\mathbb{T}\text{-}cogen^{op} & \xrightarrow{\ \simeq\ } & \mathbb{D}^{op}
\end{array}
$$

PROOF. Let's construct a functor $\mathbb{T}$-cogen $\to \mathbb{D}$. Given a T-cogenerated object A, we associate the triple $(|A|, |\widehat{A}|, \mathrm{ev})$. (Note: the assignment $\mathrm{Ab} \to \mathbb{D}$ sending $A \mapsto (A, \mathrm{Hom}(A, \mathbb{T}^\delta), \mathrm{ev})$ is a fully faithful functor.) In the other direction, we have to say what the functor $\mathbb{D} \to \mathbb{T}$-cogen does, i.e. we have to say what happens to (A, B, η). Give A the subspace topology in $\widehat{B} = \mathrm{Hom}(B^\delta, \mathbb{T})$. This is not $\mathbb{T}$-cogenerated, but instead send $(A, B, \eta) \mapsto \tau_{-\infty}(A)$. $\qquad\square$

Let A be an LCA group, and define $\mathscr{M}(A)$ to be the set of regular countably additive complex Borel measures. Also define $\mathscr{H}(A)$ to be the collection of actual Haar measures. (These do not embed into $\mathscr{M}(A)$ because Haar measures are infinite, and complex measures are never infinite.)

Fact 7.7. $\mathscr{H}(A)$ is an $\mathbb{R}_{>0}$-torsor.

Definition 7.8. If $\mu \in \mathscr{M}(A)$ then define $\widehat{\mu} : \widehat{A} \to \mathbb{C}$ as follows:

$$
\widehat{\mu}(\chi) = \int_A \overline{\chi(a)}\, d\mu(a).
$$

Fact 7.9. $\widehat{\mu}$ is bounded and uniformly continuous.

This is usually called the Fourier-Stieltjes transform.

We have a functor

$$
\widehat{(-)} : \mathscr{M}(A) \to C_u(\widehat{A})
$$

(where C_u means uniformly bounded continuous functions). If $\lambda \in \mathscr{H}(A)$, and if μ is absolutely continuous w.r.t. λ, then $d\mu = f\, d\lambda$ for some $f \in L^1(A)$. We write $\widehat{f}$ for $\widehat{\mu}$, the Fourier transform.

There's an inverse Fourier transform which is sort of dual to this thing. Suppose $\mu \in \mathscr{M}(\widehat{A})$. Then define

$$\check{\mu}(a) = \int_{\widehat{A}} \chi(a) d\mu(\chi).$$

If $d\mu = f d\lambda$ for $\lambda \in \mathscr{H}(\widehat{A})$, then $\check{f} = \check{\mu}$, and this is the inverse Fourier transform.

Aside: the image of

$$L^1(A) \xrightarrow{\widehat{(-)}} C_0(\widehat{A})$$

is called the Wiener algebra, written $\mathscr{W}_0(\widehat{A})$. (Here C_0 is compactly supported continuous functions.)

Theorem 7.10. *There are isomorphisms*

$$\mathscr{M}(A) \xrightarrow{\cong} C_u(\widehat{A})$$

$$\mathscr{M}(\widehat{A}) \xrightarrow{\cong} C_u(A)$$

Technical fact 7.11. If $\lambda \in \mathscr{H}(A)$, then there exists a Haar measure $\mu \in \mathscr{H}(A)$ such that for all $f \in L^1(A, \lambda) \cap L^2(A, \lambda)$, we have

- $\widehat{f} \in C_0(\widehat{A}) \cap L^2(\widehat{A}, \mu)$
- $\|\widehat{f}\|_2 \leq \|f\|_2$ (relative to μ)

LECTURE 8: MARCH 16

At some point in your career you might feel that you're just not famous enough. Here is a recipe for becoming more famous.

(1) Locate A and B (maybe classes of objects) that appear to be "in duality" (a procedure that turns an A thing into a B thing, and essentially the same procedure turns it back into an A thing)

(2) Discover C, a thing that contains A and B and is self-dual (and the duality should extend the duality between A and B).

(3) Call the duality a Fourier-someone transform. (It should be someone relatively classical, not someone who actually worked on this stuff.)

(4) ???

(5) Profit.

A lot of the big hits in number theory work like this. (E.g. Scholze's theorem.)

This is a recipe that you see if you think about the relationship between discrete and compact groups – the common generalization is LCA groups. We're going to try to run this more globally, working with Fourier transforms on more general LCA groups (on the adèles).

Now and forever A means some LCA group.

Lemma 8.1. *If $\lambda \in \mathscr{H}(A)$ is a Haar measure, there's a Haar measure $\mu \in \mathscr{H}(\widehat{A})$ such that:*

(1) For all $f \in L^1(A) \cap L^2(A)$, then $\widehat{f} \in L^1(\widehat{A}) \cap L^2(\widehat{A})$, and furthermore $\|\widehat{f}\|_2 \leq \|f\|_2$.

(2) (dual sentence) For all $\varphi \in L^1(\widehat{A} \cap L^2(\widehat{A}))$, then $\check{f} \in L^1(A) \cap L^2(A)$ and $\|\check{f}\|_2 \leq \|\varphi\|_2$.

If λ and μ are as over there, $L^1(A) \cap L^2(A) \subset L^2(A)$ is dense, and similarly $L^1(\widehat{A}) \cap L^2(\widehat{A}) \subset L^2(\widehat{A})$. We have maps

$$
\begin{array}{ccc}
L^1(A) \cap L^2(A) & \subset & L^2(A) \\
\updownarrow & & \updownarrow \\
L^1(\widehat{A}) \cap L^2(\widehat{A}) & \subset & L^2(\widehat{A})
\end{array}
$$

and them being dense produces maps $\widehat{(-)} : L^2(A) \rightleftarrows L^2(\widehat{A}) : \widecheck{(-)}$.

Theorem 8.2 (Fourier Inversion/ Plancherel). *For all $\lambda \in \mathscr{H}(A)$, there exists a unique $\mu \in \mathscr{H}(\widehat{A})$, as previously, such that*

(1) For all $f \in L^2(A)$, $\widecheck{\widehat{f}} = f$;

(2) For all $\varphi \in L^2(\widehat{A})$, $\widehat{\widecheck{\varphi}} = \varphi$.

(3) The assignments $\widehat{(-)}$ and $\widecheck{(-)}$ give an isometric (rel $\|-\|_2$) isomorphism $L^2(A) \cong L^2(\widehat{A})$.

Notation: λ will always be a Haar measure, and $\widehat{\lambda} = \mu$ will be the dual.

This is supposed to be background; if you've never seen this stuff, it's in Hewitt and Ross, *Abstract Harmonic Analysis*.

Theorem 8.3 (Parseval's identity). *For all $f, g \in L^2(A)$,*

$$
\int_A f\bar{g} d\lambda = \int_{\widehat{A}} \widehat{f}\widehat{\bar{g}} d\widehat{\lambda}.
$$

This is really the same as point (3) above.

We started with a function that was its own Fourier transform, put a bunch of them together into a θ-transformation, which satisfies a functional equation that comes from the Poisson summation formula; when you Mellin it up, you get a functional equation for zeta functions, L-functions, etc.

What we need is a Poisson summation formula. It required some intense growth conditions on our functions – we need an analogue of Schwartz functions. This class is the Schwartz-Bruhat functions. In the literature, these things are effectively described using a classification theorem. I'm not going to do that (but I'll tell you enough to be able to relate them to the definition you might already know).

Definition 8.4. A function $f \in L^\infty(A, \lambda)$ is *of brisk decay* if there exists a compact subset $K \subset A$ such that, for every $n \geq 1$, there exists a constant $C_n > 0$ such that for all $m \geq 1$, one has

$$\|f|_{(A \setminus K^m)}\|_\infty < \frac{C_n}{m^n}.$$

(Here K^m means the set of all m-fold products.)

Observations:

(1) f vanishes almost everywhere outside $\langle K \rangle$ (this means the group generated by K).

(2) Functions of brisk decay are translation-invariant.

(3) Functions of brisk decay are closed under convolution.

(4) (Exercise!) A function of brisk decay is L^p for all $p > 1$.
   (Protip: bound $\int_{K^m \setminus K^{m-1}} |f|$.)

Functions of brisk decay are in $L^1(A) \cap L^2(A)$, so their duals are in $L^1(\widehat{A} \cap L^2(\widehat{A}))$. The Fourier transform is a map $L^1(A) \to C_0(\widehat{A})$, so the duals are also in $C_0(\widehat{A})$.

Definition 8.5. A *Schwartz-Bruhat function* (or *function of rapid decay*) is a briskly decaying function f whose Fourier transform f is also briskly decaying. I'll write $\mathscr{S}(A)$ for the set of all Schwartz-Bruhat functions.

Fact 8.6. $\mathscr{S}(A)$ is a Fréchet space, and is dense in $L^2(A)$. I suspect it is also nuclear. (What's the topology? We'll see soon.)

Notice that $\mathscr{S}(A)$ and $\mathscr{S}(\widehat{A})$ are in duality by definition.

Examples 8.7.
- If F is a finite abelian group, $\mathscr{S}(F)$ is the set of all functions.
- $\mathscr{S}(\mathbb{Z}^m)$ is the set of functions that decay faster than any polynomial. That is, these are functions f such that for every $k \in \mathbb{Z}_{>0}^m$, $\sup_{n \in \mathbb{Z}^m} |n^k f(n)| < +\infty$.
- $\mathscr{S}(\mathbb{T}^m) = C^\infty(\mathbb{T}^m)$.
- $\mathscr{S}(\mathbb{R}^m)$ is the set of Schwartz functions.

Exercise 8.8. Characterize $\mathscr{S}(F \times \mathbb{Z}^m \times \mathbb{T}^n \times \mathbb{R}^p)$.

($\{f \in C^\infty : \|P(\delta)f\|_\infty < +\infty\}$ where $P(\delta)$ is a polynomial differential operator in the $\mathbb{Z}^m$ and $\mathbb{R}^p$ variables)

Bruhat defined Schwartz-Bruhat functions on the above classes, and then used this to approximate other ones.

If $A \in LCA$, then

$$\mathscr{S}(A) = \mathrm{colim}_{(U,K)} \mathscr{S}(U/K)$$

where the colimit is taken over $K < U < A$ with K compact, U open, and U/K a Lie group. (In particular, the Lie group has to be of the form $F \times \mathbb{Z}^m \times \mathbb{T}^n \times \mathbb{R}^p$.) And now you see what the topology has to be. It's a countable filtered colimit of Fréchet spaces, which is still a Fréchet space.

There is a topological isomorphism $\mathscr{S}(A) \cong \mathscr{S}(\widehat{A})$. Also, $\widehat{\widehat{f}}(x) = f(x^{-1})$.

Example 8.9. If A is totally disconnected, $\mathscr{S}(A)$ is the set of locally constant functions of compact support.

So we know what this is for the adèles: for the finite places, we know what to do (because it's totally disconnected), and for the infinite places, we also know what to do, because it's the original Schwartz functions.

We were looking for groups that were their own Pontryagin dual. We're looking for functions on them that are their own Fourier transform, and using them to produce our θ-functions. For this, we need the Poisson summation formula.

Here's a theorem I think is true (but can't find a proof):

Theorem 8.10 (Poisson summation). *If $0 \to A' \to A \to A'' \to 0$ is a short exact sequence of LCA groups, then*

(1) (easy – exercise) For all $\lambda \in \mathscr{H}(A)$ and $\chi' \in \mathscr{H}(A')$, then there exists a unique $\lambda'' \in \mathscr{H}(A'')$ such that

$$\int_{x \in A} f(x) d\lambda(x) = \int_{z \in A''} \int_{y \in A'} f(yz) d\lambda'(y) d\lambda''(z)$$

(this is like a Fubini theorem).

(2) For all Schwartz functions $f \in \mathscr{S}(A)$, define (for all $z \in A''$)

$$\pi_*(f)(z) = \int_{y \in A'} f(xy) d\lambda'(y)$$

where x is a lift of z. Then $\pi_* f \in \mathscr{S}(A'')$ and

$$\widehat{\pi_* f} = \widehat{f}|_{(A')^\perp}$$

under the identification $\widehat{A''} \cong (A')^\perp$.

(3) For any $x \in A$,

$$\int_{y \in A'} f(xy) d\lambda'(y) = \int_{\chi \in \widehat{A''}} \widehat{f}(\chi) \chi(x) d\widehat{\lambda''}(\chi).$$

I'll prove it next time. But I'll give the most interesting case: if $\Lambda \subset A$ is discrete, then

$$\sum_{x \in \Lambda} f(x) = \frac{1}{\operatorname{covol}(\Lambda)} \sum_{\chi \in \Lambda^\perp} \widehat{f}(\chi).$$

LECTURE 9: MARCH 21

PROOF OF THE POISSON SUMMATION FORMULA FROM LAST LECTURE. (2) If $\chi \in (A')^{\perp}$ and $x \in A$, $y \in A'$, then you'll always have $\chi(xy) = \chi(x)$ (by definition of $\perp$). So:

$$\widehat{\pi_* f}(\chi) = \int_{A''} \pi_* f(z) \overline{\chi(x)} d\lambda''(z)$$

$$= \int_{A''} \int_{A'} f(yz) \overline{\chi(yz)} d\lambda'(y) d\lambda''(z) \qquad\qquad x \text{ is a lift of } z$$

$$= \int_A f(x) \overline{\chi(x)} d\lambda(x) \qquad\qquad\qquad\qquad \text{using (1)}$$

(3) We're going to use the Fourier inversion formula. I won't bother checking the thing is still Schwartz – it is. If $x \in A$, $z = \pi(x) \in A''$ then

$$\int_{y \in A'} f(xy) d\lambda'(y) = (\pi_* f)(z) = \overline{\widehat{(\pi_* f)}}(z)$$

$$= (\widetilde{\widehat{f}|_{(A')^{\perp}}})(z)$$

$$= \int_{\chi \in (A')^{\perp}} \widehat{f}(\chi) \chi(x) d\widehat{\lambda}(\chi)$$

For the special case, use the counting measure for the discrete groups. $\qquad\qquad\qquad$ $\square$

The theory behind $\mathbb{F}_1$ is that all of these constructions (functional equation for the ζ function) come from structures that are not specialized to the field involved. In fact, it seems independent of the characteristic of the local field, and of the completion of the local field. $\mathbb{F}_1$ is telling you why that's true.

Let k be a local field. We define $|-|$ on k by choosing a Haar measure λ and a measurable set U of finite measure. Then write

$$|x| = \frac{\lambda(xU)}{\lambda(U)}.$$

This doesn't depend on λ or U.

Examples:

(1) If $k = \mathbb{R}$, this is just the ordinary absolute value.

(2) If $k = \mathbb{C}$, $|z| = z\overline{z}$.

(3) If k is non-archimedean but characteristic zero, let $\mathfrak{o} \subset k$ be the ring of integers. Let $\mathfrak{p} \subset \mathfrak{o}$ be the maximal ideal, and π a uniformizing parameter. Let $F \cong \mathbb{F}_q$ be the residue field. Then $|\pi| = \frac{1}{q}$.

Let k^+ refer to the additive (as opposed to multiplicative) group. We know we have $\widehat{k^+} \cong k^+$. How many isomorphisms are there? You can specify one by specifying any nontrivial character χ of k^+: the map is $x \mapsto (y \mapsto \chi(xy))$. Let's just pick a character for the moment and think about Haar measures – the set is an $\mathbb{R}_{>0}$-torsor. We can choose a unique Haar measure μ_k that is self-dual w.r.t. the chosen χ.

Exercise 9.1. This does not depend on the choice of χ.

So we have a canonical choice of Haar measure for a local field. We're going to do a multiplicative Fourier transform (Mellin transform) to something that's self dual.

In his thesis, Tate says he's not going to identify $\widehat{k^\times}$ because he didn't know how. We have canonical exact sequence

$$1 \to U_k \to k^\times \xrightarrow{|-|} V_k \to 1$$

where

$$V_k = \{t \in \mathbb{R}_{>0} \; : \; \exists x \in k^\times, t = |x|\}$$
$$U_k = \{x \in k^\times \; : \; |x| = 1\}$$

This is split, but the splitting (in the non-archimedean characteristic zero case) depends on π.

$V_\mathbb{R} = \mathbb{R}_{>0}$	$U_\mathbb{R} = \{\pm 1\}$
$V_\mathbb{C} = \mathbb{R}_{>0}$	$U_\mathbb{C} = S^1$
$V_{\mathbb{Q}_p} = p^\mathbb{Z}$	$U_{\mathbb{Q}_p} = \mathbb{Z}_p^\times$
	$U_k = \mathfrak{o}^\times$

To tell the whole story, we need a multiplicative Haar measure.

If $g \in C_c(k^\times)$ is a compactly supported continuous function, then $\frac{g(x)}{|x|} \in C_c(k^+ \backslash 0)$. The idea is to use the additive Haar measure μ^+ to get a multiplicative Haar measure. Define

$$\Phi(g) := \int_{k^+ \backslash 0} \frac{g(x)}{|x|} d\mu^+.$$

This is translation-invariant (and a positive nontrivial functional) on $C_c(k^\times)$, so there exists a unique corresponding Haar measure, which we will write as $\log |\mu^+|$. (I am reserving $\mu^\times$ for something later.)

Here I'm going to use a characteristic zero assumption. We're going to renormalize; you'd expect the normalization would make U_k have measure 1, but that's not the choice we make. I don't understand why Tate made this choice.

If k is archimedean, then define $\mu^\times = \log |\mu^+|$. If k is nonarchimedean, then define $\mu^\times = \frac{q}{q-1} \log |\mu^+|$. (where q is the size of the residue field). (These definitions don't require characteristic zero, but the stuff that follows does, and I'm worried these definitions don't agree with later characteristic > 0 definitions.)

Exercise 9.2. Compute $\mathrm{vol}(U_k, \mu^\times)$.

The answer is $\frac{1}{\sqrt{d_k}}$.

Definition 9.3. A *quasicharacter* on $k^\times$ is a homomorphism $\psi : k^\times \to \mathbb{C}^\times$.

Any quasicharacter ψ on $k^\times$ factors as

$$\psi(x) = \chi(x)|x|^s.$$

How canonical is this? χ is determined by ψ. In the archimedean case, s is determined by ψ; in the nonarchimedean case, s modulo $\frac{2\pi i}{\log q}\mathbb{Z}$ is determined by ψ. Its real part $\sigma = \mathrm{Re}(s)$ (called the *exponent*) is determined by ψ.

Definition 9.4. A quasicharacter is *unramified* if its restriction to U_k is the trivial character (i.e. $\chi = 1$).

We have

$$S := \begin{cases} \mathrm{QChar/Unram} \cong \mathbb{C}\big/\frac{2\pi i}{\log q}\mathbb{Z} & \text{nonarchimedean} \\ \mathbb{C} & \text{archimedean.} \end{cases}$$

So it's a Riemann surface.

Definition 9.5. If $f \in \mathscr{S}(k^+)$ and $\psi \in \mathrm{QChar}$ has exponent $\sigma > 0$, then

$$z(f, \psi) = \int_{k^\times} f(x)\psi(x)d\mu^\times(x)$$

is the *local zeta function*.

Define

$$D = \{\psi \in S : \sigma > 0\}.$$

Lemma 9.6. $z(f, \psi)$ *is a regular function on* D.

Theorem 9.7 (Functional equation). *Fix* $\chi \in \widehat{k^+}$.[1] *Then* $\psi(f, \psi)$ *admits an analytic continuation to all quasicharacters by means of the following functional equation:*

$$z(f, \psi) = \rho(\psi)z(\widehat{f}, \widehat{\psi})$$

where $\widehat{\psi}(x) = \frac{|x|}{\psi(x)}$ *and* $\rho(\psi)$ *is independent of* f *and analytic on* $\sigma \in (0, 1)$ *(to be defined more precisely next time).*

LECTURE 10: MARCH 23

I want to try to go through most of Tate's thesis today if I can. We were working on the local story – we imagined our global field completed at some place. Remember k is the local field. We had an identification $k^+ \cong \widehat{k^+}$, with one identification for every character. Fix a nontrivial character:

(1) If $k \cong \mathbb{R}$, take $\chi_\mathbb{R} : \mathbb{R} \xrightarrow{(-1)} \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$.

(2) If $k \cong \mathbb{Q}_p$, take $\chi_{\mathbb{Q}_p} : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z} \subset S^1$.

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[1]unclear if we need this? The statement relies on having a chosen identification $k^+ \cong \widehat{k^+}$. We know the identification $\mathscr{S}(k^+) \cong \mathscr{S}(\widehat{k^+})$ without choosing χ...

(3) If $k \cong \mathbb{F}_p((t))$, then take $\chi_{\mathbb{F}_p((t))} : \mathbb{F}_p((t)) \to \mathbb{F}_p \subset S^1$ where the map $\mathbb{F}_p((t)) \to \mathbb{F}_p$ takes the coefficient on t^{-1} and the embedding $\mathbb{F}_p \subset S^1$ hits the primitive roots of unity.

(4) In the remaining case, k is a finite extension of some k_0 of type (1), (2), or (3). Then define $\chi_k = \chi_{k_0} \circ \mathrm{tr}_{k/k_0}$.

The answer to Exercise 9.2 was $\mu^\times(U_k) = \frac{1}{\sqrt{d_k}}$. Then for almost all places v, you have $\mu^\times(U_{k_v}) = 1$.

Let Q_k be the set of quasicharacters, i.e. continuous homomorphisms $\psi : k^\times \to \mathbb{C}^\times$. Given a character $\chi \in \widehat{U}_k$, you can always write $\psi(x) = \chi(x)|x|^s$. This s is uniquely determined if our field is archimedean; if it's nonarchimedean, s is determined up to an ambiguity of $\frac{2\pi i}{\log q}\mathbb{Z}$ where q is the size of the residue field. (You can think of Q_k (in the nonarchimedean case) as a bunch of cylinders (copies of S_k for the discrete points of $\widehat{U}_k$)).

Write S_k for the set of quasicharacters modulo unramified quasicharacters. We saw that

$$S_k = \begin{cases} \mathbb{C} & k \text{ archimedean} \\ \mathbb{C}\big/ \frac{2\pi i}{\log q}\mathbb{Z} & k \text{ nonarchimedean.} \end{cases}$$

Then we have a map $Q_k \to \widehat{U}_k \times S_k$ sending $\psi \mapsto (\chi, s)$. U_k is compact so $\widehat{U}_k$ is discrete.

This is the space on which our zeta function is defined. First we define the zeta function using something that might not converge anywhere, and then we have an analytic continuation. We need to be able to specify where it does(n't) converge. We will be exceptionally coy and write $\mathrm{Re}(s) =: \mathrm{Re}(\psi)$. Now we can talk about our old friend the strip $\rangle a, b \langle$ but this time it's

$$\rangle a, b \langle = \{\psi \in Q_k \ : \ \mathrm{Re}(\psi) \in]a, b[\}.$$

Definition 10.1. If $f \in {}^+$ and $\psi \in \rangle 0, +\infty\langle$, define the local zeta integral

$$z(f, \psi) = \int_{k^\times} f(x)\psi(x)d\mu^\times(x).$$

We claim this is well-defined and holomorphic on that region.

For the functional equation, we need to see the analogue of $s \mapsto 1 - s$, but for the space of quasicharacters. We'll define an involution $\psi \mapsto \widehat{\psi}$, defined by

$$\widehat{\psi}(x) = \frac{|x|}{\psi(x)}.$$

Check that $\mathrm{Re}(\widehat{\psi}) = 1 - \mathrm{Re}(\psi)$.

Exercise 10.2. For $\psi \in \rangle 0, 1\langle$, use Fubini to show

$$z(f, \psi)z(\widehat{g}, \widehat{\psi}) = z(g, \psi)z(\widehat{f}, \widehat{\psi}).$$

This isn't hard.

Recall given the data (χ, s) we're making the quasicharacter $\chi(x)|x|^s$.

Fix $\chi \in \widehat{U}_k$ (so we're restricting to one cylinder). If we can find one Schwartz function $f \in \mathscr{S}(k^+)$ such that

(1) $\psi \mapsto z(\widehat{f}, \widehat{\psi})$ is not identically zero on $\rangle 0, 1\langle$,

(2) $\rho(\psi) = \frac{z(f, \psi)}{z(\widehat{f}, \widehat{\psi})}$ admits a meromorphic continuation to $\{\chi\} \times S_k$,

then for any $g \in \mathscr{S}(k^+)$, $z(g, \psi)$ admits a meromorphic continuation to all of $\{\chi\} \times S_k$, and
$$z(g, \psi) = \rho(\psi) z(\widehat{\gamma}, \widehat{\psi}).$$

Case 1a: $k \cong \mathbb{R}$ and $\chi = 1$. Then take $f_1 = \exp(-\pi x^2)$. Then $\psi(x) = |x|^s$ and s is uniquely determined by ψ. We proved in the first week that
$$z(f_1, \psi) = \pi^{-s/2} \Gamma(\tfrac{s}{2}).$$

Moreover,
$$z(\widehat{f}, \widehat{\psi}) = \pi^{(s-1)/2} \Gamma(\tfrac{1-s}{2}).$$
(I'm being sloppy about whether I'm writing ψ or s, but there's no difference so it's OK.) We've also seen before
$$\rho(s) = 2^{1-s} \pi^{-s} \cos(\tfrac{\pi}{2}s) \Gamma(s).$$

Case 1b: $k \cong \mathbb{R}$ and $\chi = -1$. Take $f_{-1} = x \exp(-\pi x^2)$. Then $\psi(x) = \mathrm{sgn}(x)|x|^s$, and
$$\rho(s) = i 2^{1-s} \pi^{-s} \sin(\tfrac{\pi}{2}s) \Gamma(s).$$

Case 2a: $k \cong \mathbb{Q}_p$ and $\chi = 1$ (unramified case). Choose $f_1 = $ the indicator function of $\mathbb{Z}_p \subset \mathbb{Q}_p$. Then
$$\widehat{f_1}(x) = \int_{\mathbb{Z}_p} \exp(-2\pi i \{xy\}) d\mu(y) = f_1(x).$$

(So this is Fourier self-dual.) In order to get our zeta function, we're supposed to do a Mellin transform:
$$z(f_1, \psi) = \int_{\mathbb{Q}_p^\times} f_1(x) |x|^s d\mu^\times(x)$$
$$= \frac{p}{p-1} \int_{\mathbb{Z}_p \backslash 0} |x|^{s-1} d\mu(x)$$
$$= \sum_{r=0}^{+\infty} p^{-rs} = \frac{1}{1 - p^{-s}}.$$

where $\{-\}$ is the fractional part of a p-adic number. Because f_1 is Fourier self-dual, we also have
$$z(\widehat{f_1}, \widehat{\psi}) = \frac{1}{1 - p^{s-1}}.$$

This shows that
$$\rho(s) = \frac{1 - p^{s-1}}{1 - p^{-s}}$$
which has a meromorphic continuation.

Case 2b: $k \cong \mathbb{Q}_p$ and $\chi \neq 1$ (ramified case). Then you need to modify f:
$$f_\chi(x) = \begin{cases} 0 & \text{if } |x| > p^n \\ e^{2\pi i \{x\}} & \text{if } |x| \leq p^n. \end{cases}$$

Here n is the conductor, the smallest n such that χ factors through $(\mathbb{Z}/p^n)^\times \to S^1$. Then the Fourier transform is

$$\widehat{f_\chi}(x) = \begin{cases} 0 & |1-x| > p^n \\ p^n & |1-x| \leq p^n. \end{cases}$$

Then the Gauss sum appears in the zeta function

$$z(f_\chi, \psi) = \frac{p^{ns+1-n}}{p-1} \sum_{r=1}^{p^n-1} \chi(r) \exp(\tfrac{2\pi i r}{p^n})$$

and $\rho(\psi) = p^{n(s-1)} \sum_{r=1}^{p^n-1} \chi(r) \exp(\tfrac{2\pi i r}{p^n})$ so

$$z(\widehat{f_\chi}, \widehat{\psi}) = \frac{p}{p-1}.$$

Exercise 10.3. Do this for $\mathbb{F}_p((t))$. (You get a similar thing with an indicator function.)

For a finite extension, the trace gets involved and the numbers get uglier but it's not too hard. So the theorem holds.

Now K is a global field (eventually I'll be lazy and make K a number field). We have $\mathbb{A}_K \cong \widehat{\mathbb{A}}_K$. There's a norm on the LHS that is canonical because it came from the specified norms on our finite things; these mostly agree that the volume of the unit thing is 1, and so there aren't convergence issues preventing you from applying that here. You can product together all your chosen Haar measures to get $\widehat{\mathbb{A}}_K$.

On the multiplicative part of the story, we had $\mathbb{I}_K$. Given μ and $|-|$ on $\mathbb{A}_K$, we have $\mu^\times$ on $\mathbb{I}_K$. We have a short exact sequence $0 \to K \to \mathbb{A}_K \to \mathbb{A}_K/K \to 0$. Put the counting measure on K (it's discrete so we can do this). Then the first step of the Poisson summation formula gives a measure on $\mathbb{A}_K/K$, and this has the property that $\mu(\mathbb{A}_K/K) = 1$. (This is a good reason to choose that weird normalization from before.)

Define V_K using the exact sequence $1 \to \mathbb{I}_K^1 \to \mathbb{I}_K \to V_K \to 1$ (here $\mathbb{I}_K^1$ is the compact piece – it's the difference between $\mathbb{R}\backslash 0$ and $\{\pm 1\}$). There is also an exact sequence $1 \to K^\times \to \mathbb{I}_K^1 \to \mathbb{I}_K^1/K \to 1$. Then $K^\times$ is discrete and $\mathbb{I}_K^1/K$ is compact (this is called the idèle class group). We have

$$V_K = \begin{cases} \mathbb{R}_{>0} & \text{number field} \\ p^{\mathbb{Z}} & \text{function field.} \end{cases}$$

In the first case, use $d \log t$ as a measure. In the second case, use $\log$ where q is the order of the biggest extension of the finite field contained in K.

We have a theory of quasicharacters here, as in the local case.

Definition 10.4. A quasicharacter (or Hecke character, or größencharakter, or idèle class character) is a map

$$\chi : \mathbb{I}_K/K^\times \to \mathbb{C}^\times.$$

The set of these is called $\mathbb{H}$.

We have $\mathbb{H} \cong (K^\times)^\perp \times \mathbb{C}$. (This is a topological decomposition, not something group-theoretic.)

We have to think about $s \mapsto 1 - s$ on this set of complex planes. The analogue of this sends $\psi \mapsto \widehat{\psi}$ where
$$\widehat{\psi}(x) = \frac{|x|}{\psi(x)}.$$

Definition 10.5. The global zeta function is
$$Z(f, \psi) = \int_{\mathbb{I}_K} f(x)\psi(x)d\mu^\times(x).$$

Main theorem 10.6. This integral converges for $\mathrm{Re}(\psi) > 1$, and $Z(f, \psi)$ extends to a meromorphic function on $\mathbb{H}$ with poles only at:

- $(1, 0)$, with residue
$$\mathrm{res}_{\psi=(1,0)} Z(f, \psi) = -\mathrm{covol}(K^\times)f(0),$$

- $(1, 1)$, with residue
$$\mathrm{res}_{\psi=(1,1)} Z(f, \psi) = \mathrm{covol}(K^\times)\widehat{f}(0).$$

The functional equation is
$$Z(f, \psi) = Z(\widehat{f}, \widehat{\psi}).$$

PROOF. I'm going to skip all the parts of the story that are of the form "take all the local pieces and glom them together". There's just one piece of this story (really, the main point) not of this form. Recall the proof for the Dedekind zeta function. (Instead of integrating over the whole region, we restricted to the norm one piece.) Everything works in both cases, but I'll write this out just in the number field case. We're going to decompose
$$Z(f, \psi) = \int_{\mathbb{I}_K} f(x)\psi(x)d\mu^\times(x)$$
$$= \int_0^{+\infty} Z_t(f, \psi)d\log t$$
where $Z_t(f, \psi) = \int_{\mathbb{I}_K^1} f(ty)\psi(ty)d\mu^1(y)$ (here μ^1 is the measure on $\mathbb{I}_K^1$). We're not going to analyze Z_t by breaking it into local pieces.

Let E be a fundamental domain for the lattice $K^\times \subset \mathbb{I}_K^1$. Then
$$Z_t(f, \psi) = \int_E \Big(\sum_{\alpha \in K^\times} f(\alpha ty)\Big)\psi(ty)d\mu^1(y).$$

We're really close to having something we can use Poisson's summation formula on – if that sum were over K, instead of $K^\times$, we could use Poisson in the additive world. Do this anyway:
$$\int_E \Big(\sum_{\alpha \in K} f(\alpha ty)\Big)\psi(ty)d\mu^1(y) = \int_E \sum_{\alpha \in K} \widehat{f}(\tfrac{\alpha y}{t})\widehat{\psi}(\tfrac{y}{t})$$

So now we have
$$Z_t(f, \psi) + f(0)\int_E \psi(ty)d\mu^1(y) = Z_{1/t}(\widehat{f}, \widehat{\psi}) + \widehat{f}(0)\int_E \widehat{\psi}(\tfrac{y}{t})d\mu^1(y)$$

There are two cases to contemplate: ψ is either unramified or it isn't. If it's unramified, then this integral ends up being

$$\int_E \psi(ty)d\mu^1(y) = t^s\mu^1(E).$$

In the ramified case, it's 1 (exercise – definition chase).

The unramified case is harder, but not in any meaningful way, so let's just do the ramified case. We have

$$Z(f,\psi) = \int_1^\infty Z_t(f,\psi)d\log t + \int_1^{+\infty} Z_u(\widehat{f},\widehat{\psi})d\log u$$
$$= Z(\widehat{f},\widehat{\psi})$$

and victory is ours.

We've already computed $\mu^1(E)$ in terms of the regulator (though we didn't say it out loud).

Next week I'll start introducing things relevant to $\mathbb{F}_1$. $\qquad\qquad\qquad\square$

LECTURE 11: APRIL 4

I'm going to talk about the field with one element $\mathbb{F}_1$. There's a tendency not to take the mathematics too seriously; none of the things had good definitions for 50 years, and the names are kind of playful, but there is plenty of meaning. For X a scheme, you're supposed to make sense out of $\lim_{q\to 1} X(\mathbb{F}_q)$, which works better or worse depending on the situation. The first place in print I know of is J. Tits' article from the 50's, where he does this with algebraic groups. The attempt to prove the Riemann hypothesis with this involves imitating Deligne's proof (for curves) over $\operatorname{Spec}\mathbb{Z}$ instead, but not sure how it relates to this older story.

Idea: if I want to give you an $\mathbb{R}$-vector space, then a good way to do that is to give you a $\mathbb{C}$-vector space and a C_2-semilinear action (it acts by complex conjugation on $\mathbb{C}$, and also acts on the vector space, giving a canonical real structure). Instead of doing this for the finite extension $\mathbb{R}\to\mathbb{C}$, we'll try to do this descent story for a much bigger extension along the lines of $\mathbb{F}_q\to\mathbb{F}_q[t]$. Actually, we'll look at the extension $\mathbb{F}_1\to\mathbb{Z}$. (This is not a finitely presented map – it's big.)

But there's a big difference between these – in the $\mathbb{F}_q[t]$ case, you have the rationality of the zeta function as proved by Deligne, but you're not going to get that in the $\mathbb{F}_1$ case.

I'm going to present this following work of Jim Borger. The data you need to descend from $\mathbb{Z}$ to $\mathbb{F}_1$ is a Λ-structure (here this means the structure of a module over Λ, which will be defined shortly; this is the same Λ-structure that shows up in representation theory etc.).

Definition 11.1. An $\mathbb{F}_1$-algebra S is a ring (commutative with unit) along with a Λ-structure.

My goal is to tell you what these words mean, and I'm going to do this in a strange way.

Definition 11.2. Suppose k is a ring. Then a *k-algebra affine scheme* is a functor $\mathrm{CAlg}_k \to$ CAlg (CAlg means commutative algebras) such that the composite

$$\mathrm{CAlg}_k \xrightarrow{X} \mathrm{CAlg}_k \to \mathrm{Set}$$

is corepresentable: it is $\mathrm{Hom}_k(R, -)$ for some R.

This is a k-algebra object in the category of affine schemes. But purely algebraically, you have a k-algebra R and a big co-k-algebra structure: there is

- a co-zero $\varepsilon^+ : R \to k$,
- a co-addition $\Delta^+ : R \to R \otimes_k R$,
- a co-unit $\varepsilon^\times : R \to k$,
- a co-multipilication $\Delta^\times : R \to R \otimes_k R$,
- an antipode $\sigma : R \to R$

subject to some axioms. There is also an algebra map $k \to \mathrm{Hom}_k(R, k)$. (This is now a k-algebra because of all the previous structure.)

You can extract a Hopf algebra by just taking the additive structure (it's an additive antipode). You can also take the (co)-invertible things and then you get a Hopf algebra structure using the multiplicative structure. This is more structure than a Hopf ring.

This is an incredibly inefficient way to think about all this structure.

Example 11.3. The constant functor on the zero ring $\mathrm{CAlg}_k \to \mathrm{CAlg}_k$ is a k-algebra affine scheme. This is co-representable by the initial object, namely k.

Example 11.4. The identity map $\mathrm{CAlg}_k \xrightarrow{I} \mathrm{CAlg}_k$ is a k-algebra affine scheme [from now on, "$k$aas"] co-represented by $k[t]$. (The analogy is that this is like the integers.)

Example 11.5. Given a group (or even monoid) G, I can talk about the functor $\mathrm{CAlg}_k \to \mathrm{CAlg}_k$ sending $R \mapsto R^G$ (this is G many copies of R). This is a $k$aas co-represented by $k[G]$ (this is the polynomial ring generated by elements of G, not the group ring).

This is a monoidal category – you can compose the functors.

Definition 11.6. A *comonad* is a coalgebra in $\mathrm{End}(\mathrm{CAlg}_k)$. This is the data:

- (counit) $X \xrightarrow{\varepsilon} \mathbb{1}$
- (comultiplication) $X \to X \circ X$

satisfying the usual sort of axioms (so it corresponds to a monoid structure that is associative and unital).

Definition 11.7. A $k$aas $X : \mathrm{CAlg}_k \to \mathrm{CAlg}_k$ is a *plethory* if X is a comonad.

The comonad corresponds to a noncommutative product on R (the co-representing object), often called a *plethysm.*

What happens to your $k$aas when you upgrade it to a plethory? You get an additional structure map $\circ : R \times R \to R$ called a plethysm.

Example 11.8. The identity map is a comonad, so there must be an operation $k[t] \times k[t] \to k[t]$. This is just composition of polynomials.

Example 11.9. Back in Example 11.5, the co-representing object is the polynomial ring generated by the elements of G (note this is not the group ring, but rather the symmetric algebra on the group ring kG). This has a comonad structure.

Example 11.10. Set $k = \mathbb{Z}$. Consider CAlg $\to$ CAlg sending $S \mapsto \mathbb{W}(S)$ (where $\mathbb{W}(S)$ is the big Witt vectors). This is a plethory, corepresented by Λ, the subring of $\mathbb{Z}[[x_1, x_2, \ldots]]^{\mathrm{Aut}(\mathbb{N})}$ consisting of power series with bounded degree monomials. (This is in pretty much every Hazewinkel paper.)

Theorem 11.11 ("Fundamental theorem of symmetric function theory"). *We have $\Lambda = \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, \ldots]$ where the λ_i's are the elementary symmetric functions:*

$$\lambda_1 = x_1 + x_2 + x_3 + \ldots$$
$$\lambda_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 + \ldots$$
$$\vdots$$

This can be found in MacDonald's book. Of course, there are other choices of generators. For example, there are Adams generators

$$\psi_n := x_1^n + x_2^n + x_3^n + \ldots$$

which are only a generating set over $\mathbb{Q}$, but over $\mathbb{Z}$ it's true that $\Lambda = \mathbb{Z}[w_1, w_2, \ldots]$ where the w_i's are uniquely determined by:

$$\psi_n = \sum_{d \mid n} d w_d^{n/d}.$$

These are the *Witt* components of $\mathbb{W}(R)$. To believe this is a plethory, you need a map $\mathbb{W}(R) \to \mathbb{W}(\mathbb{W}(R))$; this is the Artin-Hasse map. The plethory structure is composition of symmetric functions.

We'll think of CAlg$_k$ as modules, and plethories as the ring acting on them.

Definition 11.12. Given a plethory P, a P-algebra is a k-algebra A which is a coalgebra for the comonad P.

P applied to A is $\mathrm{Hom}_k(R, A)$. A coalgebra is a map $A \to \mathrm{Hom}_k(R, A)$.

Example 11.13. Every k-algebra is an I-algebra (here I is the identity map). (This is just a fancy way of saying the following statement: "giving a map $k[x] \to R$ is the same thing as giving an element of R.")

Example 11.14. If G is a group, then we had a plethory $F^G : \mathrm{CAlg}_k \to \mathrm{CAlg}_k$ sending $R \mapsto R^G$. Then F^G-algebras are the same as G-k-algebras.

Example 11.15. A $\mathbb{W}$-algebra is a special λ-ring.

This picture is general enough that it didn't depend on working in algebras – you just need algebras in some category.

Suppose K is a symmetric monoidal category (not an ∞-category) that has all finite colimits and the tensor product preserves those colimits separately in each variable. (Nobody needs this much generality; we'll apply this to the category of finite-dimensional $\mathbb{Q}$-vector spaces.)

Now look at algebras over K (not algebras *in* K!): $\mathrm{CAlg}(K)$ is the category of symmetric monoidal categories B as above, along with a symmetric monoidal functor $K \to B$. (If $K = \mathrm{Vect}(\mathbb{Q})$, and X is a $\mathbb{Q}$-variety, then $\mathrm{Coh}(X)$ (coherent sheaves) is in $\mathrm{CAlg}(K)$.)

(I'm categorifying the whole thing.)

I can do the whole plethory story again.

Definition 11.16. A K-plethory is a comonad $\mathrm{CAlg}(K) \to \mathrm{CAlg}(K)$ such that $\mathrm{CAlg}(K) \to \mathrm{CAlg}(K) \to \mathrm{Cat}$ is co-representable (it's $\mathrm{Fun}_K^\otimes(R, -)$ for some R, i.e. symmetric monoidal K-linear functors out of R).

The comonad is a (2,1)-functor. (I don't care about the non-invertible morphisms.)

But the only example I care about is the identity functor.

Example 11.17. Suppose $K = \mathrm{Vect}(\mathbb{Q})$. Then $I : \mathrm{CAlg}(\mathrm{Vect}(\mathbb{Q})) \to \mathrm{CAlg}(\mathrm{Vect}(\mathbb{Q}))$ is co-representable by the categorical analogue of polynomials over $\mathbb{Q}$, which we'll call $\mathrm{Vect}(\mathbb{Q})[x]$. This is the category of functors $\Sigma^{op} \to \mathrm{Vect}(\mathbb{Q})$ that are eventually zero. Here Σ is the category of finite sets and bijections, and $P : \Sigma^{op} \to \mathrm{Vect}(\mathbb{Q})$ is "eventually zero" if there is some N for which P is zero on finite sets of cardinality $> N$.

(Why not the category of natural numbers? A map of rings $k[x] \to R$ is the same as specifying some $r \in R$. Our thing is also supposed to be free on one generator: a functor $\mathrm{Vect}(\mathbb{Q})[x] \to C$ should be the same as specifying an object. This says you're writing down a functor $\Sigma \to C$ sending the disjoint union of finite sets to the sum inside C. What is the free symmetric monoidal category on one generator? It's Σ, not $\mathbb{N}$.)

What's the de-categorification procedure? Take the Grothendieck group $K_0 : \mathrm{CAlg}(\mathrm{Vect}(\mathbb{Q})) \to \mathrm{CAlg}(\mathbb{Z})$. It turns out that $K_0(\mathrm{Vect}(\mathbb{Q})[x])$ is the algebra Λ of symmetric functions, with all of its structure. (The map is just taking characters.) Exercise – use Maschke's theorem.

There is also a version of this story with $\mathbb{Z}$ instead of $\mathbb{Q}$, but it's harder. In general, $K_0(\mathrm{Coh}(X))$ is a Λ-algebra.

Definition 11.18 (Borger). An $\mathbb{F}_1$-algebra is a commutative ring R along with a Λ-structure (the structure of a Λ-algebra).

The idea is that the Λ-structure is providing you with the descent data to go down to $\mathbb{F}_1$. The stupid equation to have in your head is

$$\text{``}\Lambda = \mathbb{Z}[\mathrm{Gal}(\mathbb{Z}/\mathbb{F}_1)].\text{''}$$

But $\mathbb{Z}$ is not really a field extension of $\mathbb{F}_1$. Another moral statement:

$$\text{``}\Lambda = \mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}\text{''}$$

(but we'll define things so this one is actually true.) This is Frobenius descent with all Frobenii simultaneously. For something flat over $\mathbb{Z}$, an $\mathbb{F}_1$ structure is the data of a whole bunch of Frobenius maps.

We want a forgetful functor $\mathrm{CAlg}(\mathbb{F}_1) \to \mathrm{CAlg}(\mathbb{Z})$ which is a left adjoint; this functor should be like $- \otimes_{\mathbb{F}_1} \mathbb{Z}$. This has a right adjoint: $\mathbb{W} : \mathrm{CAlg}(\mathbb{Z}) \to \mathrm{CAlg}(\mathbb{F}_1)$ (Witt vectors). It seems like the forgetful map is on the wrong side, but if you go back to thinking about the adjunction

$$\mathrm{Forget} : \mathrm{CAlg}(\mathbb{C}) \rightleftarrows \mathrm{CAlg}(\mathbb{R}) : - \otimes_{\mathbb{R}} \mathbb{C},$$

you notice that the identification $\mathrm{CAlg}(\mathbb{C})^{\sigma} \cong \mathrm{CAlg}(\mathbb{R})$ allows you to think of the "tensor up" map as forgetting as well.

LECTURE 12: APRIL 6

Definition 12.1 (Borger). An $\mathbb{F}_1$-algebra is a Λ-algebra over $\mathbb{Z}$.

We explained that this is a coalgebra for the comonad of big Witt vectors. The thing that represents it is Λ, the algebra of symmetric functions.

Recall

$$\Lambda = K_0(\mathrm{Fun}(\Sigma^{op}, \mathrm{Vect}(\mathbb{Q}))^{\mathrm{fin}}).$$

(Here Σ is the category of finite sets and bijections, and "fin" means it's zero on big enough sets.) Various kinds of functors are giving you elements in $\Lambda \subset \mathbb{Z}[[x_1, x_2, \dots]]^{\mathrm{Aut}(\mathbb{N})}$. The operations $\lambda_n = \sum_{i_1 < \dots < i_n} x_{i_1} x_{i_2} \dots x_{i_n}$ correspond to Λ^n, the nth exterior power (it's a representation of $\Sigma_n \dots$ think of this acting on $\mathrm{Fun}(-,-)$). Similarly, $\sigma_n = \sum_{i_1 \le i_2 \le \dots \le i_n} x_{i_1} x_{i_2} \dots x_{i_n} \in \Lambda$ acts as Sym^n on the RHS.

(In the literature, λ_n is sometimes called e_n ("elementary"), and σ_n is called h_n (for "homogeneous"?).) We also had the Adams symmetric functions $\psi_n = x_1^n + x_2^n + \dots$.

It is a standard fact that Λ is generated by the λ_n's. But you could also write $\Lambda \cong \mathbb{Z}[w_1, w_2, \dots]$ where the w_i's are defined by

$$\psi_n = \sum_{d|n} d w_d^{n/d}.$$

If R is a Λ-algebra, the Witt components are given by the map
$$w : R \to R^{\mathbb{N}}$$
given by the w_i.

There is a subring $\Psi \subset \Lambda$ generated by ψ_n (for $n \in \mathbb{N}$). This co-represents a plethory $\mathbb{W}_\Psi : \mathrm{CAlg} \to \mathrm{CAlg}$ that takes $R \mapsto R^{\mathbb{N}}$ (here $R^{\mathbb{N}}$ is a ring, as opposed to the Witt stuff above, where we talked about $R^{\mathbb{N}}$ as a set that you then put a weird ring structure on).

What's the comonad structure? Define $\psi_n : R^{\mathbb{N}} \to R^{\mathbb{N}}$ that sends to $\underline{a} = (a_1, a_2, \dots) \mapsto (a_n, a_{2n}, a_{3n}, \dots)$. A comonad structure is a map $\mathbb{W}_\Psi(R) \to \mathbb{W}_\Psi \mathbb{W}_\Psi(R)$, and we define this to send $\underline{a} \mapsto (\psi_1(\underline{a}), \psi_2(\underline{a}), \dots)$.

You can check this is compatible with the comonad structure on Λ.

Definition 12.2. A Ψ-*algebra* is a coalgebra for $\mathbb{W}_\Psi$.

This is actually really simple: it's an action of $\mathbb{N}^\times$ on a ring. It's equivalent to specifying a collection of ψ_n's compatible with multiplication; alternatively, you can just specify the ψ_p's for p prime that commute. (This thing about just specifying ψ_p works in the 1-categorical setting, but in the world of E_∞-rings you want to keep track of the whole $\mathbb{N}^\times$ structure.)

Lemma 12.3 (Newton formula). *For all $k \geq 1$,*
$$\sum_{m=1}^{k} (-1)^{k-m} \lambda_{k-m} \psi_m = (-1)^{k+1} k \lambda_k.$$

    PROOF. Exercise. (Use induction.)   □

Corollary 12.4. *If R is flat over $\mathbb{Z}$, then any Ψ-algebra structure lifts to at most one Λ-algebra structure on R.*

What are the conditions required for there to exist a lift?

Theorem 12.5 (Wilkerson). *If R is flat over $\mathbb{Z}$ then a Ψ-algebra structure on R in which*
$$\psi_p(x) \equiv x^p \pmod{pR}$$
lifts uniquely to a Λ-algebra structure (i.e. an $\mathbb{F}_1$-algebra structure).

Actually, this is an iff – you get all the $\mathbb{F}_1$-algebra structures this way. So if R is flat over $\mathbb{Z}$, a Λ-algebra structure is the same as a family of compatible lifts of Frobenius maps.

    PROOF. We define a Λ-algebra structure on $R \otimes \mathbb{Q}$ as follows:
$$-t \frac{d}{dt} \log\left(\sum_{m \in \mathbb{N}_0} \lambda_m(x) t^m \right) = \sum_{n \in \mathbb{N}} (-1)^n \psi_n(x) t^n.$$

(Exercise (or… go look it up) – show this defines a Λ-algebra structure.) For every p, we'll show the λ_m's carry $R_{(p)}$ to $R_{(p)}$ – that is, for any prime p and any $x \in R \otimes_{(p)}$, $\lambda_k(x) \in R \otimes \mathbb{Z}_{(p)}$.

Do this by induction. This is true for $k = 1$ because $\lambda_1 = \mathbb{1}$. Now assume it's true for $i < k$.

Case 1: $p \nmid k$. We can divide by k, and then Newton's formula + the induction hypothesis wins the day.

Case 2: $k = p$. Note that
$$\psi_p(x) - x^p = p((-1)^{p+1}\lambda_p(x) + P(\lambda_1(x), \ldots, \lambda_{p-1}(x)))$$
where P is some polynomial with integer coefficients. (This is something for you to check. I'm using flatness, that $R \to R \otimes \mathbb{Q}$ is injective.)

Case 3: $k = mp$ for $m \geq 2$. Then note that
$$\lambda_k(x) = (-1)^{(p+1)(m+1)}\lambda_m(\lambda_p(x)) + Q(\lambda_1(x), \ldots, \lambda_{p-1}(x)).$$
(Again, Q is a polynomial with integer coefficients.) Now we're done by the induction hypothesis. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Examples of $\mathbb{F}_1$-algebras:

(1) $\mathbb{F}_1$ itself is $\mathbb{Z} = K_0(\mathbb{Q})$ with its unique Λ-structure. (Or alternatively just write $\psi_p = \mathbb{1}$.) This has $\lambda_k(n) = \binom{n}{k}$ and $\sigma_k(n) = \binom{n+k-1}{k-1}$. To get these, take the vector space with dimension n and do the corresponding operation, e.g. $\otimes$.

(2) Given a monoid M, $\mathbb{Z}M$ is an $\mathbb{F}_1$-algebra with the following descent data: for $x \in M$, define $\psi_p(x) = x^p$. For example, $\mathbb{Z}[x]/(x^m - 1)$ with this structure is called $\mathbb{F}_{1^m}$. You're supposed to think of this as a cyclotomic extension of $\mathbb{F}_1$. You can write $\mathbb{Z}[x]/(x^m - 1) \cong \mathbb{F}_{1^m} \otimes_{\mathbb{F}_1} \mathbb{Z}$.

Categorifying $\mathbb{A}^1$. Look at $K_0(\mathrm{Rep}(SL(2,\mathbb{C})))$. This contains the standard (2-dimensional) representation which we'll call $[V]$. Because this is the ring of a symmetric monoidal Ψ-linear idempotent-complete category, this comes with a Λ-structure. Look at representations via the characters on $\begin{pmatrix} a & \\ & -a \end{pmatrix}$; this gives a map $K_0(\mathrm{Rep}(SL(2,\mathbb{C}))) \to \mathbb{Z}[a, a^{-1}]$ sending $[V] \mapsto a + a^{-1}$. You get an isomorphism
$$K_0(\mathrm{Rep}(SL(2,\mathbb{C}))) \cong \mathbb{Z}[a, a^{-1}]^{C_2} \cong \mathbb{Z}[x]$$
where the C_2 action sends $a \mapsto a^{-1}$ and $x = a + a^{-1}$.

This is *not* the $\mathbb{F}_1$-structure of $\mathbb{Z}\mathbb{N}_0$ using the monoid example. I claim that $\mathbb{Z}[x]$ is the interesting one. For the $(n+1)$-dimensional representation $\mathrm{Sym}^n(V)$ we get
$$[\mathrm{Sym}^n(V)] = a^n + a^{n-1} + a^{n-4} + \cdots + a^{4-n} + a^{2-n} + a^{-n} \in \mathbb{Z}[a, a^{-1}].$$
If $a = \exp(i\theta)$ then $x = a + a^{-1} = 2\cos\theta$. Then $[\mathrm{Sym}^n(V)] = U_n(x/2)$ where the U_n's are the Chebyshev polynomials of the second kind. This is σ_n. This is a much more interesting $\mathbb{F}_1$-structure on the affine line than the usual one.
$$\psi_1(x) = x$$

$$\psi_2(x) = x^2 - 2$$

Exercise: compute the rest of the $\psi_p(x)$'s. To get the higher ones, use $\psi_p(x) = a^p + a^{-p} \equiv (a + a^{-1})^p \pmod{p}$.

$\mathbb{Z}[a, a^{-1}] = \mathbb{Z}\mathbb{Z}$ (using that horrendous group ring notation).

Remark 12.6. $- \otimes_{\mathbb{F}_1} \mathbb{Z} : \mathrm{CAlg}(\mathbb{F}_1) \to \mathrm{CAlg}(\mathbb{Z})$ has a right and left adjoint and so this is closed under limits and colimits.

I think there are only 2 Λ-structures on $\mathbb{Z}[x]$ – boring or Chebyshev.

Non-examples:

(1) Curves of genus $g \geq 1$

(2) Flag varieties that aren't $\mathbb{P}^n$. (No Grassmannians.) This is a hard theorem of Paranjape-Srinivas: if a flag variety admits a single Frobenius lift of any sort, it's $\mathbb{P}^n$ for some n. Idea: flag varieties involve some bit of linear algebra over a specific field in a non-trivial way.

Next time: more about the geometry of these things, including the module theory for these things. (There's a notion of field extensions and étale maps, and $\mathbb{F}_1$ is purely inseparable (because $\mathbb{Z}$ is purely inseparable – there aren't any nontrivial étale extensions of $\mathbb{Z}$).)

LECTURE 13: APRIL 11

I'll talk more about $\mathbb{F}_1$-algebras, and then start talking about the module theory of these things.

Last time, Denis asked why we didn't define the affine line over $\mathbb{F}_1$ as the free $\mathbb{F}_1$-generator on one variable. Suppose $F \subset E$ is a G-Galois extension. There is a forgetful functor $\mathrm{Alg}(F) \to \mathrm{Set}$ and there is a left adjoint $F[-] : \mathrm{Set} \to \mathrm{Alg}(F)$ (the free algebra functor). The point is that we don't know about F; we know about $\mathrm{Alg}(E)$ with some Galois descent information. Note $\mathrm{Alg}(F) \simeq \mathrm{Alg}(E)^G$ where G denotes a semi-linear action.

What does this equivalence mean? There is a forgetful functor $\mathrm{Alg}(E)^G \to \mathrm{Alg}(E)$. The "tensor-up-to-$E$" map $\mathrm{Alg}(F) \to \mathrm{Alg}(E)$ factors through this.

$$\mathrm{Alg}(F) \underset{\simeq}{\longleftrightarrow} \mathrm{Alg}(E)^G \longrightarrow \mathrm{Alg}(E)$$

with $F[-]$ and Set below.

In the $\mathbb{F}_1$ case, the analogous thing is $\mathbb{F}_1 \subset \mathbb{Z}$ (a "Λ-Galois extension"). We have

$$\mathrm{Alg}(\mathbb{F}_1) = \mathrm{Alg}(\mathbb{Z})^\Lambda \xLeftarrow{\quad \mathbb{W} \quad} \mathrm{Alg}(\mathbb{Z})$$

$$\Big\Updownarrow {\scriptstyle ?}$$

$$\mathrm{Set}$$

It's not clear what the forgetful functor does – you can't just forget the Λ-algebra structure and then forget the $\mathbb{Z}$-algebra structure. The idea is that it should be like G-fixed points, but it's not clear what the G is. You could try to restrict attention to the flat things[2] (i.e. when you tensor up to $\mathbb{Z}$ it's free). Then we're looking at things with an $\mathbb{N}^\times$ action subject to certain relations. Then you could try to take the $\mathbb{N}^\times$-fixed-points functor as the forgetful functor $\mathrm{Alg}^{\mathrm{flat}}(\mathbb{F}_1) \to \mathrm{Set}$.

We had a comonad $\mathrm{CAlg}(\mathbb{Z}) \xrightarrow{\mathbb{W}} \mathrm{CAlg}(\mathbb{Z})$. Dually, there is a monad $\mathrm{Aff} \xrightarrow{\mathbb{W}} \mathrm{Aff}$ (where Aff means affine schemes), and algebras for this monad are Spec of things that are coalgebras for the comonad. Importantly, you can extend this

$$
\begin{array}{ccc}
\mathrm{Aff} & \xrightarrow{\quad \mathbb{W} \quad} & \mathrm{Aff} \\
\cup \downarrow & & \cup \downarrow \\
\mathrm{Shv}(\mathbb{Z}) & \underset{\widetilde{\mathbb{W}}}{\overset{\mathbb{M}}{\leftrightarrows}} & \mathrm{Shv}(\mathbb{Z})
\end{array}
$$

where $\mathrm{Shv}(\mathbb{Z})$ (also written $\widetilde{\mathrm{Aff}}_{\text{ét}}$) is the big étale topos, the category $\mathrm{Shv}^{\text{ét}}(\mathrm{Aff}, \mathrm{Set})$ (feel free to mentally work with space-y things like ∞-topoi). (You can even call $\mathrm{Shv}^{\text{ét}}(\mathrm{Aff}, \mathrm{Set}) = \mathrm{Sp}_{\mathbb{Z}}$, the "category of spaces", but this is not good notation.) So I Kan extended my monad and got a new monad. There is a canonical equivalence of categories

$$\mathrm{Alg}(\widetilde{\mathbb{W}}) \cong \mathrm{Coalg}(\mathbb{M}).$$

(There's a set-theoretic issue: $\mathbb{W}$ doesn't take finitely presented things to finitely presented things. There exists a regular cardinal κ with the property that $\mathbb{W}$ carries κ-generated things to κ-generated things.)

$\mathbb{M}$ preserves filtered colimits. This implies:

Proposition 13.1. $\mathrm{Alg}(\widetilde{\mathbb{W}}) =: \mathrm{Shv}(\mathbb{F}_1)$ *is a topos. Moreover, the adjunction*

$$\mathrm{Shv}(\mathbb{F}_1) \underset{v_*}{\overset{v^*}{\rightleftarrows}} \mathrm{Shv}(\mathbb{Z})$$

is a geometric morphism. Here v_* is application of $\mathbb{M}$.

(These functors are coming from the cofree/forgetful adjunction on $\mathbb{M}$, not the free/forgetful adjunction from $\widetilde{\mathbb{W}}$.)

---

[2]If A is a reduced $\mathbb{Z}$-algebra with a Λ-structure, then A is flat over $\mathbb{Z}$.

PROOF. Look up Marc Hoyois' answer about this on mathoverflow (it's for ∞-topoi but works in this setting just fine). $\qquad\square$

Definition 13.2. $\mathrm{Shv}(\mathbb{F}_1)$ is the *big étale topos of $\mathbb{F}_1$*.

Furthermore, there is a third adjoint $v_! : \mathrm{Shv}(\mathbb{Z}) \to \mathrm{Shv}(\mathbb{F}_1)$.

Sidebar 13.3. Given a morphism of rings $f : A \to B$, you get a morphism $\mathrm{Spec}\, B \xrightarrow{f} \mathrm{Spec}\, A$. Then you get functors

$$\mathrm{\acute{E}t}(B) \xleftrightarrow[f_*]{f^*} \mathrm{\acute{E}t}(A)$$

where $\mathrm{\acute{E}t}(A)$ is the big étale topos (and f_* is called a Weil restriction). There is an additional adjunction $\mathrm{\acute{E}t}(B) \xrightarrow{f_!} \mathrm{\acute{E}t}(A)$. This produces an identification $\mathrm{\acute{E}t}(B) \simeq \mathrm{\acute{E}t}(A)/\mathrm{Spec}\, B$ with

$$\begin{array}{ccc} \mathrm{\acute{E}t}(B) & \xrightarrow{\quad f_! \quad} & \mathrm{\acute{E}t}(A) \\ {\wr}{\mid} & \nearrow & \\ \mathrm{\acute{E}t}(A)/\mathrm{Spec}\, B & & \end{array}$$

This is called an étale morphism of topoi (as opposed to a geometric morphism (f^*, f_*)). A geometric morphism (f^*, f_*) is étale if

- $f_!$ exists ("essential")
- $f_!$ is conservative
- (some base-change condition) $f_!(M \times_{f^*P} f^*N) \simeq f_! M \times_P N$

In our case,

$$v_*(\mathrm{Spec}\, A) = \mathrm{Spec}(\Lambda \odot A)$$
$$v_!(\mathrm{Spec}\, A) = \mathrm{Spec}\, \mathbb{W}(A)$$

Exercise 13.4. Is $v_! \dashv v^* \dashv v_*$ étale? *Clark thought it wasn't because (2) fails, but maybe it's OK?*

I'm going to use the Beck/ Quillen definition of module. If $\mathcal{C}$ is a category with all finite limits and $A \in \mathcal{C}$, then define

$$\mathrm{Mod}(A) := \mathrm{Ab}(\mathcal{C}_{/A})$$

to be the category of abelian group objects in the slice category of $\mathcal{C}$ over A. (Prototypical example: $\mathcal{C}$ is the category of commutative algebras.)

For example, this is where derivations come from. If $M \in \mathrm{Mod}(A)$, then a *derivation* $d : A \to M$ is a map in $\mathcal{C}_{/A}$. Call the set of these $\mathrm{Der}(A, M)$; it has an abelian group structure.

Proposition 13.5. *[Beck] If $\mathcal{C} = \mathrm{CAlg}$, then*

$$\mathrm{Mod}(A) = \mathrm{Mod}^{cl}(A).$$

(That is, the above category $\mathrm{Mod}(A)$ is the same as the usual category $\mathrm{Mod}^{cl}(A)$ of modules over A.)

PROOF. Exercise. (There is a functor $\mathrm{Mod}^{cl}(A) \to \mathrm{Mod}(A)$ sending $M \mapsto A \oplus M$ where $A \oplus M$ is given a commutative algebra structure by giving it the square-zero multiplication (i.e. $(a, m) \cdot (a', m') = (aa', am' + a'm)$). In the other direction, the functor $\mathrm{Mod}(A) \to \mathrm{Mod}^{cl}(A)$ sends a morphism $f : B \to A$ to $\ker(f)$. You just have to check it's an equivalence of categories.) $\square$

If $\mathcal{C}$ is a presentable category, then there is a free-forgetful adjunction $\mathrm{Ab}(\mathcal{C}_A) \leftrightarrows \mathcal{C}_{/A}$. In particular, there exists a universal derivation $d : A \to \Omega_A$ that induces an isomorphism

$$\mathrm{Mor}_{\mathrm{Mod}(A)}(\Omega_A, M) \cong \mathrm{Der}(A, M).$$

In the example, this is exactly the usual module of Kähler differentials.

The objective is to write down $\mathrm{Mod}(A)$ when $\mathcal{C}$ is the category of $\mathbb{F}_1$-algebras.

Fact 13.6. In the comonad $\mathbb{W} : \mathrm{CAlg} \to \mathrm{CAlg}$, the polynomials co-representing the comonad structure have zero constant term. This means that this comonad extends to a comonad $\mathbb{W} : \mathrm{CAlg}^{nu} \to \mathrm{CAlg}^{nu}$ (here CAlg^{nu} denotes non-unital coalgebras).

So if M is an abelian group, then you can talk about $\mathbb{W}(M)$ (with the zero multiplication).

Lemma 13.7. *We have*
$$\mathbb{W}(A \oplus M) \cong \mathbb{W}(A) \oplus \mathbb{W}(M)$$
(where $A \oplus M$ has a square-zero multiplication).

This is a square zero extension of $\mathbb{W}(A)$.

You can probably do this with lots of kinds of plethories, but you need to know something about $\mathbb{W}$ to be able to push this through.

We have
$$\mathbb{W}(M) \cong M^{\mathbb{N}}$$
in Ab. (We're just taking the components of the Witt vector.) But I'd like to give this a $\mathbb{W}(A)$-module structure. Given $a \in \mathbb{W}(A)$ and $m \in \mathbb{W}(M)$, the action is
$$(a \cdot m)_k = \psi_k(a)m_k.$$
(This comes from some general fact.)

Definition 13.8. If A is a Λ-ring, I have a unit $\lambda_A : A \to \mathbb{W}(A)$. Then a Λ-*module over A* is
- an A-module M
- an A-linear map $\lambda_M : M \to \mathbb{W}(M)$ (where the A-module structure comes from the $\mathbb{W}(A)$-module structure above and the map λ_A)

such that

(1) $M \xrightarrow{\lambda_M} \mathbb{W}(M) \xrightarrow{\varepsilon} M$ is the identity (here ε comes from the counit for the comonad $\mathbb{W}$)

(2)

$$
\begin{array}{ccc}
M & \longrightarrow & \mathbb{W}(M) \\
\downarrow & & \downarrow{\scriptstyle \mathbb{W}(\lambda_M)} \\
\mathbb{W}(M) & \xrightarrow{\Delta} & \mathbb{W}(\mathbb{W}(M))
\end{array}
$$

(Here Δ is from the comonad structure on $\mathbb{W}$.) This ensures that $A \oplus M$ has a Λ-ring structure.

(This just means that M is a nonunital Λ-ring.)

Proposition 13.9. *If A is an $\mathbb{F}_1$-algebra, then*
$$
\mathrm{Mod}^\Lambda(A) = \mathrm{Mod}^{\mathbb{F}_1}(A)
$$
where the first $\mathrm{Mod}(A)$ is Λ-modules on A and the second one is Beck-Quillen modules.

Moreover, you get

$$
\begin{array}{ccc}
\mathrm{Mod}^\Lambda(A) & =\!=\!= & \mathrm{Mod}^{\mathbb{F}_1}(A)^{Beck} \\
\downarrow & & \downarrow \\
\mathrm{Mod}^{cl}(A) & =\!=\!= & \mathrm{Mod}(A)^{Beck}
\end{array}
$$

PROOF. This is completely formal. Lift the proof of Proposition 13.5.   □

So now I can use $\mathrm{Mod}(A)$ unambiguously.

Corollary 13.10. *Under this equivalence, a derivation from an $\mathbb{F}_1$-algebra to a Λ-module M over A, is a map $d : A \to M$ such that*

(1) $d(a + b) = d(a) + d(b)$

(2) $d(ab) = a\, db + b\, da$

(3) $\lambda_n(da) = \sum_{k|n} \lambda_k(a)^{n/k-1} d(\lambda_k(a))$

I could have taken this as a definition of an $\mathbb{F}_1$-derivation. But I'm showing that this comes from a universal machine.

Suppose $X \in \mathrm{Shv}(\mathbb{F}_1)$ (an object of the étale topos on $\mathbb{F}_1$). We might want to define quasi-coherent sheaves $\mathrm{QCoh}(X)$. You can do this extremely formally:
$$
\mathrm{QCoh}(X) = \lim(((\mathrm{Aff}\ /\mathbb{F}_1)_{/X})^{op} \to \mathrm{Cat})
$$

where the functor we're taking the limit over sends $\operatorname{Spec} A \to X$ to $\operatorname{Mod}(A)$. Here $\operatorname{Aff}/\mathbb{F}_1$ is the opposite to the category of Λ-rings. So you're just right-Kan-extending $\operatorname{Mod}(A)$:

$$(\operatorname{Aff}/\mathbb{F}_1)^{op} \xrightarrow{\operatorname{Mod}} \operatorname{Cat}$$

$$\operatorname{Shv}(\mathbb{F}_1) \qquad \operatorname{QCoh}$$

Aside: the Zariski topology is not the right thing to use. There are no problems defining it, but some things you want to be sheaves, aren't.

LECTURE 14: APRIL 13

I'll define the (big) de Rham-Witt complex (which is actually not a complex), and then see that it arises naturally from TR, a thing you get out of THH and friends. We'll see that the de Rham-Witt complex behaves as a cotangent complex in the $\mathbb{F}_1$-world.

Today I'll try to write out some examples of $\mathbb{F}_1$-rings and modules over them.

Example 14.1. Recall the Chebyshev line; this was defined as $K_0(\operatorname{Rep}(SL(2,\mathbb{C}))) \cong \mathbb{Z}[x]$, with a certain Λ-structure. We expressed

$$\sigma_n(x) = u_n(x/2)$$
$$\psi_p(x) = 2T_p(x/2)$$

(This is a monic version of the usual Chebyshev polynomials.)

Recall: a module over an $\mathbb{F}_1$-algebra A is

- an A-module (in the classical sense)
- $\lambda_M : M \to \mathbb{W}(M)$

Recall the λ_M's had to be A-linear (recall $\mathbb{W}(M)$ is naturally a $\mathbb{W}(A)$-module). What is this really? Look at $A^\Psi \mathbb{N}^\times$, the twisted monoid ring (when you commute one of the monoid generators past one of the elements of A it picks up a ψ). This is a noncommutative ring.

Exercise 14.2. $\operatorname{Mod}_{\mathbb{F}_1}(A) = \operatorname{LMod}(A^\Psi \mathbb{N}^\times)$.

$n \in \mathbb{N}^\times$ acts by picking out the nth component of λ_M.

If $A = \mathbb{F}_1$, then an $\mathbb{F}_1$-module is an abelian group with an action of $\mathbb{N}^\times$. (Don't you need some compatibility with the Adams operations for $\mathbb{F}_1$? Those are trivial!) So the functor $\operatorname{Mod}(\mathbb{F}_1) \to \operatorname{Mod}(\mathbb{Z})$ is just forgetting the $\mathbb{N}^\times$-structure.

Note that this isn't the same $\mathbb{F}_1$ story as the one where $\mathbb{F}_1$-vector spaces are just pointed sets.

Let X be a commutative monoid. Then $\mathbb{F}_1 X$ is just $\mathbb{Z}X$ with $\psi_p(x) = x^p$. This is giving an action $\psi : \mathbb{N}^\times \to \mathrm{End}(X)$. Then

$$\{\mathbb{F}_1 X\text{-modules}\} \cong \{\text{abelian groups with an } X \ltimes_\psi \mathbb{N}^\times\text{-action}\}.$$

In particular, $\mathbb{F}_1 \mathbb{N}_0$-modules are the same as abelian groups M with endomorphisms $\sigma : M \to M$ and $\lambda_p : M \to M$ such that $\lambda_p \circ \sigma = \sigma^p \circ \lambda_p$. And $\mathbb{F}_{1^n}$-modules are abelian groups M with σ, λ_p as above such that σ has order dividing n. We have

$$\mathbb{F}_{1^n} \cong \mathbb{F}_1 C_n$$
$$\overline{\mathbb{F}}_1 \cong \mathbb{F}_1 \mathbb{Q}/\mathbb{Z}$$

If you believe that $\mathbb{F}_1 \mathbb{N}_0$ is the correct affine line, then this is really the algebraic closure (you have a notion of algebraic extensions $\mathbb{F}_1[x]/(f)$ (where f is irreducible and monic), and the system of all of these has $\overline{\mathbb{F}}_1$ as the biggest one).

The map from $\mathbb{F}_1$-modules to $\mathbb{Z}$-modules is just the "forget the Λ-structure" map. To see this, interpret these things as Beck modules:

$$
\begin{array}{ccc}
\mathrm{Ab}(\mathbb{F}_1\text{-Alg}/\mathbb{F}_1) \longrightarrow \mathrm{Ab}(\mathbb{Z}\text{-Alg}/\mathbb{Z}) & \qquad & (\mathbb{Z} \oplus M, \lambda) \longmapsto \mathbb{Z} \oplus M \\
\Vert \qquad\qquad\qquad \Vert & & \downarrow \qquad\qquad\qquad \downarrow \\
\mathbb{F}_1\text{-Mod} \longrightarrow \mathbb{Z}\text{-Mod} & & (M, \lambda_M) \longmapsto M
\end{array}
$$

We can denote this "forget the Λ-structure" functor as $- \otimes_{\mathbb{F}_1} \mathbb{Z}$.

Considering $\mathbb{Z}$ as an $\mathbb{F}_1$-algebra, we can consider $\mathbb{W}(\mathbb{Z})$. What are $\mathbb{W}(\mathbb{Z})$-modules? The claim is that this "had better be" just $\mathbb{Z}$-modules. There's a functor from $\mathbb{Z}$-modules to $\mathbb{W}(\mathbb{Z})$-modules that sends $M \mapsto M \otimes_{\mathbb{Z}} \mathbb{W}(\mathbb{Z})$ where the $\mathbb{N}^\times$ action acts on $\mathbb{W}(\mathbb{Z})$. (This is the free object; the cofree object would be $\mathbb{W}(M)$.) The functor in the opposite direction should be the formula from faithfully flat descent (or, more generally comonadic descent – see a paper by Hess): $N \mapsto \mathrm{coeq}(N \underset{\mathrm{triv}}{\overset{\lambda_N}{\rightrightarrows}} \mathbb{W}(N))$.

$\mathbb{F}_1$-**points.** Let's go back to the case where X is a commutative monoid. We can ask about $\mathrm{Spec}\,\mathbb{F}_{1^n} \to \mathrm{Spec}\,\mathbb{F}_1 X$. These are maps

$$\mathrm{Alg}_{\mathbb{F}_1}(\mathbb{F}_1 X, \mathbb{F}_1 C_n) = \mathrm{Mon}(X, C_{n,+}).$$

For example, $\mathrm{Alg}_{\mathbb{F}_1}(\mathbb{F}_1 \mathbb{N}_0, \mathbb{F}_1 C_n) = C_{n,+}$. This is one way to define $\mathbb{F}_{1^n}$.

What about the other affine line? (Write $\mathbb{A}^1_{Ch}$ to denote it's the Chebyshev one.)

$$\mathbb{A}^1_{Ch}(\mathbb{F}_{1^n}) = \left\{ f \in \mathbb{Z}[t]/(t^n - 1) \ : \ \forall p, f(t^p) = 2T_p(\tfrac{f(t)}{2}) \pmod{t^n - 1} \right\}.$$

I don't know how many of these there are.

Next time, I'll talk about the de Rham-Witt complex and how it relates to the cotangent complex in the $\mathbb{F}_1$ world. Later, I'll relate this to THH and friends.

LECTURE 15: APRIL 20

We've been talking about the functor $R \mapsto \mathbb{W}(R)$ (I mean the big Witt vectors here); this is the comonad corepresented by Λ. A Λ-algebra (a coalgebra for this comonad) is the same thing as an $\mathbb{F}_1$-algebra; we were thinking of this as descent data. But there are other flavors of Witt vectors out there, e.g. p-typical Witt vectors, or p-typical of length n.

There was an awful lot of structure on Λ – a ring structure, a co-ring structure, and a plethysm. But there's even more structure. There's a lot of nice combinatorics that drive this picture, and a lot of it is not written down correctly in the literature.

Hesselholt has the following view of these things – instead of thinking of the single ring $\mathbb{W}(R)$ attached to R, he has a whole family of rings.

Definition 15.1. A *truncation set* is a sieve in the divisibility poset Φ. That is, it's a set $S \subset \Phi$ of natural numbers such that if $n \in S$ and $d \mid n$ then $d \in S$.

Given the multiplication map $n : \Phi \to \Phi$, I can pull back $S \subset$ (the second copy of) Φ to get a sieve we call S/n:

$$
\begin{array}{ccc}
\Phi & \xrightarrow{\ n\ } & \Phi \\
\uparrow & & \uparrow \\
S/n & \longrightarrow & S
\end{array}
$$

S/n is literally the pullback in sets; explicitly, $S/n = \{k \in \mathbb{N} \ : \ kn \in S\}$.

Hesselholt defines $\mathbb{W}_S(R)$ for each S: as a set, $\mathbb{W}_S(A) = A^S$, but with the unique ring structure, natural in A, such that the ghost map $\mathbb{W}_S(A) \xrightarrow{w} A^S$ sending $(a_n)_{n \in S} \mapsto (\sum_{d \mid n} d a_d^{n/d})_n$ is a ring homomorphism (i.e. using the product ring structure on A^S). It is a theorem that these things exist.

Example 15.2. If $S = \{1, p, p^2, \ldots\}$, then this leads to the p-typical Witt vectors. I could stop at p^n, and then you get the n-truncated p-typical Witt vectors.

The construction $\mathbb{W}_S(R)$ comes with the following maps:
- (Restriction) If $T \subset S$, then I have a restriction map $R_T^S : \mathbb{W}_S(R) \to \mathbb{W}_T(R)$.
  This is literally restriction.

- (Frobenius) Given $n \in \mathbb{N}$, there is a ring map $F_n : \mathbb{W}_S(R) \to \mathbb{W}_{S/n}(R)$.
  Recall we have ghost components $\mathbb{W}_S(R) \xrightarrow{w} R^S$. There is a map $F_n^w : R^S \to R^{S/n}$ where $F_n^w(x) = (x_{nd})_{d \in S/n}$. Then F_n is the unique ring map making the following diagram commute:
  $$
  \begin{array}{ccc}
  \mathbb{W}_S(R) & \longrightarrow & \mathbb{W}_{S/n}(R) \\
  \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\
  R^S & \xrightarrow{\ F_n^w\ } & R^{S/n}
  \end{array}
  $$

- (Verschiebung) For every $n \in \mathbb{N}$, there is a (not necessarily ring) map $V_n : \mathbb{W}_{S/n}(R) \to \mathbb{W}_S(R)$.
  Explicitly, $V_n((a_d)_{d \in S/n})$ has m^{th} component a_d if $m = dn$ and 0 otherwise.

There are also a bunch of compatibilities (e.g. what happens when you compose). Hesselholt writes all of this down.

I'd like to package up all this information in a coherent way, by getting an indexing category. I want to minimize the abstract nonsense while getting the naturality across.

Idea: we have $\langle n \rangle = \{d \in \Phi \: : \: d \mid n\}$, the sieve generated by n. Regard $\langle n \rangle$ as the "basic opens" of some space. Then general truncation sets are the general opens. The idea is, if I know $\mathbb{W}_{\langle n \rangle}(R)$ for all n, I should be able to recover $\mathbb{W}_S(R)$ for all S. When $m \mid n$, I have a Frobenius map $F_{m\mid n} : \mathbb{W}_{\langle n \rangle}(R) \to \mathbb{W}_{\langle m \rangle}(R)$. In the other direction we have $V_{m\mid n} : \mathbb{W}_{\langle m \rangle}(R) \to \mathbb{W}_{\langle n \rangle}(R)$.

Given $m \mid n$ and $k \mid n$, you have to make sense of the composite

$$\mathbb{W}_{\langle m \rangle}(R) \overset{V_{m\mid n}}{\to} \mathbb{W}_{\langle n \rangle}(R) \overset{F_{k\mid n}}{\to} \mathbb{W}_{\langle k \rangle}(R).$$

You can write down a diagram
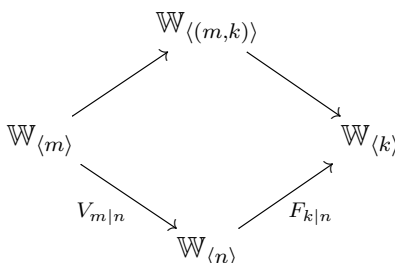
but it doesn't commute. But, the claim is that it almost commutes.

$$F_{k\mid n} V_{m\mid n} = \frac{n}{[m,k]} V_{(m,k)\mid k} F_{(m,k)\mid m}$$

This looks like the restriction-induction formulas in representation theory. Suppose G is a finite group and H_1, H_2 are finite subgroups of K, a finite subgroup of G. Then we have maps of orbits

$$
\begin{array}{ccc}
 & & [G/H_1] \\
 & & \downarrow \\
[G/H_2] & \longrightarrow & [G/K]
\end{array}
$$

The pullback (in the category of G-sets) is not a single orbit; it's a disjoint union of them:

$$\bigsqcup_{g \in H_1 \backslash K / H_2} [G/H_1 \cap {}^g H_2].$$

(Here ${}^g H_2$ means conjugate, but I haven't told you which one...)

If you think of the Witt vectors for each R as an assignment of cyclic groups to abelian groups *(i.e. for every R, to $\langle n \rangle$ you assign $\mathbb{W}_{\langle n \rangle}(R)$)*, then you want to see this kind of decomposition.

The idea is:

$$
\begin{array}{ccc}
\bigsqcup [G/H_1 \cap {}^g H_2] & \overset{Frobenius}{\longrightarrow} & [G/H_1] \\
{\scriptstyle Verschiebung} \uparrow & & \uparrow {\scriptstyle Verschiebung} \\
[G/H_2] & \underset{Frobenius}{\longrightarrow} & G[K]
\end{array}
$$

where the red maps (Verschiebung) are the "wrong way" maps. We want to achieve a similar diagram

$$\begin{array}{ccc} \bigsqcup ??? & \longrightarrow & \mathbb{W}_{\langle k \rangle} \\ \uparrow & & \uparrow {\scriptstyle F} \\ \mathbb{W}_{\langle m \rangle} & \xrightarrow[V]{} & \mathbb{W}_{\langle n \rangle} \end{array}$$

Let $C = \mathbb{Q}/\mathbb{Z}$. Let $\mathcal{O}_C$ be the category of C-orbits of the form $\mathbb{Q}/\frac{1}{m}\mathbb{Z} = \langle m \rangle$. Notice that I'm indexing this using the size of the stabilizer: $\mathbb{Q}/\frac{1}{m}\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})/(\frac{1}{m}\mathbb{Z}/\mathbb{Z})$.

Consider formal products of the form $N = \prod_{p \text{ prime}} p^{v_p(N)}$ where $v_p(N) \in \mathbb{N}_0 \cup \{+\infty\}$, and define $C_N = \frac{1}{N}\mathbb{Z}/\mathbb{Z}$. For example, if N is finite, you get the finite cyclic group on N elements, but we also have $C_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}_p$ and $C_\infty = C$.

In the group example, if we tried to take a pullback in the category of single orbits, we would have failed – you need general G-sets (disjoint unions of orbits). Similarly, in the $\mathbb{W}$ example we need more things than just $\langle n \rangle$.

Pass to F_{C_N}, the category of C_N-sets with finitely many orbits, all of the form $\langle m \rangle_N = C_N/(\frac{1}{m}\mathbb{Z}/\mathbb{Z})$ (this is the same as saying the stabilizers are finite).
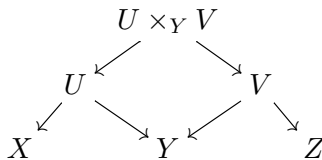
The Witt vector construction $R \mapsto (\langle n \rangle \mapsto \mathbb{W}_{\langle n \rangle}(R))$ is

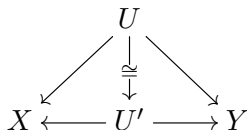$$\mathbb{W} : \text{Ring} \to \text{BiFun}(F_{C_N}, \text{Ab}).$$

(Bifunctors are both covariant and contravariant.) The Frobenius and Verschiebung blend in a way that gives a pullback diagram. We're going to construct a yet bigger category than F_{C_N} to contain our pullback.

Definition 15.3. The effective Burnside category $A^{\text{eff}}(F_{C_N})$ is described as follows:

- objects are the objects of F_{C_N}

- 1-morphisms from X to Y are spans $X \leftarrow U \to Y$

- Composition is given by pullbacks:

$$\begin{array}{ccccc} & & U \times_Y V & & \\ & \swarrow & & \searrow & \\ & U & & V & \\ \swarrow & & \searrow \swarrow & & \searrow \\ X & & Y & & Z \end{array}$$

- 2-isomorphisms from $X \leftarrow U \to Y$ to $X \leftarrow U' \to Y$ are diagrams

$$\begin{array}{ccc} & U & \\ \swarrow & {\scriptstyle \cong}\downarrow & \searrow \\ X \longleftarrow & U' & \longrightarrow Y \end{array}$$

This is a $(2,1)$-category – a category enriched in groupoids.

Ordinarily I would just quotient out by these isomorphisms. But I'm about to enlarge F_{C_N} into a 2-category, and then I'll need to use this extra data.

Now our Witt vector construction can be seen as a functor

$$\mathbb{W} : \mathrm{Ring} \to \mathrm{Fun}(A^{\mathrm{eff}}(F_C), \mathrm{Ab}).$$

But I want orbits to be running the show. A disjoint union of orbits gives a product on A^{eff}. But A^{eff} is its own opposite. So now Witt vectors are

$$\mathbb{W} : \mathrm{Ring} \to \mathrm{Fun}^{\oplus}(A^{\mathrm{eff}}(F_C), \mathrm{Ab}).$$

(I.e. these are additive functors – they preserve direct sums. This means $\mathbb{W}_{\langle m \rangle \sqcup \langle n \rangle}(A) = \mathbb{W}_{\langle m \rangle}(A) \oplus \mathbb{W}_{\langle n \rangle}(A)$.)

But I have an issue: $\mathrm{Fun}^{\oplus}(A^{\mathrm{eff}}, \mathrm{Ab})$ is too much structure – it suggests you have an action of the stabilizer on everything. There's no action like this on actual Witt vectors.

We took the orbit category $\mathcal{O}_{C_N}$ and added formal coproducts to get to F_{C_N}. Then we formed the effective Burnside category. I'm going to define $\mathcal{O}_{\mathbb{C}_N}$ such that we get analogous categories

$$
\begin{array}{ccc}
\mathcal{O}_{C_N} & \hookrightarrow & \mathcal{O}_{\mathbb{C}_N} \\
\downarrow & & \downarrow \\
F_{C_N} & \hookrightarrow & F_{\mathbb{C}_N} \\
\downarrow & & \downarrow \\
A^{\mathrm{eff}}(F_{C_N}) & \hookrightarrow & A^{\mathrm{eff}}(F_{\mathbb{C}_N})
\end{array}
$$

What is $\mathcal{O}_{\mathbb{C}_N}$? The objects and 1-morphisms are the same as those in $\mathcal{O}_{C_N}$. A 2-isomorphism between $u, v : \langle m \rangle_N \rightrightarrows \langle n \rangle_N$ is an *intertwiner*, an element $r \in \frac{1}{N}\mathbb{Z}$ such that $v(t) = r + u(t)$ (mod $\frac{1}{n}\mathbb{Z}$). This takes the generator for $\langle n \rangle_N = (\frac{1}{N}\mathbb{Z}/\mathbb{Z})/(\frac{1}{n}\mathbb{Z}/Z)$. and contracts it to the identity.

Example 15.4. Take $N = 1$. Then $\mathcal{O}_{C_1}$ has one object and one morphism. What's $\mathcal{O}_{\mathbb{C}_1}$? It has one object and one 1-morphism, and a $\mathbb{Z}$'s worth of 2-morphisms. This is sometimes known as $BB\mathbb{Z} = BS^1 = \mathbb{C}\mathrm{P}^{\infty}$. (We're identifying 2-types (spaces with no $\pi_{>2}$) with 2-groupoids, and will continue to do that without fear.)

LECTURE 16: APRIL 25

No class Thursday.

We've constructed this cyclonic orbit category, which is supposed to pick up some of the maps we're seeing in the Witt vectors. We were looking at $\mathbb{W}_{\langle n \rangle}(R)$.

We've produced this (2,1)-category $\mathcal{O}_{\mathbb{C}}$ that we call the *cyclonic orbit category*. We took a category of $\mathbb{Q}/\mathbb{Z}$-orbits and enlarged it by introducing some 2-morphisms which formally interpolated between the generator of a cyclic group and the identity. For any supernatural

number N we had the N-cyclonic category $\mathcal{O}_{\mathbb{C}_N}$. We saw last time that $\mathcal{O}_{\mathbb{C}_1} = BB\mathbb{Z}$. The picture is the following.

Idea: there's some stratified space $X_{\mathbb{C}}$ such that constructible sheaves[3] $\mathrm{Cnstr}(X_{\mathbb{C}}, \mathrm{Set}) \simeq \mathrm{Fun}(\mathcal{O}_{\mathbb{C}}^{op}, \mathrm{Set})$. Recall we had a procedure

$$\mathcal{O}_{\mathbb{C}} \hookrightarrow F_{\mathbb{C}} \hookrightarrow A^{\mathrm{eff}}(F_{\mathbb{C}})$$

where the objects of A^{eff} are finite disjoint unions of things in $\mathcal{O}_{\mathbb{C}}$ and the morphisms are spans. This is picking up some basic structure we saw on the Witt vectors.

The Witt vectors

$$\mathbb{W}_{(-)}(R) : A^{\mathrm{eff}}(F_{\mathbb{C}}) \to \mathrm{Ab}$$

is a Mackey functor for each value of $-$: it carries direct sums to direct sums. In addition to direct sums, I have try to product together cyclonic sets. Problem: there's no terminal object. But you can pretend as if the product existed: looking at $X \times Y \leftarrow U \to Z$ is the same as the collection of maps $(U \to X, U \to Y, U \to Z)$. So I don't know what it means to have a product of $X \times Y$ but I still know how to talk about maps from $X \times Y$ to Z. This gives the structure of of a symmetric promonoidal category. This is exactly the structure you get on a subcategory of a symmetric monoidal category (which might not preserve $\otimes$, but embeds fully faithfully as a multicategory into a symmetric monoidal category).

This is not all the structure on the Witt vectors, though. We're going to see an action of the monoid $\mathbb{N}$ on $\mathcal{O}_{\mathbb{C}}$ which will extend to an action on all of A^{eff}. Then you'll see that $\mathbb{W}$ is naturally fixed points for that action. Under this analogy, $\mathbb{W}$ is more than just a constructable sheaf – it's a bi-constructable sheaf. You're supposed to be rigging it so that $\mathrm{Bicnstr}(X_{\mathbb{C}}, \mathrm{Ab}) \simeq \mathrm{Mack}(F_{\mathbb{C}}, \mathrm{Ab})$.[4] But actually what we want is $\mathrm{Mack}(F_{\mathbb{C}}, \mathrm{Ab})^{\mathbb{N}^{\times}}$ (this is fixed points in the categorical sense – objects that are fixed up to some equivalence). You should think of this as

$$\mathrm{Bicnstr}(X_{\mathbb{C}}/\mathbb{N}^{\times}, \mathrm{Ab}) \simeq \mathrm{Mack}(F_{\mathbb{C}}, \mathrm{Ab})^{\mathbb{N}^{\times}}$$

where $X_{\mathbb{C}}/\mathbb{N}^{\times}$ is now some stack. The opens are the truncation sets.

The idea is that the orbit category $\mathcal{O}_{\mathbb{C}}$ is completely running the show. There is a map $\mathcal{O}_{\mathbb{C}} \to \Phi$ ($\mathbb{N}$ ordered by divisibility); you're supposed to think of this as giving rise to a stratification of a space attached to $\mathcal{O}_{\mathbb{C}}$.

If C and D are symmetric monoidal and D has all colimits and $\otimes_D$ preserves them separately in each variable, then $\mathrm{Fun}(C, D)$ has a natural symmetric monoidal structure. Suppose $X, Y \in \mathrm{Fun}(C, D)$; I need to define $(X \otimes Y)(x)$ for $x \in C$. Define

$$(X \otimes Y)(x) = \mathrm{colim}_{a \otimes b \to x} X(a) \otimes_D Y(b) = \int^{a,b \in C} \mathrm{Hom}(a \otimes b, x) \otimes X(a) \otimes_D Y(b)$$

---

[3] When you restrict to the strata, you get locally constant sheaves. I also need a convergence hypothesis.

[4] Recall $\mathrm{Mack}(F_{\mathbb{C}}, \mathrm{Ab}) = \mathrm{Fun}^{\oplus}(A^{\mathrm{eff}}(F_{\mathbb{C}}), \mathrm{Ab})$ (these are both covariant and contravariant).

(the first $\otimes$ in the coend is a co-power). Equivalently, I could take the Kan extension of

$$
\begin{array}{ccc}
C \times C & \xrightarrow{(X,Y)} & D \times D \\
\scriptstyle{\otimes_C}\downarrow & \swarrow & \downarrow\scriptstyle{\otimes_D} \\
C & \xrightarrow[X \otimes Y]{} & D
\end{array}
$$

The following exercise is genuinely unfair.

Exercise 16.1. The unit in $\mathrm{Mack}(A^{\mathrm{eff}}(F_{\circledcirc}), \mathrm{Ab})$ is $\mathbb{W}(\mathbb{Z})$. (I only know really expensive proofs of this.)

PROOF. Nothing I've done relies on the fact I'm working with Ab; I can replace it with the ∞-category of spectra and everything will be the same. There, I know by Barratt-Priddy-Quillen that the unit is the (cyclonic) sphere spectrum. I can take π_0 of this and get the Grothendieck groups that appear in $\mathbb{W}(\mathbb{Z})$. There's an old paper of Dress-Siebeneicher from the 70's that (presumably) gives a more reasonable proof. Dress invented Mackey functors. $\square$

This all seems excessive, but I'm telling a story with an analogue in spectra, and then it's related to THH.

First, for any $n \in \mathbb{N}$ define $\iota_n : \mathcal{O}_{\circledcirc} \to \mathcal{O}_{\circledcirc}$ by sending $\langle m \rangle \mapsto \langle mn \rangle$. On objects, this is pullback along $\mathbb{Q}/\mathbb{Z} \xrightarrow{-\times n} \mathbb{Q}/\mathbb{Z}$. Suppose we have a 2-morphism between two morphisms $u, v : \langle m \rangle \to \langle m' \rangle$ (a rational number that intertwines these maps); send this to the 2-morphism given by $\frac{r}{n}$.

Fact 16.2. ι_n is fully faithful, and extends to a fully faithful functor $\iota_n : F_{\circledcirc} \to F_{\circledcirc}$ preserving finite coproducts.

ι_n admits a right adjoint p_n, with

$$
p_n(\langle m \rangle) = \begin{cases} \langle \frac{m}{n} \rangle & \text{if } n \mid m \\ \emptyset & \text{otherwise.} \end{cases}
$$

(Note $\emptyset$ isn't an orbit but it exists in $F_{\circledcirc}$.) This is fully faithful; $p_n \iota_n = \mathbb{1}$. This exhibits $F_{\circledcirc}$ as a localization of itself.

Furthermore, both ι_n and p_n preserve pullbacks (p_n does because it's a right adjoint; ι_n you have to check, and this is exactly where we got the definition on 2-morphisms). They also preserve finite coproducts. Then you get

$$
A^{\mathrm{eff}}(\iota_n) : A^{\mathrm{eff}}(F_{\circledcirc}) \to A^{\mathrm{eff}}(F_{\circledcirc})
$$

$$
A^{\mathrm{eff}}(p_n) : A^{\mathrm{eff}}(F_{\circledcirc}) \to A^{\mathrm{eff}}(F_{\circledcirc})
$$

These are not adjoint anymore, but $A^{\mathrm{eff}}(p_n) A^{\mathrm{eff}}(\iota_n) = \mathbb{1}$. It's no longer a localization, but it is a retraction.

(The "effective" in A^{eff} is a reference to effective divisors in all divisors.)

Now let's pass to Mackey functors. These functors preserve direct sums, and pulling back a Mackey functor along such a functor is still a Mackey functor:

$$i_{n,*} \operatorname{Mack}(F_{\mathbb{C}}, \operatorname{Ab}) \overset{A^{\operatorname{eff}}(p_n)^*}{\to} \operatorname{Mack}(F_{\mathbb{C}}, \operatorname{Ab}).$$

I'm calling this $i_{n,*}$ because it corresponds to the pushforward along a closed immersion.

Recollement. We're thinking about the covering of a space by a closed piece and its open complement. There are a bunch of adjunctions relating constructible sheaves on the pieces and on the whole thing. Convention: $i : Z \hookrightarrow X$ is the closed piece, and $j : U = X \backslash Z \hookrightarrow X$ is the open complement.

Which things are adjoint? $i^* \dashv i_* \dashv i^!$ *and* $j_! \dashv j^* \dashv j_*$. This is a recollement if

(1) i_* and j_* are fully faithful. (Exercise: j_* is fully faithful iff $j_!$ is fully faithful.)

(2) $j^* i_* = 0$ (we're talking about sheaves of vector spaces here)

(3) $\operatorname{Sh}(Z) = \ker(j^*)$

Suppose you're in this situation and you want to specify a sheaf on X, but all you have is a sheaf $\mathcal{F}_Z$ on Z and a sheaf $\mathcal{F}_U$ on U. This is sufficient, provided you add the data of a gluing map $\mathcal{F}_Z \to i^* j_* \mathcal{F}_U$. I'm saying that $\operatorname{Sh}(X)$ is equivalent to triples consisting of a sheaf on Z, a sheaf on U, and a map as mentioned. This is sometimes called *Artin gluing*. Reference: Beilinson, Bernstein, and Deligne on perverse sheaves.

There's a wrong way recollement in algebraic geometry. Let X be a nice scheme, $Z \subset X$ a closed immersion, and $U = X \backslash Z \subset X$ the open complement. We'll be working with the derived categories of quasi-coherent sheaves:

"X on Z" means quasi-coherent sheaves on X whose set-theoretic support is on Z. There are two ways to embed this into $D_{qc}(X)$ – as torsion objects (left adjoint), or as complete objects (right adjoint). For example, the category of p-torsion abelian groups is equivalent to the category of p-complete abelian groups. In our case, j_* is fully faithful. It has a left adjoint j^* and a right adjoint $j^\times$. This is the one that shows up in K-theory (localization sequences); the other one is the one that shows up in constructible sheaves. They are Tannaka dual pictures.

I have a functor

$$i_{n,*} : \operatorname{Mack}(F_{\mathbb{C}}, \operatorname{Ab}) \to \operatorname{Mack}(F_{\mathbb{C}}, \operatorname{Ab})$$

which I want to think of as pushforward along a closed immersion $F_{\mathbb{C}} \to F_{\mathbb{C}}$; this is pullback $- \circ A^{\operatorname{eff}}(p_n)$. This has adjoints i_n^*, $i_n^!$ which are left and right Kan extensions, respectively.

In the following, if you don't like spectra you can imagine the dg category of quasi-coherent sheaves on a scheme. Let D be the ∞-category of spectra (or this replacement, if you want).

i_n^* is a very important functor; in homotopy theory, this is geometric fixed points Φ^{C_n}. (I.e. $\mathrm{Mack}(F_{\copyright}, D)$ agrees with the homotopy theory of S^1-spectra.) I need the j's to get Artin gluing off the ground.

Let p be prime. Recall $\iota_n : \mathcal{O}_{\copyright} \hookrightarrow \mathcal{O}_{\copyright}$; this is the embedding of a cosieve. There's a complementary sieve $j_p : \mathcal{O}_{\copyright_{p'}} \hookrightarrow \mathcal{O}_{\copyright}$, where $\mathcal{O}_{\copyright_{p'}}$ is the full subcategory spanned by $\langle m \rangle$'s such that $p \nmid m$.

We can extend this to a functor $j_p : F_{\copyright_{p'}} \to F_{\copyright}$ which is also fully faithful. This preserves pullbacks, and I defined it to preserve coproducts. So it's one of the functors we can take A^{eff} of:
$$A^{\mathrm{eff}}(j_p) : A^{\mathrm{eff}}(F_{\copyright_{p'}}) \to A^{\mathrm{eff}}(F_{\copyright}).$$
This is a direct-sum-preserving functor.

I can pre-compose

$$\mathrm{Mack}(F_{\copyright}, D) \xrightarrow[\overset{j_{p,!}}{\overset{}{\longleftarrow}}]{\underset{j_{p,*}}{\overset{j_p^*}{\longrightarrow}}} \mathrm{Mack}(F_{\copyright_{p'}}, D)$$

Theorem 16.3. *These functors define a recollement*

$$\mathrm{Mack}(F_{\copyright}, D) \xrightarrow[\overset{i_n^*}{\overset{}{\longleftarrow}}]{\underset{i_n^!}{\overset{i_{n,*}}{\longrightarrow}}} \mathrm{Mack}(F_{\copyright}, D) \xrightarrow[\overset{j_{p,!}}{\overset{}{\longleftarrow}}]{\underset{j_{p,*}}{\overset{j_p^*}{\longrightarrow}}} \mathrm{Mack}(F_{\copyright_{p'}}, D)$$

This is looking like a topology you can imagine on the divisibility poset: the basic opens are the sieves. What we'll say is that it's really the case that you can glue together Mackey functors on the pieces to get Mackey functors on the whole thing.

LECTURE 17: MAY 2

So far: we've introduced the cyclonic orbit category $\mathcal{O}_{\copyright}$ by taking the category of $\mathbb{Q}/\mathbb{Z}$-sets with finitely many orbits and finite stabilizers, and introducing some 2-morphisms. Then we enlarge this in the same way we enlarge the orbit category of a finite group, first to get the category $F_{\copyright}$ of finite cyclonic sets, and then further to the effective Burnside category $A^{\mathrm{eff}}(F_{\copyright})$. Then we talked about Mackey functors $\mathrm{Mack}(F_{\copyright}, D)$ for this category (here D is the ∞-category of spectra or the DG category of complexes of quasi-coherent sheaves over your favorite variety) – direct-sum preserving functors $A^{\mathrm{eff}}(F_{\copyright}) \to D$.

We also wanted to look at the action of $\mathbb{N}^{\times}$ on this category. This gave rise to an action on $A^{\mathrm{eff}}(F_{\copyright})$, which gave rise to an action on $\mathrm{Mack}(F_{\copyright}, D)$.

Definition 17.1. Cyclotomic objects of D are objects of $\text{Mack}(F_{\circledcirc}, D)^{h\mathbb{N}^\times}$. A cyclotomic object of D is the following data:

- a Mackey functor $M : A^{\text{eff}}(F_{\circledcirc}) \to D$

- for all primes p, an identification $i_p^* M \stackrel{\theta_p}{\cong} M$ satisfying the obvious relations (using the fact that i_p and i_q commute). ($\mathbb{N}^\times$ is freely generated as a commutative monoid by the p's, so this is all you have to write down.)

Recall we had a recollement, where the open piece came from the "p part", and the closed piece came from the "prime-to-p part"

$$\text{Mack}(F_{\circledcirc}, D) \xrightarrow[]{\overset{i_p^*}{\longleftarrow}\ \ \overset{i_{p,*}=i_{p,!}}{\longrightarrow}\ \ \underset{i_p^!}{\longleftarrow}} \text{Mack}(F_{\circledcirc}, D) \xrightarrow[]{\overset{j_{p,!}}{\longleftarrow}\ \ \overset{j_p^*=j_p^!}{\longrightarrow}\ \ \underset{j_{p,*}}{\longleftarrow}} \text{Mack}(F_{\circledcirc_{p'}}, D)$$

The fact that this is a recollement is that the following

$$j_{p,!} j_p^! M \to M \to i_{p,*} i_p^* M$$

is a fiber sequence (distinguished triangle). (This expresses the fact that M can be recovered from the pieces.) In homotopy theory, this is just the isotropy separation sequence. (This isn't hard to prove, because the i^* thing gives geometric fixed points.) Given a normal subgroup $H < G$, you can run this to get a recollement

$$\text{Mack}(G/H) \xrightarrow{\longleftarrow\ \longrightarrow\ \longleftarrow} \text{Mack}(G) \xrightarrow{\longleftarrow\ \longrightarrow\ \longleftarrow} \text{Mack}(H)$$

But this also makes sense for the derived category of abelian groups. If you take H_0 (project everything to the heart), this distinguished triangle becomes a right exact sequence. That is, if $X \in \text{Mack}(F_{\circledcirc}, \text{Ab})$, we get an exact sequence

$$j_{p,!} j_p^! X \stackrel{\lambda_p}{\to} X \to i_{p,*} i_p^* X \to 0.$$

Let's talk about $(j_{p,!} j_p^! X) \langle n \rangle$. Actually, factor $n = mp^v$ where $(m, p) = 1$. Then you have

$$(j_{p,!} j_p^! X) \langle mp^v \rangle = X \langle m \rangle.$$

Given this, $\lambda_p : X \langle m \rangle \to X \langle mp^v \rangle$ is just the Verschiebung $\varphi_{m|mp^v,*}$, the pushforward functor in the Mackey functor you get from the divisibility relation $m \mid mp^v$. So:

$$i_{p,*} i_p^* X \langle mp^v \rangle = X \langle mp^v \rangle / \varphi_*(X \langle m \rangle).$$

Exercise 17.2. A cyclotomic structure on X is the following data:

- for each prime p, an isomorphism $r_p : X \stackrel{\cong}{\to} i_p^* X$, such that for any pair of primes p_1, p_2, the following diagram commutes:

$$\begin{array}{ccc} X & \longrightarrow & i_{p_1}^* X \\ \downarrow & & \downarrow \\ i_{p_2}^* X & \longrightarrow & i_{p_1 p_2}^* X \end{array}$$

Since $i_p^* X \langle mp^v \rangle = X \langle mp^{v+1} \rangle / X \langle m \rangle$, this is giving the data of identifications $X \langle mp^v \rangle \cong X \langle p^{v+1} m \rangle / X \langle m \rangle$ compatible with both Frobenius and Verschiebung.

Write $n = mp$, where $m = ap^v$. Then I can form

$$\rho_{m|n} : X\langle n \rangle \to X\langle n \rangle / X\langle a \rangle \xrightarrow{r_p^{-1}} X\langle m \rangle.$$

This composition gives a restriction map.

Recall: an open $U \subset \mathcal{O}_{\circledcirc}$ is a sieve. (You're specifying a full subcategory (a collection of integers) which is closed under the property that if $d \mid n$ and n is in the sieve, then d is in the sieve.) These opens are the same as truncation sets in the sense of Hesselholt. For a fixed base ring R (which will eventually be $\mathbb{Z}$), look at the sheaf

$$\mathrm{Opens}(\mathcal{O}_{\circledcirc})^{op} \to \mathrm{Ab}$$

sending $S \mapsto \mathbb{W}_S(R)$. If $S' \subset S$, the corresponding map $\mathbb{W}_S(R) \xrightarrow{\rho} \mathbb{W}_{S'}(R)$ is our restriction map as long as S is some $\langle n \rangle$. But every other truncation set is a union of $\langle n \rangle$'s. Because rings of Witt vectors of rings, we can think of this sheaf as valued in rings instead of just Ab. The restrictions ρ and the Frobenius are ring maps (but the Verschiebung is not). This is most of the structure on the Witt vectors – there's also that plethysm, and that can also be injected into this picture. But for now this is enough for me to tell you what the Witt complex is. (It's not actually a complex...)

Definition 17.3. A *Witt complex* (over a ring R) is a sheaf

$$\mathrm{Opens}(\mathcal{O}_{\circledcirc})^{op} \xrightarrow{E^{\bullet}} \{\text{anticommutative}^5 \text{ graded rings}\}$$

along with a natural ring map $\eta : \mathbb{W} \to E^0$, and natural (w.r.t. restriction maps) graded abelian group maps

$$d : E^* \to E^{*+1} \qquad\qquad F_n : E^*_S \to E^*_{S/n} \qquad\qquad V_n : E^*_{S/n} \to E^*_S$$

such that:

(1) for all $x \in E^q_S$, $x' \in E^{q'}_S$,

$$d(xx') = d(x)x' + (-1)^q x d(x')$$
$$d(d(x)) = d\log \eta_S([-1]_S) d(x)$$

where $d\log \eta_S[-1]_S = \eta_S([-1]_S)^{-1} d\eta_S([-1]_S)$ (the brackets are referring to the Witt vectors so you get a Teichmüller lift).

(2) For $m, n \in \mathbb{N}$:

$$F_1 = V_1 = \mathbb{1}$$
$$F_m F_n = F_{mn} \qquad\qquad V_m V_n = V_{mn}$$
$$F_n V_n = n\mathbb{1} \qquad\qquad F_m V_n = V_n F_m \text{ if } (m,n) = 1$$
$$F_n \eta_S = \eta_{S/n} F_n \qquad\qquad \eta_S V_n = V_n \eta_{S/n}$$

(3) F_n is a ring map and V_n is a module map: there is a projection formula just like in algebraic geometry: for $x \in E^q_S$ and $y \in E^{q'}_{S/n}$ we have

$$x V_n(y) = V_n(F_n(x)y).$$

(4) $F_n d V_n(y) = d(y) + (n-1)d\log \eta_{S/n}([-1]_{S/n})y$

(5) $F_n d\eta_S([a]_S) = \eta_{S/n}([a]^{n-1}_{S/n}) d\eta_{S/n}([a]_{S/n})$

The idea is that this relates to cyclotomic things.

Cyclotomic objects	Witt complexes
opens of $\mathcal{O}_{\mathbb{C}}$	truncation sets
pushforwards $\varphi_{m\mid n,*}$	Verschiebung maps V_n
pullbacks $\varphi_{m\mid n}^*$	Frobenius maps F_n
restrictions $\rho_{m\mid n}$ coming from the $\mathbb{N}^\times$ action	restrictions
residual S^1 action	differential

Note that the differential is the only thing that needed structure beyond just $\mathbb{Q}/\mathbb{Z}$-sets.

Claim 17.4. *The homotopy groups of a cyclotomic spectrum inherit the structure of a Witt complex.*

Remark 17.5. A Witt complex is a complex if one of:

  (1) 2 is invertible in R

  (2) $2 = 0$ in R

  (3) if you restrict to 2-primary truncation sets (the only n's in the truncation sets are odd)

Definition 17.6. The big de Rham-Witt complex over R is the initial object in the category of Witt complexes over R.

This is quite similar to the definition given in Deligne-Illusie. But there, they're working in a p-primary situation (e.g. the Witt vectors of a characteristic-p field) so they're not so worried about the fact that the differential isn't a differential, and there are fewer error terms.

The idea is that, via Claim 17.4, the homotopy groups of something related to THH are the de Rham-Witt complex.

LECTURE 18: MAY 4

Last time we stated a relationship between cyclotomic spectra and Witt complexes. In these last lectures my goal is to connect the $\mathbb{F}_1$ stuff with the first half of the course about Tate's thesis.

The idea is the following. If $X \to \operatorname{Spec} \mathbb{Z}$ is a smooth and projective variety, then Serre defined local L-factors for the motive $h^w(X)$ (this is just a formal piece of notation at this point). Here w is a weight. These are familiar things at the finite places, but more complicated at the archimedean primes, where these factors are given as Γ-functions. The product of all of these should be a completed L-function $\Lambda(k^w(X)) = \prod_v L_v(h^w(X))$, which should have an

analytic continuation to a meromorphic function on $\mathbb{C}$, and a nice functional equation for the replacement $s \mapsto w + 1 - s$.

This picture works perfectly well when we work with varieties over finite fields. In particular, we have expressions of the L-factors using various kinds of cohomology theories – ℓ-adic and crystalline. As well we have functional equations which come from Poincaré duality for the aforementioned cohomology theories. So we know how to deal with $\mathbb{F}_q$, and you're supposed to contemplate $q \to 1$.

This goes back to an idea of Deninger (1991), who tried to get an expression of $L_v(h^w(X))$ (in particular, the ones at infinity) in terms of a cohomology theory. The L-factors in the $\mathbb{F}_q$ case came from crystalline cohomology, the characteristic-p version of de Rham cohomology. So the question is: what is de Rham cohomology over $\mathbb{F}_1$? We now have a candidate for this; it's something extracted from THH (or, equivalently, the big de Rham Witt complex $\mathbb{W}\Omega$).

Why is this a reasonable candidate for de Rham cohomology? Last time we talked about the "complex" $\mathbb{W}\Omega$. From our point of view, $\mathbb{F}_1$-algebras are the same thing as Λ-rings, and we talked about Λ-modules, and this gave a notion of differentials. You can exteriorize this to get the de Rham complex. You get a map $\Omega^\bullet_{\mathbb{W}(\mathbb{Z})} \to \mathbb{W}\Omega^\bullet_{\mathbb{Z}}$ which is essentially a quotient map; the point is that it "forces good behaviour w.r.t. the Frobenius and Verschiebung".

The conjecture is that, just thinking about $\mathbb{F}_1$-algebras as Λ-rings on their own is not enough. This is a coalgebra for a plethory, and maybe you need more structure on the plethory. So at the moment, we don't have a picture that presents $\mathbb{W}\Omega^\bullet_{\mathbb{Z}}$ as the de Rham complex over $\mathbb{F}_1$. Instead, we have indications that this is close to being the right thing.

Recall Γ functions have infinitely many poles. That tells you that the cohomology theory has to be infinite-dimensional – otherwise there would be finitely many poles. Deninger presented a candidate for this cohomology theory by picking up a summand of Deligne cohomology. It turns out that works perfectly at complex places, but it's a little awkward at real places. To fix up this discrepancy, Connes and Consani proved:

Theorem 18.1 (Connes-Consani). *Let K be a number field. Suppose we have a smooth and projective variety $X \to \operatorname{Spec} \mathcal{O}_K$.*

$$\prod_{0 \le w \le 2d} L_v(h^w(X), s)^{(-1)^{w+1}} = \frac{\det_\infty \left(\frac{1}{2\pi}(s - \Theta)\big|_{HC^{ar}_{even}(X_v)} \right)}{\det_\infty \left(\frac{1}{2\pi}(s - \Theta)\big|_{HC^{ar}_{odd}(X_v)} \right)}$$

Here HC is cyclic homology, and $\det_\infty$ is a regularized determinant (a determinant on a countable dimensional vector space).

If this were the correct cohomology theory, we'd expect this formula, but the formula is true regardless. Being right would include having everything that you have over finite fields – including the Riemann hypothesis!

Theorem 18.2 (Hesselholt). *Let X be smooth and proper over $\mathbb{F}_q$. Choose an embedding $\mathbb{W}(\mathbb{F}_q) \hookrightarrow \mathbb{C}$.*

$$\zeta(X,s) = \frac{\det_\infty\left(\left.(s-\Theta)\right|_{TP_{odd(X)}} \otimes_\mathbb{W} \mathbb{C}\right)}{\det_\infty\left(\left.(s-\Theta)\right|_{TP_{even(X)}} \otimes_\mathbb{W} \mathbb{C}\right)} = \frac{\det(\mathbb{1} - q^{-s}Fr_q^*|_{H_{cry}^{odd}})}{\det(\mathbb{1} - q^{-s}Fr_q^*|_{H_{cry}^{even}})}.$$

Here $TP_*(X)$ is the Tate cohomology (homotopy of the Tate construction for S^1, or equivalently (since we're working p-completely) the limit of the Tate constructions for all the C_{p^r}'s) of $THH(X)$.
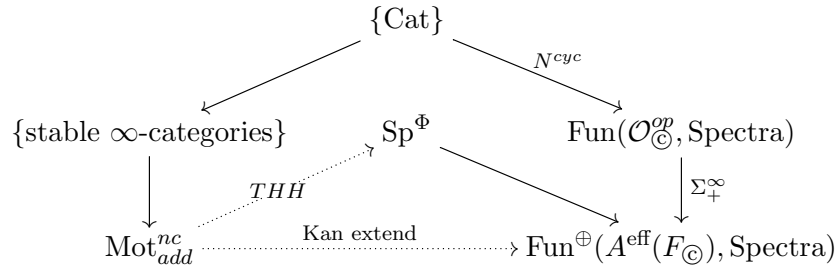
HC is like rationalized THH.

I hope this is sufficient motivation for studying THH and its friends TP etc. The presentation here is supposed to be visibly free of choices.

What is THH?

(1) $THH(X)$ is a cyclotomic spectrum.

(2) $THH(X)$ should receive cycle class maps from $K(X)$.

(3) $THH(X)$ should depend only on the "category of modules" $\mathrm{Perf}(X)$ (perfect complexes), because that's what happens to K-theory and I want to think of THH as a repository of cycle class maps from K-theory.

THH will be a cyclonic spectrum, and I'll define Tate cohomology using recollements.

Let me give you an 85% Tabuada-esque description of THH. (Aside: what I'm going to describe won't work if k-linearized – I can't work over $\mathbb{Z}$, only over the sphere spectrum.) Take an idempotent-complete stable ∞-category and attach a noncommutative motive Mot_{add}^{nc}. Given any category and make the free stable ∞-category generated by it (left adjoint to the obvious forgetful functor). Given that category I can also take the cyclic nerve N^{cyc}; this will be an object of $\mathrm{Fun}(\mathcal{O}_\mathbb{C}^{op}, \mathrm{Spaces})$. Define THH as the lift in the following diagram:

$$
\begin{array}{ccc}
& \{\mathrm{Cat}\} & \\
\swarrow & & \searrow {\scriptstyle N^{cyc}} \\
\{\text{stable } \infty\text{-categories}\} \quad \mathrm{Sp}^\Phi & & \mathrm{Fun}(\mathcal{O}_\mathbb{C}^{op}, \mathrm{Spectra}) \\
\downarrow \quad {\scriptstyle THH} \nearrow & & \downarrow {\scriptstyle \Sigma_+^\infty} \\
\mathrm{Mot}_{add}^{nc} \xrightarrow{\text{Kan extend}} & & \mathrm{Fun}^\oplus(A^{\mathrm{eff}}(F_\mathbb{C}), \mathrm{Spectra})
\end{array}
$$

Here $\mathrm{Sp}_\Phi = \mathrm{Fun}^\oplus(A^{\mathrm{eff}}(F_\mathbb{C}), \mathrm{Spectra})^{h\mathbb{N}^\times}$. Then

$$THH(\mathbb{S}[\Omega X]) = \Sigma_+^\infty \wedge X.$$

This description makes it obvious that there is a natural transformation

$$K \to THH.$$

This is called the Dennis trace.

Recall for any supernatural number N, we could replace all $\textcircled{c}$ with $\textcircled{c}_N$ above – you're only looking at cyclic groups whose order divide N. This is a way to get the p-typical version of THH, called $THH(-;p) \in \text{Sp}_{\Phi_{p^\infty}}$. There are spectra $TR^n(-;p) := THH(-;p)\left\langle p^{n-1}\right\rangle_{p^\infty}$. (These are cyclotomic spectra so you can plug in $\langle n\rangle$.) Then TR^* is the pro-system of these things.

Say X is a p^∞-cyclonic spectrum (this is the p-typical case). I want to try to appeal to the recollement story. Specifying p^r should specify a closed piece and an open piece. We have

$$\mathcal{O}_{\textcircled{c}_{p^\infty}} \overset{\text{closed}}{\hookrightarrow} \mathcal{O}_{\textcircled{c}_{p^\infty}} \overset{\text{open}}{\hookleftarrow} \mathcal{O}_{\textcircled{c}_{p^r}}$$

which gives rise to the recollement:

$$\text{Sp}_{\textcircled{c}_{p^\infty}} \underset{\substack{\longleftarrow i^* \\ \xrightarrow{\ i_* = i_{p,!}\ } \\ \longleftarrow i^!}}{} \text{Sp}_{\textcircled{c}_{p^\infty}} \underset{\substack{\longleftarrow j_! \\ \xrightarrow{\ j^* = j^!\ } \\ \longleftarrow j_*}}{} \text{Sp}_{\textcircled{c}_{p^r}}$$

This gives a cofiber sequence

$$j_! j^! X \to X \to i_* i^* X.$$

Last time I mentioned that this is the same as the isotropy separation sequence. We can compare this to another fiber sequence on the bottom:

$$
\begin{array}{ccc}
j_! j^! X & \longrightarrow X \longrightarrow & i_* i^* X \\
\| & \downarrow & \downarrow \\
j_! j^! X & \longrightarrow j_* j^* X \longrightarrow & i_* i^* j_* j^* X
\end{array}
$$

This presents $i_* i^* j_* j^* X$ as a pushout; this is the Tate construction. (Literally, it's geometric fixed points of the Borel construction.) Here i^* is geometric fixed points and i_* is forgetting.

We should analyze this map in the case where $X = THH(-, p)$. There is a restriction map $R : TR^{n+1}(-, p) \to TR^n(-, p)$. Evaluate all the cyclotomic objects on p^{n+1}.

$$
\begin{array}{ccc}
TR^{n+1}(-, p) & \overset{R}{\longrightarrow} & TR^n(-, p) \\
\downarrow & & \downarrow \\
H_\bullet(C_{p^n}, THH) \overset{N}{\longrightarrow} H^\bullet(C_{p^n}, THH) & \longrightarrow & \widehat{H}^\bullet(C_{p^n}, THH)
\end{array}
$$

This is a pullback diagram and $\widehat{H}$ is the Tate cohomology.

Clark Barwick and Saul Glasman will be putting a paper out soon about this method of defining THH.

Definition 18.3.

$$TP_i(X) = \widehat{H}^{-i}(\mathbb{T}, THH) = \pi_i(\lim_n \widehat{H}^\bullet(C_{p^n}, THH))$$

(Here $\mathbb{T} = S^1$ and lim is in the homotopy sense.)

This is the candidate for our cohomology theory.

Next time I'll write down a computation of some of these groups, and relate TR to the de Rham-Witt complex – we'll see that they're really describing the same structure.

LECTURE 19: MAY 9

THH carries the structure of a cyclotomic complex. In particular, it receives a map from the de Rham-Witt complex. We also started to construct TR (taking $C_{p^{n-1}}$-fixed points of THH). We're talking about the ζ-function of a variety over $\mathbb{F}_q$.

(Is it clear that the category of Witt complexes has an initial object? No. If you get rid of the prime 2 by looking at p-typical things, you can use an adjoint functor argument.)

Recall:
$$TR(-,p) = THH(-)\left\langle p^{n-1}\right\rangle.$$
Last time we had

$$
\begin{array}{ccccc}
H_\bullet(C_{p^n}, THH(-)) & \longrightarrow & TR^{n+1}(-,p) & \longrightarrow & TR^n(-,p) \\
\| & & \downarrow & & \downarrow \\
H_\bullet(C_{p^n}, THH(-)) & \longrightarrow & H^\bullet(C_{p^n}, THH(-)) & \longrightarrow & \widehat{H}^\bullet(C_{p^n}, THH(-))
\end{array}
$$

where $\widehat{H}$ was Tate cohomology. The bottom right term is what you take the limit of to get TP.

I need to tell you basic results (due to Hesselholt and Hesselholt-Madsen) about how THH functions.

Theorem 19.1. *Let X be smooth over a perfect field k of characteristic p. There's an isomorphism of cyclotomic $\mathcal{O}_X$-algebras:*
$$\mathbb{W}_\bullet(X) \stackrel{\cong}{\Rightarrow} TR_0^\bullet(X,p).$$

$\mathbb{W}_n$ is the n-truncated Witt vectors (i.e. you're using the truncation set $\left\langle p^{n-1}\right\rangle$) and $\mathbb{W}_\bullet$ is what you get when you put them together (the Frobenius and Verschiebung move these things around).

Formally, TR already has all the structure you need to be a p-Witt complex (only care about evaluating on $\left\langle p^r\right\rangle$) except an algebra structure over the Witt vectors, which is what this computation tells you. So we have that $TR_*^\bullet(X)$ is a p-Witt complex, and there is a map $\eta : \mathbb{W}_\bullet\Omega^* \to TR_*^\bullet(X)$ of pro-complexes that is a ring map degree-wise, and compatible with Frobenius and Verschiebung.

(Note on the indexing: you should think of $\bullet$ giving a pro-system of complexes, and $*$ is the complex degree.)

Theorem 19.2 (Hesselholt)**.** *The map $\mathbb{W}_n\Omega^j_X \to TR^n_j(X)$ is an isomorphism for $j \le 1$.*

We want to know what this looks like more generally as j varies. There's an answer where $X = \operatorname{Spec} k$:

Theorem 19.3 (Bökstedt, Hesselholt-Madsen (Periodicity theorem)). *$TR_2^n(k, p)$ is free of rank 1 over $\mathbb{W}_n(k)$. If $n \geq 1$, there is an isomorphism of graded $\mathbb{W}_n(k)$-algebras*

$$\operatorname{Sym}(TR_2^n(k, p)) \xrightarrow{\cong} TR_*^n(k, p).$$

(Hesselholt also specifies a precise "good" generator for $TR_2^n(k, p)$.)

Bökstedt's paper is still unpublished but can be found on the internet.

Theorem 19.4 (Hesselholt). *Let $f : X \to \operatorname{Spec} k$ be a smooth scheme over a perfect field k, and let $n \geq 1$. Then there is an isomorphism*

$$\mathbb{W}_n \Omega_X^* \otimes_{f^* \mathbb{W}_n(k)} f^* TR_*^n(k, p) \xrightarrow{\cong} TR_*^n(X, p).$$

This is from Hesselholt's p-typical curves paper, which you should definitely read.

Recall we're regarding $TR_*^n(X, p)$ as a sheaf of $\mathbb{W}_n(\mathcal{O}_X)$-algebras.

Remark 19.5. There is an isomorphism, due to Voevodsky and Geisser-Levine:

$$K_*^M(\mathcal{O}_X) \otimes_{f^*(\mathbb{Z}/p^n)} f^* K_*(k, \mathbb{Z}/p^n) \to K_*(\mathcal{O}_X, \mathbb{Z}/p^n).$$

Classically, you take the hypercohomology of the de Rham-Witt complex to get crystalline cohomology.

Theorem 19.6 (Illusie). $H_{crys}^i(X/\mathbb{W}_\bullet(k)) \cong H^i(X, \mathbb{W}_\bullet \Omega_X^*)$

Here H_{crys}^* is the cohomology of the crystalline site.

This gives rise to the *conjugate spectral sequence*

$$E_2^{i,j} = \lim_F H^i(X, \mathbb{W}_\bullet \Omega_X^j) \implies H_{\mathrm{crys}}^{i+j}(X/\mathbb{W}(k)).$$

This is the spectral sequence you get by filtering the crystalline cohomology by the truncated crystalline cohomology. It looks like this should be $\lim_R$, but there is an isomorphism $\lim_F \mathbb{W}_n(k) \xrightarrow{\varphi} \lim_R \mathbb{W}_n(k)$.

Fact 19.7. $U \mapsto \widehat{H}^\bullet(C_{p^n}, THH(U))$ *is an étale sheaf of $\mathbb{W}_n(\mathcal{O}_X)$-modules on X.*

Theorem 19.8 (Hesselholt). *If $n \geq 1$,*

$$\mathbb{W}_n \Omega_X^* \otimes_{f^* \mathbb{W}_n(k)} f^* \widehat{H}^{-*}(C_{p^n}, THH(k)) \xrightarrow{\cong} \widehat{H}^{-*}(C_{p^n}, THH(X)).$$

$$\mathbb{W}_n\Omega_X^* \otimes_{f^*\mathbb{W}_n(k)} f^*TR_*^n(k,p) \xrightarrow{\;\cong\;} TR_*^n(X,p)$$

$$\mathbb{W}_n\Omega_X^* \otimes_{f^*\mathbb{W}_n(k)} f^*\widehat{H}^{-*}(C_{p^n}, THH(k)) \longrightarrow \widehat{H}^{-*}(C_{p^n}, THH(X))$$

(with vertical map γ on the right)

Note that both of the things on the bottom row are 2-periodic algebras. So it suffices to prove the theorem in a range. In particular, we will prove that γ is an isomorphism in degrees $\geq \dim_k(X)$.

Everything in sight is an étale sheaf; so we can quickly reduce to the affine case.

Key Fact 19.9. If A/k is smooth of relative dimension d, there is a spectral sequence (the *Tate spectral sequence*)

$$E_{i,j}^2 = \widehat{H}^{-u}(C_{p^n}, THH_j(A)) \implies \widehat{H}^{-i-j}(C_{p^n}, THH(A))$$

which converges strongly, and $E_{i,j}^\infty = 0$ if $j \geq d + 2n$.

PROOF IDEA. We have

$$E_{**}^2 = \Lambda\{u\} \otimes S\{t^\pm, \alpha\} \otimes \Omega_A^*$$

with $|u| = (-1,0)$, $|t| = (-2,0)$, and $|\alpha| = (0,2)$. (Here Λ means exterior algebra and S means symmetric algebra, and Ω_A^* is the usual de Rham complex on A.) Here u and α are permanent cycles, and $d^{2n+1}(u) = \lambda t^{n+1}\alpha^n$ for $\lambda \in \mathbb{F}_p$. Sort of the whole point is that Ω_A^* is in degrees $(0,j)$ with $0 \leq j \leq d$. Compare this to

$$E_{**}^2 = S\{t^\pm, \alpha\} \otimes \Omega_A^* \implies \widehat{H}^{-*}(\mathbb{T}, THH(A)).$$

The nonzero differentials are all even, and so $E_{i,j}^{2n+2} = 0$ if $j \geq d+2n$. (This isn't true for formal reasons – all of this is a computation.) Here $\widehat{H}^{-*}(\mathbb{T}, THH(A)) = \lim_n \widehat{H}^{-*}(C_{p^n}, THH(A))$ (for now, just take this as a definition). $\square$

Definition 19.10. $TP_*(X) := \lim_n \widehat{H}^{-*}(C_{p^n}, THH(A)) = \widehat{H}^{-*}(\mathbb{T}, THH(A))$

(Oops, forgot to define TC. The category of cyclotomic spectra is symmetric monoidal, so it has a unit. The TC groups are the Ext groups out of the unit. This is the one that closely approximates K-theory.)

In defense of that crazy definition of THH: all it tells you is what to do with free things. But no one knows how to do a computation with THH that doesn't involve using descent, formal arguments, and knowing what happens on free things.

Here's a spectral sequence for computing TP (one might call this the *Hodge spectral sequence*, but we're getting it out of pure topology):

$$E_{i,j}^2 = \bigoplus_{m \in \mathbb{Z}} \lim_F H^{-i}(X, \mathbb{W}_\bullet\Omega_X^{j+2m}) \implies TP_{i+j}(X).$$

This is just gotten by putting together pieces we already know. This spectral sequence is concentrated in the strip $-d \leq i \leq 0$.

71

We were going to tensor TP up to $\mathbb{C}$ so we could look at the regularized determinant for some operator (if you don't use $\mathbb{C}$ you have convergence issues), and some quotient of those was our ζ function. I'm going to use some embedding $\iota : \mathbb{W}(k) \hookrightarrow \mathbb{C}$. At least in theory, everything depends on this embedding. We can tensor this entire spectral sequence up to $\mathbb{C}$:

$$\bigoplus_{m \in \mathbb{Z}} \lim_F (H^{-i}(X, \mathbb{W}_\bullet \Omega^{j+2m})) \otimes \mathbb{C} \implies TP_{i+j}(X) \otimes \mathbb{C}.$$

There's going to be an operator Θ on the RHS, and that's what we're going to take the regularized determinant of. There's also a scaling factor, which doesn't matter until you work in an archimedean setting.

$$\zeta(X, s) = \frac{\det_\infty \left(\frac{1}{2\pi}(s\mathbb{1} - \Theta)\big|_{TP_{\mathrm{odd}}(X) \otimes \mathbb{C}} \right)}{\det_\infty \left(\frac{1}{2\pi}(s\mathbb{1} - \Theta)\big|_{TP_{\mathrm{even}}(X) \otimes \mathbb{C}} \right)}$$

LECTURE 20: MAY 11

We need to define the regularized determinant, and then we'll be able to finish off Hesselholt's proof from last time.

Definition 20.1. Suppose $(\lambda_n)_{n \in \mathbb{N}}$ is a sequence of complex numbers with specified arguments $(\alpha_n)_{n \in \mathbb{N}}$. Make two assumptions:

(1) There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\lambda_0 \mathfrak{n} \neq 0$.

(2) Consider the Dirichlet series

$$\sum_{n \geq N} |\lambda_n|^{-s} \exp(-is\alpha_n).$$

This converges on some half-plane. Assume that this admits an analytic continuation to a holomorphic function $\zeta_N(s)$ on $\rangle - \varepsilon, +\infty \langle$ for $\varepsilon > 0$.

Then the *regularized product*

$$\prod_{n \in \mathbb{N}} (\lambda_n, \alpha_n) = \left\{ \prod_{n=1}^{n-1} \lambda_n \right\} \exp(-\zeta_N'(0))$$

We'll always pick $\arg(\lambda) \in (-\pi, \pi]$. The existence and value of $\square$ is invariant under rearrangement.

Example 20.2. Let $\gamma, z \in \mathbb{C}^\times$. Recall the Hurwitz ζ-function is

$$\zeta_\gamma(s, z) = \sum_{n \in \mathbb{N}_0} \frac{1}{(\gamma(z+n))^s}.$$

Then

$$\prod_{n \in \mathbb{Z}} \gamma(z+n) = (z\gamma)^{-1} \exp\left(-\frac{d}{ds}\zeta_\gamma(s, z)\big|_{s=0} \right) \exp\left(-\frac{d}{ds}\zeta_{-\gamma}(s, -z)\big|_{s=0} \right).$$

Exercise 20.3.

$$\exp\left(-\frac{d}{ds}\zeta_\gamma(s,z)\Big|_{s=0}\right) = \begin{cases} |\gamma|^{1/2-z}\exp(i\pi(\frac{1}{2}-z))(\frac{1}{\sqrt{2\pi}}\Gamma(z))^{-1} & \operatorname{im} z \leq 0 \\ \gamma|^{1/2-z}\exp(-i\pi(\frac{1}{2}-z))(\frac{1}{\sqrt{2\pi}}\Gamma(z))^{-1} & \operatorname{im} z > 0 \end{cases}$$

Definition 20.4. Suppose V is a $\mathbb{C}$-valued space of countable dimension, $\Theta : V \to V$ is an endomorphism, and assume:

(1) $V \cong \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$, where $V_\lambda = \ker(\Gamma - \lambda)^N$ for $N \gg 0$ (i.e. it stabilizes) and V_λ is finite-dimensional

(2) Let (λ_n) be the sequence of eigenvalues with multiplicity, and assume that $\boxed{}\lambda_n$ converges.

Then the *regularized determinant* is

$$\det_\infty(\Theta) = \boxed{}\lambda_n.$$

We're looking at

$$\zeta_\Theta(s) = \sum_{\lambda \in \mathbb{C}^\times} \dim_{\mathbb{C}}(V_\lambda)\lambda^{-s}.$$

Condition (2) says that this analytically continues to $\rangle - \varepsilon, +\infty\langle$. (1) is necessary because the Dirichlet series doesn't converge otherwise (so you can't even hope to get an analytic continuation necessary to define $\boxed{}$).

Note that

$$\det_\infty(\Theta) = \begin{cases} 0 & \dim V_0 > 0 \\ \exp(-\zeta'_\Theta(0)) & \text{otherwise.} \end{cases}$$

Definition 20.5. Define

$$\dim_\infty(\Theta) = \dim_{\mathbb{C}} V_0 + \zeta_{\overline{\Theta}}(0)$$

where $\overline{\Theta} : V/V_0 \to V/V_0$.

If V is finite-dimensional, then our product is a finite product, choose $N > \dim V$, and $\det_\infty(\Theta) = \det(\Theta)$ and $\dim_\infty(\Theta) = \dim(V)$.

Suppose we have a short exact sequence

$$0 \to (V', \Theta') \to (V, \Theta) \to (V'', \Theta'') \to 0.$$

Then we have $\det_\infty(\Theta) = \det_\infty(\Theta')\det_\infty(\Theta'')$ and $\dim_\infty(\Theta) = \dim_\infty(\Theta') + \dim_\infty(\Theta'')$.

Proposition 20.6 (Deninger). *If V is a graded-commutative graded $\mathbb{C}$-algebra such that $V(j) \subset V$ is finite-dimensional, and if $\Theta : V \to V$ is a graded linear derivation, and if $\beta \in V(-2)$ is a unit such that $\Theta(\beta) = \frac{2\pi i}{\log q}\beta$, then $\det_\infty(s - \Theta|_{V(2*+j)}) = \det(1 - q^{-s\Theta}|_{V(j)})$.*

(Here $V(i)$ means the ith graded piece, since we've already used V_i for the eigenspace.)

We were trying to prove:

Theorem 20.7. *Let X be smooth and proper over $\mathbb{F}_q$, and $\iota : \underbrace{W(\mathbb{F}_q)}_{W} \hookrightarrow \mathbb{C}$ an embedding. Then*

$$\zeta(X, s) = \frac{\det_\infty \left((s - \Theta) \Big| TP_{odd}(X) \otimes_W \mathbb{C} \right)}{\det_\infty \left((s - \Theta) \Big| TP_{even}(X) \otimes_W \mathbb{C} \right)}$$

where Θ is a $\mathbb{C}$-linear derivation, and $q^\Theta = Fr_q^*$.

So Θ will be some kind of logarithm of Frobenius which is independent of q.

You can replace $s - \Theta$ with $\delta(s - \Theta)$ for any $\delta > 0$ and this will still be true. Here $\det_\infty(\delta\Theta) = \delta^{\dim_\infty(\Theta)} \det_\infty(\Theta)$. This is why I don't care about the scaling factor.

Last time we had a spectral sequence

$$E_{ij}^2 = \bigoplus_{m \in \mathbb{Z}} \lim_F H^{-i}(X, \mathbb{W}_\bullet \Omega_X^{j+2m}) \otimes \mathbb{C} \implies TP_{i+j}(X) \otimes \mathbb{C}.$$

The point is that the E_{ij}^2 terms are finite-dimensional. That means we can apply the formula of Deninger once you've specified the element β. We saw that everything was generated by a class in degree 2, and we can specify a canonical generator that gives rise to an element of degree -2 in TP.

By Berthelof's thesis, we have

$$\zeta(X, s) = \frac{\det(1 - q^{-s} Fr_q^* | H_{\text{crys}}^{\text{odd}}(X/W) \otimes \mathbb{C})}{\det(1 - q^{-s} Fr_q^* | H_{\text{crys}}^{\text{even}}(X/W) \otimes \mathbb{C})}.$$

I need to be able to relate determinants on the crystalline cohomology to determinants on TP.

Assemble the conjugate spectral sequence and the Hodge spectral sequence, and the multiplicativity of $\det_\infty$ in exact sequences, to get

$$\det(1 - q^{-s} Fr_q^* | TP_j(X) \otimes_W \mathbb{C}) = \begin{cases} \det(1 - q^{-s} Fr_q^* | H_{\text{crys}}^{\text{odd}} \otimes \mathbb{C}) & j \text{ odd} \\ \det(1 - q^{-s} Fr_q^* | H_{\text{crys}}^{\text{even}} \otimes \mathbb{C}) & j \text{ even.} \end{cases}$$

LECTURE 21: MAY 16

In the Hesselholt story ($\sim$2016), we were looking at a smooth, projective scheme X over a finite field $\mathbb{F}_q$. We were interested in understanding its ζ-function; there was a formula for this involving a regularized determinant of an operator Θ, which was defined roughly as "$\log_q Fr_q^*$". It was acting on TP_*, which was related to THH. We proved that formula by comparing things like TR, TP, and THH to the hypercohomology of the de Rham-Witt complex, which was in turn related to crystalline cohomology and friends.

This was a story that came roughly a year after a result by Connes-Consani. They contemplated a smooth, projective scheme X over a number field K. (The goal, ultimately, is to imagine an arithmetic variety that combines these, but we're not there yet.) For an archimedean place v of K, they looked at

$$\prod_{0 \leq w \leq 2d} L_v(h^w(X), s)$$

(we'll define h^w later). There is an operator Θ, the analogue of the other Θ, which is defined as "a generator of all Λ_m's" (to be explained later). The idea is that THH rationalized, is just HH rationalized. You can run some of the same tricks used to get TP, to get an analogous periodic thing related to HH called HC^{an}. In the Hesselholt story, we compared to Berthelof's formula for ζ involving H_{crys}; here, we compare to Deligne cohomology and work of Beilinson.

In order to define this stuff, I need to review some Hodge theory. Write $\mathbb{S}$ for the Weil restriction of $\mathbb{G}_m$ from $\mathbb{C}$ to $\mathbb{R}$. For example, we have $w : \mathbb{G}_m \hookrightarrow \mathbb{S}$ such that on real points, we have the usual map $w(\mathbb{R}) : \mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$.

Definition 21.1. An $\mathbb{R}$- Hodge structure on a finite-dimensional vector space $H_\mathbb{R}$ (i.e. real vector space) is an action of the real group $\mathbb{S}$.

An action of $\mathbb{C}^\times \times \mathbb{C}^\times$ on $H_\mathbb{C} := H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ is the same thing as a decomposition $H_\mathbb{C} \cong \bigoplus_{p,q} H^{p,q}$ where $H^{p,q} = \{x \in H_\mathbb{C} : (u,v)x = u^{-p}v^{-q}x\}$. This is a Hodge structure iff $\overline{H^{q,p}} = H^{p,q}$. It is *pure of weight* k iff $H^{p,q} = 0$ unless $p + q = k$.

Example 21.2. If X is compact and Kähler, then $\Omega_X^\bullet \cong \mathbb{C}$. There is a "foolish filtration"

$$\cdots \to \Omega_X^{\geq n} \to \Omega_X^{\geq n-1} \to \cdots \to \mathbb{C}$$

inside the derived category of sheaves on X. This gives a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H^{p+q}(X, \mathbb{C}).$$

(We're thinking of the analytic topology now.) The theorem is that this degenerates to a filtration on $H^{p+q}(X, \mathbb{C})$, which is the same as the Hodge filtration $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega^p)$. This is a Hodge structure. It is really valued in the category of representations of $\mathbb{S}$. (Part of the Tannaka idea is that any reasonable algebraic cohomology theory isn't supposed to be valued in vector spaces, but rather in representations of some pro-group.)

Notation 21.3. Write

$$\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})$$

$$\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$$

(Some authors don't use the first 2 for $\Gamma_\mathbb{C}(s)$. You want it there because you want the identity $\Gamma_\mathbb{C}(s) = \Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1)$.)

If $H_\mathbb{C}$ is a $\mathbb{C}$-Hodge structure (a $\mathbb{C}$-vector space with an action of $\mathbb{C}^\times \times \mathbb{C}^\times$), then define

$$\Gamma_{H_\mathbb{C}}(s) = \prod_{p,q} \Gamma_\mathbb{C}(s - \min\{p,q\}) \quad \text{where } h(p,q) = \dim_\mathbb{C} H_{p,q}.$$

If $H_\mathbb{R}$ is an $\mathbb{R}$-Hodge structure, then

$$\Gamma_{H_\mathbb{R}}(s) = \prod_n \Gamma_\mathbb{R}(s-n)^{h(n,+)} \Gamma_\mathbb{R}(s-n+1)^{h(n,-)} \prod_{p<q} \Gamma_\mathbb{C}(s-p)^{h(p,q)}$$

I've got an involution on $h(n,n)$ and I'm looking at the two eigenspaces – that is, $H^{n,+} = \{x \in H^{n,n} : \overline{x} = (-1)^n x\}$ and $H^{n,-} = \{x : \overline{x} = -(-1)^n x\}$, and $h(n,+)$ and $h(n,-)$ are their dimensions respectively.

If you try to generalize away from projective things, you have to talk about mixed Hodge structures – in addition to a Hodge filtration, there's also a weight filtration. But we'll concentrate on projective things.

Given a number field K, I have some complex places and some real places. If $K_v \cong \mathbb{C}$ (i.e. we're completing at some place) then $H^w(X(K_v)^{an}, \mathbb{C})$ has a $\mathbb{C}$-Hodge structure, and we define

$$L_v(h^w(X), s) = \Gamma_{H^w(X(K_v)^{an}, \mathbb{C})}(s).$$

If $K_v \cong \mathbb{R}$, then $H^w(X(K_v(i))^{an}, \mathbb{C})$ has an $\mathbb{R}$-Hodge structure, and we define

$$L_v(h^w(X), s) = \Gamma_{H^w(X(K_v(i))^{an}, \mathbb{C})}(s).$$

Definition 21.4 (Deligne cohomology). Define $\mathbb{R}(p) = (2\pi i)^p \mathbb{R} \subset \mathbb{C}$. Form the *homotopy* pullback

$$\begin{array}{ccc} \mathbb{R}(p)_D & \longrightarrow & \Omega^{\geq p} \\ \downarrow & & \cap\downarrow \\ \mathbb{R}(p) & \longrightarrow & \mathbb{C} \end{array}$$

(I'm writing $\Omega^{\geq p} \hookrightarrow \mathbb{C}$ as an inclusion because I'm thinking of this going on in sheaves of complexes.) The homotopy pullback is the Deligne complex of weight p. Then *Deligne cohomology* is the hypercohomology of this complex:

$$H^q(X, \mathbb{R}(p)) = \mathbb{H}^q(X, \mathbb{R}(p)_D).$$

I invite you to think about this with the analytic topology.

This is computed by the complex

$$0 \to \mathbb{R}(p) \to \mathcal{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^{p-1} \to 0.$$

I can do all the above with $\mathbb{Z}$-coefficients as opposed to $\mathbb{R}$-coefficients.

Example 21.5. $\mathbb{R}(0)_D \cong \mathbb{R}$

Example 21.6.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow \\ 1 & \longrightarrow & 1 & \longrightarrow & \mathcal{O}_X^\times & \longrightarrow & 1 \end{array}$$

We have $\mathbb{Z}_X(1)_D \cong \mathcal{O}_X^\times[-1]$, and $H^q(X, \mathbb{Z}_X(1)_D) \cong H^{q-1}(X, \mathcal{O}_X^\times)$.

76

Example 21.7. In weight 2:
$$\mathbb{Z}_X(2) \cong [d\log : \mathcal{O}_X^\times \to \Omega_X^1][-1].$$
$H^{2,2}$ classifies line bundles with holomorphic connections.

There's a short exact sequence
$$0 \to J^1(X) \to H_D^2(X, \mathbb{Z}(1)) \to NS(X) \to 0.$$
$$H_D^q(X_v, \mathbb{R}(p)) = \begin{cases} H_D^q(X(K_v), \mathbb{R}(p)) & \text{complex case} \\ H_D^q(X(K_v(i)), \mathbb{R}(p))^{C_2} & \text{real case.} \end{cases}$$
(Here $(-)^{C_2}$ means taking C_2-fixed points.)

Theorem 21.8 (Beilinson). *For $m \in \mathbb{Z}$ such that $m \leq \frac{w}{2}$,*
$$\text{ord}_{s=m} L_v(h^w(X), s)^{-1} = \dim_\mathbb{R}(H_D^{w+1}(X_v, \mathbb{R}(w+1-m))).$$

Next time we'll talk about how this is all related to cyclic homology HC^{an}, which leads to the proof of:

Theorem 21.9 (Connes-Consani). *For v an archimedean place,*
$$\prod_{0 \leq w \leq 2d} L_v(H^w(X), s)^{(-1)^{w+1}} = \frac{\deg_\infty \left(\frac{1}{2\pi}(s - \Theta)|HC_{even}^{ar}(X_v)\right)}{\deg_\infty \left(\frac{1}{2\pi}(s - \Theta)|HC_{odd}^{ar}(X_v)\right)}.$$

We'll get a Λ-action on HC. We learned λ-operations are the same as compatible families of lifts of Frobenius. We'll try to do a simultaneous logarithm of all the Frobenius, which will be expressed in terms of λ-operations.

We'll define Θ by defining u^Θ for any $u > 0$. u^Θ will be an action of $\mathbb{R}_{>0}$ on HC. If $k \in \mathbb{N}$,
$$k^\Theta|_{HC_n(X_v)} = k^{-n}\lambda_k.$$
By fractions and density this can be extended to $\mathbb{R}_{>0}$. Remember these λ_k's should be thought of as lifts of Frobenius.

I'll tell you about the relationship between Deligne homology and cyclic homology. If you like, you can take that as the definition, but that might not be helpful. Recall we had

$$\begin{array}{ccccccc}
\Omega^{<p}[1] & \longrightarrow & \mathbb{R}(p)_D & \longrightarrow & \Omega^{\geq p} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \uparrow & & \downarrow \\
0 & \longrightarrow & \mathbb{R}(p) & \lhook\joinrel\longrightarrow & \mathbb{C} & \longrightarrow & \Omega^{<p}
\end{array}$$

$\Omega^{<p}[1]$ is the fiber; sometimes people call this reduced Deligne cohomology:
$$\widetilde{H}_D^n(X; \mathbb{R}(p)) = \mathbb{H}^n(X, \Omega^{<p}[1]).$$
There's a nice LES relating these things.

Here's one in a long line of HKR-type theorems (e.g. relating HH to the complex of Kähler differentials):

77

Theorem 21.10. *Let X be a finite-type variety over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Then we have:*

$$HC_n(X) \cong \bigoplus_{i \geq 0} \widetilde{H}_D^{n+1-2i}(X, \mathbb{R}(n+1-i)).$$

HC is roughly what happens when you rationalize TC.

Let

$$HC_n^{(j)}(X) = \{x \in HC_n \ : \ \lambda_m(x) = m^j x\}.$$

Then you get a decomposition

$$HC_n(X) = \bigoplus_{j \geq 0} HC_n^{(j)}(X).$$

This is a formal sentence, but the sentence in the theorem above is less formal, and the claim is that they're "the same".

$$HC_n^{(j)}(X) = \widetilde{\mathbb{H}}_D^{2j+1-n}(X, \mathbb{R}(j+1)).$$

This should remind you of the formula relating the eigenspaces of the Adams operations on algebraic K-theory to motivic cohomology. (But there's a weird degree shift here.)

We'll take a form of HC and extract HC^{an}, which is where we'll look at the regularized determinant of Θ and get our result.

LECTURE 22: MAY 18

We started talking purely analytically about the Γ-function using the Mellin transform. This allowed us to define related functions of arithmetic origin, such as ζ-functions and L-functions. We were interested in completing these using Γ-functions. This inspired us to look at Tate's thesis about functional equations for these things.

Then we seemingly dropped this story and started talking about $\mathbb{F}_1$. These things were defined as Λ-algebras. The motivation for this is that a reduced $\mathbb{F}_1$-algebra is the same thing as a $\mathbb{Z}$-algebra with compatible lifts of Frobenius. We looked at the theory of modules over $\mathbb{F}_1$-algebras, in particular Ω^1. This is like the usual Ω^1 but it has a Λ-module structure, which means that when you took the entire de Rham complex you get the de Rham-Witt complex. This led us to thinking about Witt complexes, which were related to cyclotomic spectra. We learned there was a special relationship between the de Rham-Witt complex $\mathbb{W}\Omega_X$ and THH and friends. We saw a sketch of a proof of Hesselholt's formula for $\zeta(X, s)$ in terms of regularized determinants for an operator acting on something related to THH. We compared the invariants coming from THH to the hypercohomology to the sheaf of complexes, which turns out to be crystalline cohomology, and there is a known formula for $\zeta(X, s)$ in terms of crystalline cohomology.

Finally, we're going to look at the ζ-function for a smooth, projective variety X over a number fields.

$$\prod_{0 \leq w \leq 2d} L_v(h^w(X), s)^{(-1)^{w+1}} = \frac{\det_\infty(\frac{1}{2\pi}(s - \Theta)|_{HC_{\text{even}}^{an}})}{\det_\infty(\frac{1}{2\pi}(s - \Theta)|_{HC_{\text{odd}}^{an}})}$$

The last piece is to say what HC^{an} is, in a way that shows how it relates to THH.

Definition 22.1. Let $X_{\mathbb{C}}$ be a smooth projective complex variety. Look at $C^{\infty}(X^{an}, \mathbb{C}, \mathbb{C})$ and $C^{\infty}(X^{an}_{\mathbb{C}}, \mathbb{R})$; these are Fréchet algebras. I can inject topology into my definition of THH and friends to define
$$TP^{\text{cts}}(C^{\infty}(X^{an}_{\mathbb{C}}, \mathbb{C})) \otimes \mathbb{C}.$$
Just use the cyclic bar construction. (I don't know how to do this integrally.) You can define this in such a way that you get
$$TP^{\text{cts}}(C^{\infty}(X^{an}_{\mathbb{C}}, \mathbb{C})) \otimes \mathbb{C} \simeq TP(X_{\mathbb{C}}) \otimes \mathbb{C}$$
with all the cyclic structure. I can also look at $TP^{\text{cts}}(C^{\infty}(X^{an}_{\mathbb{C}}, \mathbb{R})) \otimes \mathbb{C}$. Let Θ_0 be the generator of the λ-operations without the k^{-n} factor we had last time; that is, define $k^{\Theta_0} = \lambda_k$. There is a natural map
$$TP^{\text{cts}}(C^{\infty}(X^{an}_{\mathbb{C}}, \mathbb{R})) \otimes \mathbb{R} \xrightarrow{(2\pi i)^{\Theta_0}} TP^{\text{cts}}(C^{\infty}(X^{an}_{\mathbb{C}}, \mathbb{C})) \otimes \mathbb{C}.$$
(If this were THH instead of TP, on one side I'm taking the cyclic bar construction over $\mathbb{C}$, and on the other side I'm taking the cyclic bar construction over $\mathbb{R}$.)

I don't think anyone knows an algebraic form for this map; this is one of the obstructions to proving the Riemann hypothesis.

Define
$$\mathbb{P}^{an}(X_{\mathbb{C}}) = \left(TC(X_{\mathbb{C}}) \otimes \mathbb{C} \times^h_{TP(X_{\mathbb{C}}) \otimes \mathbb{C}} TP^{\text{cts}}(C^{\infty}(X^{an}_{\mathbb{C}}, \mathbb{R})) \otimes \mathbb{R}\right)[1].$$
This should look a lot like what we used to define Deligne cohomology. Then define
$$HC^{an}_*(X_v) = \pi_* \mathbb{P}^{an}(X_v).$$
(Here X_v is completion at some real or complex place.)

Let $X_{\mathbb{R}}$ be a smooth projective real variety. Define
$$\mathbb{P}^{an}(X_{\mathbb{R}}) := \mathbb{P}^{an}(X_{\mathbb{C}})^{C_2}.$$
where the action is the de Rham conjugation (same as for Deligne cohomology).

Proposition 22.2. Let $E_d = \{(n, j) : n \geq 0, 0 \leq 2j - n \leq 2d\}$.
$$\pi_n(\mathbb{P}^{an}(X_v)^{\Theta_0 = j}) = \begin{cases} H_D^{2j+1-n}(X_{\mathbb{C}}, \mathbb{R}(j+1)) & (n, j) \in E_d \\ 0 & (n, j) \notin E_d. \end{cases}$$

The rest is just pushing formulas around.

Proposal 22.3 (Hesselholt)**.** Prove the Riemann hypothesis.

The idea is that the completed zeta function $\widehat{\zeta}(X, s)$ should have the formula
$$\widehat{\zeta}(X, s) = \frac{\det_{\infty}(\frac{1}{2\pi}(s - \Theta)|_{TP_{\text{odd}}})}{\det_{\infty}(\frac{1}{2\pi}(s - \Theta)|_{TP_{\text{even}}})}$$

where $TP = THH(X)^{tS^2} \otimes \mathbb{C}$ (Tate spectrum with S^1-action). This is one question mark – no one knows how to get an appropriate complex vector space. In our previous setting, we could just tensor up over the Witt vectors.

In this picture, X is smooth and projective over $\mathbb{F}_1$, for example $X = \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$. In the story of a variety over a finite field, we had the conjugate spectral sequence. There's a motivic spectral sequence for algebraic K-theory (a.k.a. the AHSS). Motivic cohomology is what you get when you look at a weight filtration of the algebraic K-theory spectrum $W^{j-1}K(X) \to W^j K(X) \to \cdots \to K(X)$; then

$$\pi_{2j-i}W^{j/j-1} =: H^i(X, \mathbb{Z}(j)).$$

(You can think of this as one definition of motivic cohomology, albeit maybe not the most useful one.)

Crystalline cohomology is effectively Ext groups in some category. Motivic cohomology is Ext groups in a universal category for these things. We have a trace map $K(X) \to TR^\bullet(X)$ where $TR^\bullet(X)$ is a pro-spectrum (indexed by the divisibility poset). There should be a weight filtration $W^j TR^\bullet(X) \to \cdots \to TR^\bullet(X)$ but no one knows how to do that. You get the weight filtration on $K(X)$ using the fact that $K(X)$ is $\mathbb{A}^1$-invariant, but $TR^\bullet(X)$ is not. But if you were to make this work, you'd get a spectral sequence which is supposed to be related to the conjugate spectral sequence. That is,

$$\pi_{2j-i}W^{j/j-1} = H^i(X, W(j))$$

where $W(j)$ is this mystery complex. (Bhatt and Scholze constructed this spectral sequence at a prime p. This is the Hodge-Tate spectral sequence.)

But if you had all the above pieces, you could prove the Riemann hypothesis. Assumptions:
- $\overline{\operatorname{Spec}\mathbb{Z}}$ (some completion of $\operatorname{Spec}\mathbb{Z}$)

- a graded complex vector space $TP(\overline{\operatorname{Spec}\mathbb{Z}})$ with a Hodge $*$ operator

- Θ operator acting on $TP(\overline{\operatorname{Spec}\mathbb{Z}})$

- $\widehat{\zeta}(X, s) = \dfrac{\det_\infty(\frac{1}{2\pi}(s - \Theta)|_{TP_{\mathrm{odd}}})}{\det_\infty(\frac{1}{2\pi}(s - \Theta)|_{TP_{\mathrm{even}}})}$

As in the Hesselholt story, Θ is going to be a graded derivation. Suppose $\Theta(x) = \rho x$ (i.e. ρ is an eigenvalue). We want to show $\operatorname{Re}\rho = \frac{1}{2}$. Since Θ is a derivation, we have

$$\Theta(*x \cup x) = (*\Theta x \cup x) + (*x \cup \Theta x).$$

For weird Hodge-theoretic reasons, $\Theta(*x \cup x) = *x \cup x$. Define the inner product $\langle x, y \rangle = *x \cup y$. We could just include $\Theta\langle x, x \rangle = \langle x, x \rangle$ into our list of assumptions. But we also have $(*\Theta x \cup x) + (*x \cup \Theta x) = (\rho + \overline{\rho})\langle x, x \rangle$ and so $\operatorname{Re}\rho = \frac{1}{2}$.