algebraic topology

Lectures delivered by Michael Hopkins
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Spring 2011, Harvard

{Last updated August 14, 2012}

Contents

Introduction

Michael Hopkins taught a course (Math 231b) on algebraic topology at Harvard in Spring 2011. These are our "live-TEXed" notes from the course.

Conventions are as follows: Each lecture gets its own "chapter," and appears in the table of contents with the date. Some lectures are marked "section," which means that they were taken at a recitation session. The recitation sessions were taught by Mitka Vaintrob.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are my fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe.

Please email corrections to `ebelmont@fas.harvard.edu`.

Lecture 1
January 24, 2010

This is a second-semester course in algebraic topology; we will start with basic homotopy theory and move on to the theory of model categories.

§1 Introduction

Roughly speaking, algebraic topology can be construed as an attempt to solve the following problems:

1. Given spaces X and Y, determine the set

$$[X, Y] = Map(X, Y)/\text{homotopy}$$

   of maps between them, up to homotopy;

2. Classify spaces up to homotopy.

This is related to various *lifting problems*. Suppose we have a diagram:

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

If we knew the answers to the above questions, we would know whether there is some lifting $h : X \to Y$ making the diagram commute. Last semester, we learned some tools that might tell you when the answer is "no": if you apply homology or cohomology to the diagram, then any possible lift would have to be compatible with the new algebraic structure. Sometimes you can get a contradiction this way. Lifting properties will become one of the themes of the course.

For now, let's focus on "good" spaces (for example, CW complexes).

§2 Homotopy groups.

Let X be a space with a base point. For $n = 0, 1, 2 \ldots$ define $\pi_n X = [S^n, X]$ to be the space of base-point-preserving maps $S^n \to X$, modulo base-point preserving homotopy. Here are some facts:

- $\pi_0 X$ is the set of path components. (It is the set of maps out of S^0, which is a point, so choosing an element of $\pi_0 X$ amounts to choosing a destination for this point. Two maps are equivalent if their destination points are path-connected.)

- $\pi_1 X$ is the fundamental group of X

- $\pi_n X$ is an abelian group for $n \geq 2$. Why?

Proof of the last assertion. $I^n/\partial I \equiv S^n$ so we can define $\pi_n = [I^n/\partial I^n, X]$. The group operation defines another map $f * g : I^n \to X$ that is best represented pictorially:

You can check that everything works out at the boundaries.

To show it's abelian, draw homotopies as follows:

$$\mathscr{Q}.\mathscr{E}.\mathscr{D}.$$

How does this help with the fundamental problems? Suppose X is a CW complex, and suppose we know $\pi_n Y$ for all n. We can try to describe $[X, Y]$ by induction on the n-skeletons of X. It is easy to understand the maps $X^{(0)} \to Y$, as you just have to name a path component of Y for every point in $X^{(0)}$. In the general case we can understand $X^{(k)}$ by recalling that there is a natural structure of spheres that map to it:

Up to homotopy, each map f_α is an element of $\pi_{k-1} X^{(k-1)}$; composition with g_{k-1} gives a map $\pi_{k-1} X^{(k-1)} \to \pi_{k-1} Y$. However, if the map marked ? exists, then the elements of $\pi_{k-1} Y$ arising in this way are homotopically zero, because commutativity of the diagram shows that they factor through $D^k \simeq 0$. So we can translate our problem about $[X, Y]$ into a problem about the structure of homotopy groups.

We can apply a similar inductive approach to translate the classification problem into homotopic language.

Conclusion: *Knowing how to compute homotopy groups should lead to a solution to our two fundamental problems.* The goal of this course is to calculate rational homotopy groups $\pi_* X \otimes \mathbb{Q}$. We will do this by studying Quillen's model categories, which are "places to do homotopy theory." We will show that there is an equivalence

$$
\begin{array}{ccc}
\text{$\mathbb{Q}$-homotopy theory} & \Longleftrightarrow & \text{Homotopy theory} \\
\text{of CW complexes} & & \text{of differential graded algebras}
\end{array}
$$

Then we can describe $\mathbb{Q}$-homotopy theory using purely algebraic tools.

Lecture 2
1/26

The next lecture or two will be a little formal, but a few ideas will need to be covered before we move forward. We are going to be a little breezy about this, and you can read more details in Hatcher.

§1 Introduction

In the last lecture, we defined something called the **homotopy groups.** (Last semester, we studied the **homology groups**.) Recall,

2.1 Definition. For a pointed space $(X, *)$, the homotopy groups $\pi_n(X)$ are the sets of pointed homotopy classes of maps $S^n \to X$; equivalently, homotopy classes of maps $(I^n, \partial I^n) \to (X, *)$. Yet another way of defining this is to consider homotopy classes of maps

$$(D^n, S^{n-1}) \to (X, *),$$

which is clearly equivalent. This relies on the fact that $I^n / \partial I^n = S^n$.

The groups $\pi_n(X)$, unlike $H_n(X)$, do depend on a basepoint.

The following is obvious:

2.2 Proposition. *The π_n are covariant functors (on the category of pointed spaces).*

Last time, we showed how each $\pi_n X$ is in fact an abelian group, for $* \geq 2$. ($\pi_1 X$ is a group, but not necessarily abelian.)

Recall that homology is very easy to compute. There were two tools that let us compute it:

1. We defined homology groups $H_*(X, A)$ for a pair (X, A) and had a long exact sequence
$$H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \ldots.$$
   This was completely formal. Here, the groups $H_n(X, A)$ were defined completely formally, as the homology groups of some quotient chain complexes.

2. There was a **Mayer-Vietoris sequence**, or, equivalently, an **excision theorem.** One consequence of this was that
$$H_n(X, A) = H_n(X \cup CA, *)$$
   for any pair (X, A), and under good circumstances, this is even
$$\widetilde{H_n}(X/A).$$

These properties, plus homotopy invariance, gave the homology of spheres, and the cellular homology theory.

§2 Relative homotopy groups

We want to do the same for homotopy groups. In particular, we want **relative homotopy groups** $\pi_n(X, A)$ for a pair (X, A), and a long exact sequence. We'll in fact identify $\pi_n(X, A)$ with the π_{n-1} of some other space.

This makes it look like we have the same basic setup. However, it turns out that excision doesn't work. So while we will make some computations of homotopy groups, it will require a fair bit of algebra.

Notation. Let I^n be the n-cube, and ∂I^n the boundary as usual. Let J^{n-1} be the boundary minus the interior of one of the faces. So it contains points $(x_1 \cdots x_n)$, where some $x_i = 0$ for $i \neq 1$.

2.3 Definition. Suppose (X, A) is a pair (with common basepoint $*$). Then $\pi_n(X, A)$ is the set of homotopy classes
$$[(I^n, \partial I^n, J^{n-1}), (X, A, *)].$$

So we are looking at maps $I^n \to X$ that send the boundary into A, but most of the boundary into $*$.

Here $\pi_1(X, A)$ is only a set. This just consists of homotopy classes of curves that start at the basepoint $*$ and end in A. *However,* for $n \geq 2$, $\pi_n(X, A)$ is a *group*. This can be seen geometrically by splicing squares together. In fact, $\pi_n(X, A)$ is abelian if $n \geq 3$.

2.4 Proposition. *For a pair (X, A), there is a long exact sequence*

$$\pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \dots.$$

Proof. Omitted. This is completely straightforward to check, but you should do it. The maps

$$\pi_n(A) \to \pi_n(X), \quad \pi_n(X) \to \pi_n(X, A)$$

are obvious (e.g. the first is functoriality). The connecting homomorphism is given as follows: if $(I^n, \partial I^n, J^{n-1}) \to (X, A, *)$, just restrict to the first face (the one missing in J^{n-1}) to get a map $(I^{n-1}, \partial I^{n-1}) \to (A, *)$. $\mathcal{Q.E.D.}$

It should be noted that the map $\pi_n(X) \to \pi_n(X, A)$ is generally not an injection, because we are working with homotopy classes. Namely, it is *easier* for two maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, *)$ to be homotopic than for two maps $(I^n, \partial I^n) \to (X, *)$ to be homotopic.

Remark. In general, the sequence consists of abelian groups for $n \geq 3$. But the end of it is an exact sequence of pointed *sets* only.

§3 Relative homotopy groups as absolute homotopy groups

Consider a pair (X, A). We want to build a new space F such that the homotopy groups of (X, A) are related to those of F. In homology, we showed that

$$H_n(X, A) \simeq \widetilde{H}_n(X \cup CA),$$

and we want something like this for homotopy groups.

The first thing we would like is

$$\pi_1(X, A) = \pi_0(F)$$

for some F. Here, an element of $\pi_1(X, A)$ is a map from $[0, 1] \to X$ that carries $\{0, 1\}$ into A and $\{0\}$ into $*$, with two elements declared equivalent if they are homotopic. If we consider the pointed space

$$F = \{\gamma : [0, 1] \to X : \gamma(0) = *, \gamma(1) \in A\}$$

as a subspace of X^I (where this has the compact-open topology), with the basepoint the constant curve, then we see immediately that

2.5 Proposition. $\pi_1(X, A) = \pi_0(F)$.

It follows from the definition in addition that

2.6 Proposition. $\pi_n(X, A) = \pi_{n-1}(F)$.

Proof. $\pi_{n-1}(F)$ consists of equivalence classes of maps $I^{n-1} \to F$. Such a map gives a map

$$I^{n-1} \times I \to X$$

because $F \subset X^I$, by the adjoint property. It is easy to see that such maps have to send $\partial I^{n-1} \times I$ into A, and so forth; from this the result is clear. We leave this to the reader.

$$\mathcal{Q.E.D.}$$

There is a subtle point-set topology issue that we are not going to care about. Namely, we have to check that a continuous map $I^{n-1} \to X^I$ is the same thing as a continuous map $I^n \to X$. We'll ignore this.

More generally, consider an arbitrary map (not necessarily an inclusion)

$$A \xrightarrow{f} X$$

and consider the space

$$F_f = \left\{ (\gamma, a) \in X^I \times A : \gamma(0) = *, \gamma(1) = a \right\}.$$

This space is the analog of the above construction for a map which isn't an inclusion. It is to be noted that any map $X \xrightarrow{f} Y$ is homotopy equivalent to an inclusion: there is a canonical inclusion of X into the mapping cylinder $(X \times [0,1]) \cup_f Y$, and this space is homotopy equivalent to Y.

Lecture 3
1/28

Last time, we constructed the relative homotopy groups, and showed that they were really special absolute homotopy groups.

§1   Fibrations

3.1 Definition. A map $p : E \to B$ has the **homotopy lifting property** if in any diagram of solid arrows

$$
\begin{array}{ccc}
A \times \{0\} & \longrightarrow & E \\
\downarrow & {\scriptstyle g} \nearrow & \downarrow \\
A \times [0,1] & \longrightarrow & B
\end{array} \;,
$$

there is a lift g. Such a map is called a **Hurewicz fibration.**

If a map has the homotopy lifting property as above whenever A is an n-cube, then it is a **Serre fibration.**

We will discuss these more carefully in the theory of model categories. There is a dual notion of a **cofibration** (a homotopy extension property), which will be explored in the homework.

If there is a diagram

$$
\begin{array}{ccc}
 & & E \\
 & & \downarrow {\scriptstyle p} \\
B' & \xrightarrow{\;f\;} & B
\end{array}
$$

then one can form a **fibered product** $E' = B' \times_B E$ fitting into a commutative diagram

$$
\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow {\scriptstyle p'} & & \downarrow \\
B' & \longrightarrow & E
\end{array}
$$

where $E' = \{(b,e) : f(b) = p(e)\}$.

Remark. E' is the categorical fibered product (limit) of the above diagram.

3.2 Definition. The map $p' : E' \to B'$ as above is said to have been gotten by **base-change** of p along f.

3.3 Proposition. *If p is a Serre (resp. Hurewicz) fibration, so is any base-change.*

Proof. Obvious from the following write-out of categorical properties: to map into E' is the same as mapping into E and B' such that the compositions to B are equal. $\mathcal{Q.E.D.}$

§2   Long exact sequence

Suppose $p : E \to B$ is a Serre fibration of pointed spaces. Suppose F is the **fiber** $p^{-1}(*) \subset E$.

3.4 Theorem. *There is a long exact sequence of homotopy groups*

$$\pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \dots.$$

Proof. We will define the maps involved, and leave exactness for the reader to verify. The maps $\pi_n(F) \to \pi_n(E), \pi_n(E) \to \pi_n(B)$ just come from functoriality. We need to define the connecting homomorphisms.

An element of $\pi_n(B)$ is a map $I^n \to B$ sending the boundary ∂I^n to the basepoint. We can write $I^n = I^{n-1} \times I$. We can lift $I^{n-1} \times \{0\} \to B$ to $I^{n-1} \times \{0\} \to E$; it is the constant map. Thus we can lift $I^n \to B$ to $I^n \to E$ such that $I^{n-1} \times \{0\}$ is sent to a point by the lifting property. We can even do this such that J^{n-1} (which is homeomorphic to I^{n-1}) is sent to a constant map to the basepoint.

We can consider the restriction of this to $I^{n-1} \times \{1\} \to F \subset E$. This is the element of $\pi_{n-1}(F)$.   *Q.E.D.*

This is supposed to remind you of the long exact sequence in *homology* for a cofiber sequence $A \hookrightarrow X$.

At the end, one should clarify what "exactness" of the sequence means. This means that they are exact sequences of *pointed sets*.

Remark. $\pi_1(B)$ acts on $\pi_0(F)$. Indeed, take a path $\gamma \in \pi_1(B)$ and a point x in the fiber F. There is a lifting of γ to E such that $\gamma_0 = x$. Then the endpoint γ_1 belongs to some path component of F, and this component is $\gamma \cdot x$. We used an arbitrary lifting of γ, and an arbitrary representative x of a path component in $\pi_0(F)$; you can show that the path component of $\gamma \cdot x$ is independent of these choices.

Something else happens, though we don't quite have the techniques to prove it yet. **Construction.** Suppose $p : E \to B$ is a Hurewicz fibration of pointed spaces. Suppose $\gamma \in \pi_1(B)$, represented by a map $\gamma : [0,1] \to B$. Consider a diagram

where the bottom map is just $F \times [0,1] \to [0,1] \xrightarrow{\gamma} B$. There is thus a lift $F \times [0,1] \to E$, and thus a map

$$\mu_\gamma : F \to F$$

given by restricting to $F \times \{1\}$. This is independent (up to homotopy) of the homotopy class of γ, so $\pi_1(B)$ acts on the homotopy classes fibers. (This requires proof.)

3.5 Corollary. *$\pi_1(B)$ acts on $H_*(F)$ and $H^*(F)$ and $\pi_k(F)$.*

§3 Replacing maps

We saw that any map

$$X \xrightarrow{f} Y$$

could be replaced by an inclusion

$$X \to M_f \to Y$$

where $X \hookrightarrow M_f$ is an inclusion and $M_f \to Y$ is a homotopy equivalence. This inclusion is in fact a cofibration (a special type of inclusion).

3.6 Proposition. *Any map is, up to homotopy, a fibration. In fact, any map can be functorially factored as a homotopy equivalence and a Hurewicz fibration.*

Proof. We use:

3.7 Lemma. *A composition of Hurewicz fibrations is a Hurewicz fibration.*

Proof. Obvious from the definition via lifting properties.   *Q.E.D.*

3.8 Lemma. *The projection $X \times Y \to X$ is a Hurewicz fibration.*

Proof. Again, obvious.   *Q.E.D.*

3.9 Lemma (Homework)**.** *If X is a space and X^{D^n} is the function space, then the map*

$$X^{D^n} \to X^{S^{n-1}}$$

given by restriction is a Hurewicz fibration.

We now prove the proposition. Let $E \to B$ be a map. Take $n = 1$ in the last lemma, so consider the **path space.** We find that

$$B^I \to B \times B$$

is a Hurewicz fibration. There is a map $E \times B \to B \times B$ (the obvious one) and consider the pull-back of $B^I \to B \times B$; we get a map

$$\widetilde{E} \to E \times B,$$

which is a Hurewicz fibration. Then compose that with $E \times B \to B$. This composition is a Hurewicz fibration.

Now we need to get a map $E \to \widetilde{E}$: we just send each $e \in E$ to the pair $((e, p(e)), \gamma_e)$ for γ the constant path at $p(e)$. We leave it as an exercise to check that this is a homotopy equivalence. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Lecture 4
1/31

§1 Motivation

A *fiber bundle* $p : E \to B$ is a map with the property that every $x \in B$ belongs to some neighborhood $U \subset B$ such that the restriction $p^{-1}(U) \to U$ is the trivial bundle $U \times F \to F$. The *fiber* is F.

Now suppose B was a CW complex. Recall the skeleton fibration $B^{(0)} \subset B^{(1)} \subset \ldots$, where each $B^{(n)}$ is a union of cells of dimension $\leq n$. Consider restricting the map $p : E \to B$ to a fibration over the n-skeleton: $p : p^{-1}B^{(n)} \to B^{(n)}$.

We can get the n-skeleton from the $(n-1)$-skeleton by attaching n-cells. Namely,

$$B^{(n)} = B^{(n-1)} \cup_{f_\alpha} D_\alpha^n$$

so that

$$E^{(n)} = E^{(n-1)} \cup p^{-1}(D_\alpha^n).$$

However, it turns out that the bundle E restricted to D^n is trivial, so $E^{(n)}$ is obtained from $E^{(n-1)}$ by pushing out by a product of $D^n \times F$. In particular:

4.1 Proposition. *The pair* $(E^{(n)}, E^{(n-1)})$ *is relatively homeomorphic to* $(\sqcup F \times D_\alpha^n, \sqcup F \times S_\alpha^{n-1})$.

By excision:

4.2 Corollary.
$$H_*(E^{(n)}, E^{(n-1)}) = H_*(F) \otimes C_n^{\text{cell}}(B).$$

We are trying to relate the homology of E, F, and B. We will do this by studying a bunch of long exact sequences.

This is how we calculated the homology of $SO(n)$, for instance. Anyway, this gives a relation between the homology of F, that of E, and that of B. This week, we will set up the algebraic way of depicting that relationship.

§2 Exact couples

In general,

4.3 Definition. An **exact couple** is a long exact sequence sequence of abelian groups that goes
$$D \to D \to E \to D \to D \to E$$
that repeats periodically, or an exact triangle

$$D \xrightarrow{\quad i \quad} D$$
$$k \nwarrow \qquad \swarrow j$$
$$E$$

Spectral sequences come from exact couples.

We define the **differential** $d : E \to E$ to be $d = j \circ k$. Note that $d^2 = 0$. This is because $k \circ j = 0$ by exactness.

From an exact couple, we will make something called the **derived couple.**

$$D' \xrightarrow{\quad i' \quad} D' \ .$$
$$k' \nwarrow \qquad \swarrow j'$$
$$E'$$

1. D' is the image of $i : D \to D$. $i' = i|_{D'}$.

2. $E' = \ker d / \operatorname{im} d$.

3. Now we have to get j' and k'. We set $j'(i(x)) = j(x)$, so that $j' = j \circ i^{-1}$; one can check that this makes sense (i.e. the choice of x is irrelevant, because if $i(x_1) = i(x_2)$ then $x_1 - x_2 \in \operatorname{im} k$ and consequently $j(x_1) - j(x_2) \in \operatorname{im} j \circ k = \operatorname{im} d$). Note that anything in the image of j is killed by d.

4. $k'(x) = k(x)$. More precisely, if $\widetilde{x} \in E'$, lift to some $x \in \ker d \subset E$, and set $k'(\widetilde{x}) = k(x)$. This in fact lies in D' because $j(k'(\widetilde{x})) = j(k(x)) = 0$ as $x \in \ker d$,

and the original triangle was exact. Moreover, if we had a different $x' \in \ker d$ lifting $\widetilde{x}$, then x, x' differ by something in the image of j'.

4.4 Proposition. *The derived couple is in fact an exact couple.*

Proof. One has to check that the sequence is actually exact! This is very tedious. It is also not illuminating. $\mathcal{2.E.D}$.

So given an exact couple $D \to D \to E \to D \to \ldots$, we have constructed a *derived couple*. We construct a sequence as follows. We let (E_1, d_1) be the pair of E together with its differential (that is, $j \circ k$). We can construct (E_r, d_r) more generally by deriving the exact couple $r - 1$ times and doing the same construction.

Thus we get a sequence
$$(E_1, d_1), (E_2, d_2), \ldots$$
of differential objects, such that

$$H(E_i) = E_{i+1}$$

for each i.

4.5 Definition. Such a sequence of groups is called a **spectral sequence.**

So we have seen that an exact couple gives rise to a spectral sequence. Most people get through their lives without having to know the precise definitions. You can use the theory and the formalism without understanding all this. Calculating is more important than the technical dry details.

§3 Important examples

4.6 Example. Consider a filtered chain complex C_*, d with a filtration

$$C_*^{(n)} \subset C_*^{(n+1)} \subset \ldots$$

For instance, we could get one from a filtered space by taking the singular chain complexes of the filtration.

We are going to define an exact couple. **THIS IS ALL A BIG LIE**

1. $D = \bigoplus H_*(C^{(n)})$. The map $D \to D$ is the map induced by the inclusions $C^{(n)} \to C^{(n+1)}$.

2. $E = \bigoplus H_*(C^{(n)}/C^{(n-1)})$.

3. The boundary maps $E \to D$ can be obtained using general nonsense.

We are getting this from the long exact sequence from the exact sequence

$$0 \to \bigoplus C^{(n)} \to \bigoplus C^{(n+1)} \to \bigoplus C^{(n+1)}/C^{(n)} \to 0$$

From this we get an exact triangle

which gives an exact couple, hence a spectral sequence.

But this is the wrong picture, since we have a ton of data bundled into each E_r. Namely, you have a direct sum of infinitely many homologies.

Suppose $x \in E_1 = \bigoplus H_*(C^{(n)}/C^{(n-1)})$. So x is represented by $\widetilde{x} \in C^{(n)}$ such that $d\widetilde{x} \in C^{(n-1)}$. More generally, one can show

4.7 Proposition. $d_r x = 0$ if one can find $\widetilde{x}$ representing it such that $d\widetilde{x} \in C^{(n-r)}$.

4.8 Example. Spectral sequences arise from double complexes. A **double complex** is what you would think it would be: a bigraded complex $\{C_{pq}, p, q \in \mathbb{Z}\}$ with horizontal and vertical differentials d_h, d_v (such that $d^h \circ d^h = 0, d^v \circ d^v = 0, d^h \circ d^v = d^v \circ d^h$, the last one to make the 2-dimensional diagram commute). From this one can construct a **total complex** C_* where

$$C_n = \bigoplus_{i+j=n} C_{ij}$$

with differentials being given by the sums of the horizontal and vertical differentials, *with a sign trick* to make it a complex.

Namely, we define

$$C_{ij} \to C_{i-1,j} \oplus C_{i,j-1}$$

by sending

$$x \mapsto d^h(x) + (-1)^i d^v(x).$$

The main theorem we are shooting for is

4.9 Theorem. *If $F \to E \to B$ is a Serre fibration (F the fiber), there is a spectral sequence whose E_2 term is $H_p(B, H_q(F))$ and which converges (in a sense to be defined) to $\underline{H}_{p+q}(E)$.*

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§4 Spectral sequences

<div align="center">

Lecture 5
February 2, 2010

</div>

Let's start with an example. We want to make a double chain complex:

$$
\begin{array}{ccc}
\mathbb{Z} & & \\
\downarrow{\scriptstyle 2} & & \\
\mathbb{Z} \xleftarrow{\ 1\ } \mathbb{Z} & & \\
& \downarrow{\scriptstyle 2} & \\
& \mathbb{Z} \xleftarrow{\ 1\ } \mathbb{Z} & \\
& & \downarrow{\scriptstyle 2} \\
& & \mathbb{Z} \xleftarrow{\ 1\ } \mathbb{Z} \\
& & \quad\downarrow{\scriptstyle 2} \\
& & \quad\mathbb{Z} \xleftarrow{\ 1\ } \mathbb{Z}
\end{array}
$$
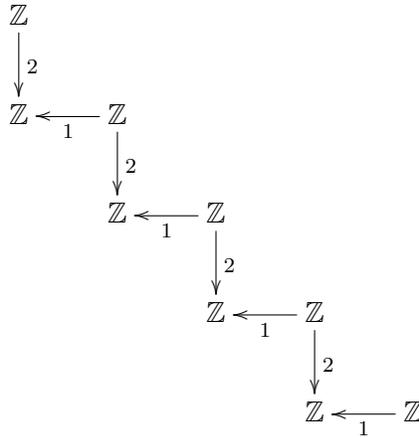
The numbers are the differential: $2 : \mathbb{Z} \to \mathbb{Z}$ is multiplication by 2. Let's filter this complex by vertical maps. [Writing out homology groups, note that everything on the top diagonal except the last group has no kernel.]

$\mathbb{Z}/2$					
	$\mathbb{Z}/2$				
		$\mathbb{Z}/2$			
			$\mathbb{Z}/2$	$\mathbb{Z}$	

On the first page, the differential d_1 has only a horizontal component. There is only one place where there can be a horizontal differential, and it is $\mathbb{Z} \to \mathbb{Z}/2$. It is surjective here (the image of the generator of $\mathbb{Z}$ is not in the kernel of $2 : \mathbb{Z} \to \mathbb{Z}$). To get the E_2 term, take the kernel of the differential, and mod out by the image. The kernel is the ideal generated by 2.

$$\mathbb{Z}/2 \qquad . \qquad . \qquad . \qquad . \qquad .$$

$$. \qquad \mathbb{Z}/2 \qquad . \qquad . \qquad . \qquad .$$

$$. \qquad . \qquad 0 \qquad . \qquad . \qquad .$$

$$. \qquad . \qquad . \qquad 0 \qquad (2) \qquad .$$

Apply the differential at (2). Take the double chain complex, and chase right to left: $2 \mapsto 2 \mapsto 1 \mapsto 1$. So there is only one E_2 differential, sending the generator of (2) onto the generator of $\mathbb{Z}/2$. The kernel this time is (4). The E_3 term looks like:

$$\mathbb{Z}/2 \qquad . \qquad . \qquad . \qquad . \qquad .$$

$$. \qquad \mathbb{Z}/2 \qquad . \qquad . \qquad . \qquad .$$

$$. \qquad . \qquad 0 \qquad . \qquad . \qquad .$$

$$. \qquad . \qquad . \qquad 0 \qquad (4) \qquad .$$

Chase right-to left again. Start with $4 \mapsto 4 \mapsto 2 \mapsto 2 \mapsto 1 \mapsto 1$ so d_3 is surjective. So the last one starts with (8).

$$\mathbb{Z}/2 \qquad . \qquad . \qquad . \qquad . \qquad .$$

$$. \qquad 0 \qquad . \qquad . \qquad . \qquad .$$

$$. \qquad . \qquad 0 \qquad . \qquad . \qquad .$$

$$. \qquad . \qquad . \qquad 0 \qquad (8) \qquad .$$

Keep going. The last page is all zeroes, where the last row is:

$$0 \quad 0 \quad 0 \quad 0 \quad (16)$$

All the spectral sequences we're going to look at come from a double chain complex. So the E_r term is just a doubly-graded group. Call the rth page E^r, and the rth differential moves you from $E^r_{p,q}$ to $E^r_{p-r,q+r-1}$.

For r sufficiently large, and fixed p and q, the incoing and outgoing arrows come from so far away they are outside of the first quadrant, so we call them zero. In this case $E_{p,q}^r = E_{p,q}^{r+1}$. We call this value $E_{p,q}^\infty$ the stable value. Actually, we need $r = \max\{p+1, q\}$. What we're interested in is the homology of the total chain complex. Claim that this is what the E^∞-term tells you. The original double complex is

$$C_n^{total} = \oplus_{p+q=n} C_{p,q}$$

where $d = d^h + (-1)^? d^v$. Note that you could write the original total chain complex as a (sparse) matrix. What does homology of this mean? There is a filtration of $H_n(C_*^{total})$ whose associated graded group is

$$\oplus_{p+q=n} E_{p,q}^\infty$$

Take the descending diagonal from $(0, n)$ to $(n, 0)$. These all contribute to the nth homology. The top one is the subgroup, and the last is the quotient. In the example, our E^∞-term was

$$0 \qquad 0 \qquad 0 \qquad \mathbb{Z}$$

It's not hard to see that $C_4^{total} \to C_3^{total}$, which is $\mathbb{Z}^5 \to \mathbb{Z}^4$, is surjective. Suppose the E^∞-term is

$\mathbb{Z}$			
	$\mathbb{Z}/2$		
		$\mathbb{Z}/2$	

The total group might be a copy of $\mathbb{Z}$: this has a subgroup $2\mathbb{Z}$, which has a subgroup $4\mathbb{Z}$, which has a subgroup $8\mathbb{Z}$. Or it could be $\mathbb{Z} \oplus \mathbb{Z}/2$, or $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Finally, it could be $\mathbb{Z} \oplus \mathbb{Z}/4$. So knowing the E^∞ groups helps to calculates the original complex, up to some information you have to know how to deal with.

§1   Serre spectral sequence, special case

Consider a Serre fibration $F \to E \to B$, and assume that $\pi_1 B = 0$ and $\pi_0 B = *$ (i.e. B is simply connected).

5.1 Theorem. *There is a double chain complex $C_{p,q}$ with the property that the total complex has the same homology as E^∞, and $E_{p,q}^2 = H_p(B; H_q(F))$.*

5.2 Example. Take $B = S^{n+1}$, $E = PS^{n+1}$, $F = \Omega S^{n+1}$ (path space/ loop space, respectively). (The kth homotopy group of B is the $(k-1)^{st}$ homotopy group of F.) So we know the homology of B, and E is contractible; let's try to calculate $H_*(F)$. Let's make $n = 4$.

Let's draw the E^∞-term. $H_i(E) = 0$ when $i > 0$ and $\mathbb{Z}$ when $i = 0$. So the diagonals (except for the corner) should be successive quotients of... zero.

0	0	0	0	
0	0	0	0	
A	0	0	0	
$\mathbb{Z}$				

A is a subgroup of 0, so its zero. Similarly, everything else is zero. $E^2_{p,q}$ is $H_p(S^{n+1}; H_q \Omega S^n)$. When $q = 0$ (the bottom row), we get the $\mathbb{Z}$-homology of the n-sphere. (The other rows q are also the homology of the sphere, but with respect to the unknown group $H_q(\Omega S^n)$.) When $p = 0$ (the left column) we are getting the homology of ΩS^n (since $H_0(S^{n+1}, \mathbb{Z}) = \mathbb{Z}$ we are getting "$\mathbb{Z}$ in terms of the groups $H_q(\Omega S^n)$").

D				$\mathbb{Z}$
C				
A				B
$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

A is unchanged by everything, so it has to be zero. B is zero because it should be the same as A. C has differentials from $0, 2$, etc. that are all zero, and C needs to go to zero eventually. The entire row is the homology of the n-sphere with coefficients in zero, so it's zero. But D is hit by the second $\mathbb{Z}$ in the first row, by d_4. The kernel is 0, because that's all that's left. It has to be onto for C to be zero. So this is an isomorphism, and $D = \mathbb{Z}$. So the fourth row is $H_p(S^{n+1}; \mathbb{Z})$.

A note: suppose F is connected, and B is simply connected. What is in the bottom row of E^2? The homology groups of the base: $H_p(B; H_0\mathbb{Z})$. The first column is $H_0(B; H_qF) = H_qF$. This pattern continues:

0	0	0	0	0
0	0	0	0	0
$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

stacked on top of one another. (There is a $\mathbb{Z}$ in rows 0, 3, 6, ...). So $H_i(\Omega S^{n+1})$ is $\mathbb{Z}$ when $i = k - n$, and 0 else.

We'll do another example, but it needs a theorem:

5.3 Theorem. *A fiber bundle is a Serre fibration.*

We have fiber bundles $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. We know the homology of S^k, and pretend we forgot $H_*(\mathbb{C}P^n)$. Let's draw the E^2 page. Because $H_k(S^1)$ vanishes above degree 1, there are only two rows, and everything else is trivial. Suppose $n = 3$.

$$\begin{array}{ccccccc} \mathbb{Z} & 0 & \mathbb{Z} & B & D & & \cdots \\ \mathbb{Z} & 0 & \mathbb{Z} & A & C & & \cdots \end{array}$$

On the bottom is $H_*(\mathbb{C}P^n)$ and on the side is $H_*(S^1)$. On the S^{2n+1} sequence, everything is zero, except possibly in $(0, 2n + 1)$ or $(1, 2n)$. A has to be zero, which makes B zero, and C has to be $\mathbb{Z}$ in order for things to kill themselves. This makes $D = \mathbb{Z}$. More later.

Lecture 6
2/4

Consider the double chain complex that looked like

$$
\begin{array}{ccc}
\mathbb{Z} & & \\
\downarrow 2 & & \\
\mathbb{Z} & \xleftarrow{1} & \mathbb{Z} \\
& & \downarrow 2 \\
& & \mathbb{Z}
\end{array}
$$

etc. Here the vertical maps are 2 and the horizontal maps are 1.

We talked about this earlier. Note however that there are two ways of filtering a double complex: by columns and by rows. We will always assume that the complex is filtered by columns. However, let's look at what happens when it is filtered by rows. In this case, the spectral sequence turns out to be rather uninteresting.

It turns out that one way of filtering the double complex gives you a trivial answer while another way gives you an interesting journey at arriving at the answer. This will happen, for instance, in the Serre spectral sequence to be covered next week.

Next, recall last time that we were illustrating the Serre spectral sequence with the bundle

$$S^1 \to S^{2n-1} \to \mathbb{CP}^n.$$

OK...I have no clue how to liveTEXa spectral sequence.

The E_∞-term is basically known: it is the homology of S^{2n+1}. Since the homology of $\mathbb{CP}^n$ is zero outside a finite set (greater than $2n$), one can directly proceed.

§1 More structure in the spectral sequence

Last semester, we learned that there was a ring structure in the cohomology. It turns out that there is a Serre spectral sequence in cohomology. It comes from a double *cochain* complex.

In a double cochain complex, one can find a spectral sequence $E_r^{s,t}$ as above. One starts by taking the cohomology of the horizontal differentials. Then one takes the cohomology of that, and so on. We omit the details.

6.1 Theorem (Serre spectral sequence in cohomology)**.** *Consider a Serre fibration*

$$F \to E \to B$$

for B simply connected. Let R be a commutative ring. There is a spectral sequence whose E_2 term is

$$H^r(B, H^t(F, R))$$

and whose rth differential goes

$$E_r^{s,t} \to E_r^{s+r, t-r+1}$$

which converges to

$$H^*(E, R).$$

The other advantage of the cohomology spectral sequence is that the differentials interact with the ring structure. Note that E_2 is a doubly graded ring thanks to various cup products. Namely, there are maps

$$H^s(B, H^t(F, R)) \times H^{s'}(B, H^{t'}(F, R)) \to H^{s+s'}(B, H^{t+t'}(F, R)).$$

We have used the fact that there is a map $H^t(F, R) \times H^{t'}(F, R) \to H^{t+t'}(F, R)$.

6.2 Theorem. *For all r, we have*

$$d_r(xy) = d_r x \times y + (-1)^{|x|} x . d_r y,$$

where for $x \in E_r^{pq}$, $\dim x = p + q$.

This is good. If you know the differential on x, and that on y, you get it on the product xy. This actually lets us compute cohomology *rings*.

Consider the fibration

$$\Omega S^{n+1} \to P S^{n+1} \to S^{n+1}$$

and let's study this in cohomology.

I will type this up *after class*.

§2   The cohomology ring of ΩS^{n+1}

In cohomology, it is conventional to write not groups in the spectral sequence, but their generators. Let $\iota_{n+1} \in H^{n+1}(S^{n+1})$ be the generator. Note that the E_∞ term must be trivial.

We find that there is $x \in H^5(\Omega S^{n+1})$ generating the group which maps to ι_{n+1}. Then $x\iota_{n+1} \in H^5(S^{n+1}, H^n(\Omega S^{n+1}))$ generates. Then there is something that hits $x\iota_{n+1}$. There is $x^{(2)}$ whose differential makes it hit $x\iota_{n+1}$.

"One thing would be to pretend that five is equal to four."

This is easier in even dimensions. Now

$$d_{n+1}(x^2) = d_{n+1}x \times x + x \times \dim x$$

by evenness and this is

$$2x\iota_{n+1}.$$

The conclusion is that $x^2 = 2x^{(2)}$; in other words, squaring sends a generator x to twice a generator. Similarly we get $x^{(3)}$ and one finds that

$$x^3 = 6x^{(3)}.$$

In general, the cohomology of $H^{kn}(\Omega S^{n+1}, \mathbb{Z})$ is generated by some $x^{(k)}$ such that

$$x^{(k)}x^{(l)} = \binom{k+l}{l} x^{(k+l)}.$$

You are supposed to think that

$$x^{(k)} = x^k/k!.$$

6.3 Definition. This is called a **divided power algebra** $\Gamma[x]$.

There is a certain list of rings that comes up a lot in algebraic topology. This is one of them.

When n is odd, you get an *exterior algebra* tensored a divided polynomial ring. Namely

$$\mathbb{Z}[x]/x^2 \otimes \Gamma[y].$$

§3 Complex projective space

We had to work pretty hard to get the ring structure of $\mathbb{CP}^n$. In fact, we only deduced it from Poincaré duality. The spectral sequence makes it easy. Consider the fibration

$$S^1 \to S^{n+1} \to \mathbb{CP}^n$$

and let's study that in cohomology.

```
a ax ax^2
1 x x^2
```

There is x such that $d_2(a) = x$. Now use Leibniz rule over and over. You get the product structure on the cohomology ring of $\mathbb{CP}^n$ from this for free.

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Remark. Here's a typical thing you do with the Serre spectral sequence. If R is a field, then

$$H^r(B, H^s(F, R)) = H^r(B, R) \otimes_R H^s(F, R).$$

So when you get the cohomology of the base and the fiber, you get the E_2 term.

Suppose we have a fibration

$$F \to E \to B$$

with B simply connected such taht $H_i(B) = 0$ for $i \le n$ and $H_j(F) = 0$ for $j \le m$. Then it is clear that the spectral sequence has an E_2 page that is empty for $[0, n] \times [0, m]$ (except at $(0,0)$). It follows that the E_∞ term is zero there too. From this we find: two groups only contribute to the homology of the total space. This is right now sketched, and will be expanded on later. You get an les

$$H_k(F) \to H_k(E) \to H_k(B) \to H_{k-1}(F) \to \dots$$

that lasts for a while $k < m + n$.

This is called the **Serre exact sequence.** It tells you that B looks like the mapping cone of $F \to E$ in a range of dimensions that is pretty large. Cofibrations and fibrations look the same.

Lecture 7
January 7, 2010

We have a sequence of abelian groups E_{pq}^r where r is the page, and p, q are graphically represented as coordinates on a grid, and differentials $d^r : E_{p,q}^r \to E_{p-r,q+r-1}^r$. The next page is always obtained from the previous as ker/im. The snake lemma gives you a long exact sequence of abelian groups. One way to obtain spectral sequences: the snake lemma allows us to take an inclusion $E_{0*} \subset E_{1*}$ of chain complexes (there is an E_{00}, E_{01} group, etc.) and obtain a long exact sequence in homology. If you want to take more than one inclusion, you will be led to a spectral sequence. Consider

$$E_{0*} \subset E_{1*} \subset E_{2*} \subset \cdots \subset E_{N*}$$

which we will say is finite for convergence issues. Ultimately we want to know the homology of E_{N*}. We can approximate this by looking at things that go to some smaller E_{j*}, instead of strictly things that go to zero. This gives a spectral sequence: the "cycles" on the rth page are

$$Z_{i*}^r = \{e \in E_{i*} \ : \ de = E_{i-r,*}\}$$

and

$$E_{i*}^{(r)} = S_{i*}^r / (Z_{i-1,*}^{r-1} + dZ_{i+r-1,*}^{r-1})$$

We say that $E_{i,*}^{(1)}$ is the associated graded $gr(E_{i*}) = \bigoplus_i E_{i*}/E_{i-1,*}$ Finally, the infinity term is $E_{i,*}^{\infty} = gr(H_*(E_{N*}))$

The Serre spectral sequence comes about by taking $F \to X \to B$ where $X \to B$ is a fibration and F is the fiber. Then $E_{pq}^{(2)} = H_p(B, H_q(F))$. To interpret this, we need local coefficients; to avoid this assume B is simply connected. This converges at the E^{∞} page to $H_{p+1}(X)$. This is a special case of the $E_{i*} \subset E_{i+1,*}$ construction where $E_{i*} = C_*(p^{-1}(B^{(1)}))$ (the 1-skeleton). The associated graded is nicer: a good exercise is to show that $E_{i*}^{(1)}$ is a shift of the homology by the chains of B: $E_{**}^{(1)} = C_*(B) \otimes H_*(F)$.

RELATIVE SERRE SPECTRAL SEQUENCE: Take some subspace $A \subset B$.

$$
\begin{array}{ccccc}
F & \longrightarrow & X & \overset{p}{\longrightarrow} & B \\
\downarrow & & \uparrow & & \uparrow \\
F & \longrightarrow & p^{-1}(A) & \longrightarrow & A
\end{array}
$$

We can make another fibration $F \to p^{-1}(A) \to A$. Again assume B and A are simply connected. Then we get a spectral sequence in relative homology, where the E^2 term is $H_p(B, A; H_q(F))$, and the E^{∞} term represents $H_p(X, p^{-1}(A))$.

§1 Application I: Long exact sequence in H_* through a range for a fibration

Let $F \to X \overset{p}{\to} B$ be a fibration, and let B and F be such that $H_i(B) = 0$ for $i < n$ and $H_i(F) = 0$ for $i < m$. Then there is a sequence

$$H_{n+m-1}(F) \to H_{n+m-1}(X) \to H_{n+m-1}(B) \to H_{n+m-2}(F) \to \cdots \to H_0(B) = 0$$

The map $H_{n+m-1}(B) \to H_{n+m-2}(F)$ is called the *transgression*, which is also just d^{n+m-1}

Where could we possibly have nonzero terms in the Serre spectral sequence? The bottom row is $0 \cdots 0, H_n(B), H_{n+1}(B) \cdots$ where the first nonzero group is at n. The first column is $\mathbb{Z}, 0, \cdots, 0, H_m(F), H_{m+1}(F), \cdots$. There can be nonzero groups in these two places, and in the infinite box with corner (n, m), but not in the middle. Starting at the group in the $(i, 0)$ spot for $i > 0$, the only nonzero differentials land in the first column at $(0, i-1)$. The only nonvanishing groups on E_{pq}^{∞} are for p, q such that $E_{pq}^2 \neq 0$. For $j < n + m$ there are only two things on the diagonal that are nonzero. Because it converges to the associated graded, $E_{0,j}^{\infty}$ is a subgroup of $H_j(X)$, and the quotient is E_{j0}^{∞}. We can conclude from degree considerations that E^{∞} also has $coker(d^{j+1})$ for the first column starting above $(0, m)$, has $ker(d^j)$ for the first row starting after $(n, 0)$, and is only nonzero in these places and in the box above (n, m).

Consider $H_j(B) \xrightarrow{d^j} \ldots$ There is a sequence

$$H_{j+1}(B) \xrightarrow{d^{j+1}} H_j(F) \to H_j(X) \to H_j(B) \xrightarrow{d^j} H_{j-1}(F)$$

§2 Application II: Hurewicz Theorem

7.1 Definition. Let X be a connected topological space. Then there is a map (the Hurewicz map) $h_j : \pi_j(X) \to H_j(X)$ where $[f : S^n \to X]$ is sent to $[f_*[S^n]]$, the induced map on homology. (Note $[S^n]$ gives a choice of generator for $H_n(S^n)$; this is well-defined because homology is homotopy-invariant.)

7.2 Theorem. *Let X be a connected topological space such that $\pi_j(X) = 0$ for $j < n$. Then $H_j(X) = 0$ for $j < n$; moreover the Hurewicz map $h_n : \pi_n(X) \to H_n(X)$ is an isomorphism, and h_i is a surjection for all $i > 1$.*

Proof. By induction. We will consider the $n = 1$ case as known. Recall the path space fibration

$$\Omega X \to PX = \{\gamma : [0,1] \to X : \gamma(0) = *\} \xrightarrow{ev_1} X$$

So PX is contractible. Also $\pi_j(\Omega X) = 0$ for $j < n - 1$, because $\pi_j(\Omega X) = \pi_{j+1}(X)$.

$$
\begin{array}{ccc}
\pi_j(\Omega X) & \xrightarrow{h_j} & H_j(\Omega X) \\
\uparrow & & \uparrow{\scriptstyle d^{j+1}} \\
\pi_{j+1}(X) & \xrightarrow{h_{j+1}} & H_{j+1}(X)
\end{array}
$$

One needs to check that this diagram commutes. This *is* the boundary map of a simplex; $d^{(j+1)}$ is always the original boundary in the original chain complex.

[Note that the induction starts with $j = 1$. Also note that the homotopy groups of a loop space are abelian.] By induction applied to X we can conclude $H_j(X) = 0$ for $j < n - 1$ and h_{n-1} is an isomorphism implies $H_j(X) = 0$ for $j < n$. By induction applied to ΩX, h_{n-1} is an isomorphism. It suffices to see that d^m is an isomorphism. This is because d^m controls the homology of the total space, which is contractible.

The previous long exact sequence for $n \geq 2$ gives:

$$H_n(PX) \to H_n(X) \xrightarrow{d^n} H_{n-1}(\Omega X) \to H_{n-1}(PX)$$

But $H_{n-1}(PX) = 0 = h_n(PX)$ so d^n is an isomorphism. $\mathscr{Q}.\mathscr{E}.\mathscr{D}.$

7.3 Definition. Given $X \supset A$, we have $h_j : \pi_j(X, A) \to H_j(X, A)$. There is a tautological element of the cohomology of the base face of the cube. That is, there is a class $[(I^j, \partial I^j)] \in H_j(I^j, \partial I^j)$. (You want all of the boundary to map into A; let J^{n-1} denote all of the boundary except one face. Consider homotopy classes $f : (I^j, \partial I^j, J) \to (X, A, *)$.) So $h_j([F]) = F_*[(I^j, I^{j-1})]$.

7.4 Theorem (Relative Hurewicz Theorem)**.** *Let X and A be simply connected topological spaces such that $\pi_j(X, A) = 0$ for $j < n$. Then $H_j(X, A) = 0$ for $j < n$ and h_n is an isomorphism.*

Proof. We will use induction applied to (X, A). Again $H_j(X, A) = 0$ for $j < n - 1$ and h_{n-1} is an isomorphism so $H_j(X, A) = 0$ for $j < n$. Now take $\Omega X \to PX \xrightarrow{p} X$, with $X \subset X$ and $Y = p^{-1}(A) \subset PX$. For any fibration $F \to X \to B$ and inclusion $A \subset B$, $p_* : \pi_j(X, p^{-1}(A)) \to \pi_j(B, A)$ is an isomorphism, based on using the homotopy lifting property. An element is something that sends J to the base point... view the cube as a homotopy in one dimension lower. You can lift that by the homotopy lifting property. So the relative homotopy we want is the same as the relative homotopy of the paths of X relative to Y. So $\pi_j(PX, Y) \cong \pi_j(X, A)$.

To be continued later. $\mathscr{Q}.\mathscr{E}.\mathscr{D}.$

Lecture 8
2/9

We are now interested in proving the relative form of the Hurewicz theorem. This has several consequences. One of them is that for a pair of simply connected spaces, a map is a homology isomorphism if and only if it is a homotopy isomorphism (i.e. weak homotopy equivalence).

Let us now recall the *relative Serre spectral sequence*. Consider a fibration $p : X \to B$ and a subspace $A \subset B$. Let $Y \subset X$ be the preimage of A, or the pull-back $X \times_B A$. Then there is a spectral sequence

$$H_i(B, A; H_j(F)) \implies H_{i+j}(X, Y).$$

More precisely, there exists abelian groups E_{ij}^r and differentials $d_r : E_{ij}^r \to E_{i-r, j+r-1}^r$ such that:

1. $E_{ij}^{r+1} = \ker d_r / \operatorname{im} d_r$.

2. There exists $N = N(i, j)$ such that the E_{ij}^r for $r \geq N$ are the same, to some group $E^\infty)_{ij}$.

3. There is a filtration $F_0 \subset F_1 \subset \cdots \subset F_n = H_n(X, Y)$ such that the associated graded of this is the $\{E^\infty_{j,n-j}\}$.

§1 The relative Hurewicz theorem

Now we can finish the relative Hurewicz theorem.

8.1 Theorem. *Let A, X be simply connected spaces (with $A \subset X$). Suppose that $\pi_j(X, A) = 0$ for $j < n$. Then $H_j(X, A) = 0$ for $j < n$ and the relative Hurewicz map*

$$h_n : \pi_n(X, A) \to H_n(X, A)$$

is an isomorphism.

Proof. Consider the fibration
$$\Omega X \to PX \xrightarrow{p} X.$$

Consider $\pi_n(X, A)$. Via the homotopy lifting property, this is isomorphic to

$$\pi_n(PX, p^{-1}(A)).$$

Since PX is contractible, the long exact sequence of the pair in homotopy shows

$$\pi_n(PX, p^{-1}(A)) \simeq \pi_{n-1}(p^{-1}(A)).$$

Similarly we find for $j < n$, using the isomorphism $\pi_j(PX, p^{-1}(A)) \simeq \pi_j(X, A) = 0$,

$$0 = \pi_j(PX, p^{-1}(A)) \simeq \pi_{j-1}(p^{-1}(A))$$

and we can apply the absolute Hurewicz theorem to $p^{-1}(A)$ (in dimension $n - 1$). It follows that

$$\pi_{n-1}(p^{-1}(A)) = H_{n-1}(p^{-1}(A)).$$

Combining these, we get that

$$\pi_n(X, A) \simeq H_{n-1}(p^{-1}(A)).$$

Now we do the same in homology. The long exact sequence in homology and contractibility of PX show that

$$H_{n-1}(p^{-1}(A)) \simeq H_n(PX, p^{-1}(A)).$$

So finally we want to show that $H_n(PX, p^{-1}(A)) = H_n(X, A)$. For this we apply the relative Serre spectral sequence to

$$\Omega X \to PX \to X$$

with the subspace $A \subset X$. By the inductive hypothesis, when one draws the spectral sequence, we have $H_j(X, A) = 0$ for $j < n$ by the inductive hypothesis. For n,

The point of the Serre spectral sequence is that we have a spectral sequence

$$H_i(X, A; H_j(\Omega X)) \implies H_{i+j}(PX, p^{-1}(A)).$$

We get a lot of zeros. In the nth row, we have $H_n(X, A)$.

We know that there is a filtration on $H_n(PX, p^{-1}(A))$ whose associated graded is given by the E^∞ terms. However, a look at this shows that

$$H_n(PX, p^{-1}(A)) = H_n(X, A).$$

Putting all this together, we find the conclusion of the relative Hurewicz theorem. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

8.2 Corollary. *Let $F : X \to Y$ be a continuous map between simply connected spaces. Then F induces a homotopy isomorphism[1] if and only if it induces a homology isomorphism.*

Proof. This is a direct application of the relative Hurewicz together with the fact that any map can be viewed as an inclusion. Replace F by the inclusion of X into the mapping cylinder $M_F = X \times I \cup_F Y$. Homotopically, this inclusion $X \hookrightarrow M_F$ is indistinguishable from F. So we reduce to the case where F is an inclusion of a closed subspace, and there we have to show that the relative homotopy vanishes precisely when the relative homology does. Then this is a corollary of the relative version of the Hurewicz theorem. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

§2   Moore and Eilenberg-Maclane spaces

Now we will talk about Moore and Eilenberg-Maclane spaces.

Choose a group A, abelian, and an integer $n \geq 2$. We construct a connected space X (which will be a CW complex) $M(A, n)$ such that

$$H_i(M(A, n)) = \begin{cases} A & \text{if } i = n \\ 0 & \text{if } i \neq n, i > 0 \end{cases}.$$

---

[1]That is, a **weak homotopy equivalence.**

8.3 Definition. $M(A, n)$ is called a **Moore space.**

We use the following algebraic fact.

8.4 Proposition. *A subgroup of a free abelian group is free.*

Proof. Omitted.   *Q.E.D.*

8.5 Corollary. *Given any abelian group A, there is a short exact sequence*

$$0 \to \bigoplus_{S} \mathbb{Z} \xrightarrow{\theta} \bigoplus_{R} \mathbb{Z} \to A \to 0.$$

We construct such a resolution of our group A. Consider the wedge $\bigwedge_{R} S^n$, and attach to this S copies of $(n+2)$-cells via the attaching maps given by morphism $\bigwedge_{S} S^{n+1} \to \bigwedge_{R} S^n$ whose degree matrix is in correspondence with the map

$$\theta : \bigoplus_{S} \mathbb{Z} \to \bigoplus_{R} \mathbb{Z}$$

as above. This is our $M(A, n)$. It is easy to see that this works because the cellular chain complex is so constructed as to be precisely

$$\ldots \to 0 \to \bigoplus_{S} \mathbb{Z} \xrightarrow{\theta} \bigoplus_{R} \mathbb{Z} \to 0 \to \ldots$$

and so its homology is as claimed. We have thus constructed the Moore spaces.

By the Hurewicz theorem, we have

$$\pi_n(M(A, n)) = H_n(M(A, n)) = A,$$

so:

8.6 Proposition. *Any abelian group can arise as a π_n ($n \geq 2$) of some space.*

In fact, for homotopy groups we can do better. We can arrange get a space that has precisely one homotopy group, which can be whatever we please.

8.7 Proposition. *For any $n \geq 2$ and abelian group A, there is a space $K(A, n)$ whose n^{th} homotopy group is A and whose other homotopy groups vanish.*

8.8 Definition. $K(A, n)$ is called an **Eilenberg-Maclane space.**

§3 Postnikov towers

To prove the existence of Eilenberg-Maclane spaces, we develop Postnikov towers. These are a good way of recording the information of homotopy below a given level. (Notice that taking the n-skeleton is *not* a homotopy invariant. Example: there is a CW structure on S^n whose $n-1$-skeleton is S^{n-1}, and one where it is $*$.)

The Postnikov operation is a truncation operation on spaces, and it will respect homotopy in a way that skeletons do not.

8.9 Definition. Let X be a connected CW complex. Consider a diagram

$$
\begin{array}{c}
X_3 \\
\nearrow \downarrow \\
X_2 \\
\nearrow \downarrow \\
X \longrightarrow X_1
\end{array}
$$

such that

1. $X \to X_n$ induces isomorphisms on π_j for $j \leq n$.

2. $\pi_j(X_n) = 0$ for $j > n$.

3. $X_{n+1} \to X_n$ is a fibration.

This is called a **Postnikov tower** for X.

8.10 Proposition. *Postnikov towers of any space exist.*

Proof. X will sit inside X_n, and then we will add more things to kill higher homotopy groups. Its $n + 1$-skeleton will be declared to be X. Choose generators for $\pi_{n+1}(X)$. Then glue $n+2$-cells to the $n+1$-skeleton via the attaching maps from these generators. By cellular approximation, we find that the newly constructed space has the same homotopy groups as X and zero π_{n+1}. Choose generators for the π_{n+2} of this new space and add $n+3$-cells to kill the $(n+2)^{nd}$ homotopy group. Keep doing this forever and let X_n be the colimit of this. By cellular approximation, the maps

$$\pi_j(X) \to \pi_j(X_n)$$

induce isomorphisms for $j \leq n$, and the homotopy groups $\pi_j(X_n), j > n$ are all zero.

We now want to get maps

$$X_{n+1} \to X_n,$$

while all we have right now is a diagram

$$
\begin{array}{ccc}
 & & X_{n+1} \\
 & \nearrow & \vdots \\
X & & \vdots \\
 & \searrow & \downarrow \\
 & & X_n
\end{array}
$$
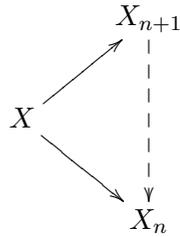
which we want to complete. Since X_{n+1} is obtained from gluing $n+3$ (and higher)-cells to X, and $\pi_j(X_n) = 0$ for $j \geq n+2$, we can extend the maps $X \to X_n$ to maps

$$X_{n+1} \to X_n$$

filling in the diagram above.

Finally, we just convert all the maps to fibrations (which we can do up to homotopy).

$$\mathscr{Q}.\mathscr{E}.\mathscr{D}.$$

By the long exact sequence in homotopy of a fibration, note that:

> The fiber of the map $X_{n+1} \to X_n$ is an Eilenberg-Maclane space $K(\pi_{n+1}(X), n+1)$.

Thus we have finally proved

8.11 Corollary. *Eilenberg-Maclane spaces exist.*

It turns out that $K(A, n)$ is functorial in A, but by a result of Carlsson, $M(A, n)$ is not functorial.

Lecture 9
February 11, 2010

§1 Eilenberg-Maclane Spaces

9.1 Definition. A space X is said to be Eilenberg-Maclane if there is some $n > 0$ such that $\pi_i(X, x) \simeq *$ for $i \neq n$.

Let $A = \pi_n(X)$ (note that A is abelian if $n > 1$). We write $X = k(A, n)$. It turns out that these are uniquely determined up to homotopy (this is not obvious!).

We have a Hurewicz map $\pi_n : K(A, n) \to H_m(K(A, n); \mathbb{Z})$. The Hurewicz theorem says that if your space is $(n-1)$-connected then the first one of these maps [for h_n] is an isomorphism and the next ones are surjective. So we know [some of] the ith homology groups of $K(A, n)$:

$$H_i(K(A, n)) = \begin{cases} 0 & \text{for } i = n+1 \\ A & \text{for } i = n \\ 0 & \text{for } i < n \end{cases}$$

We don't know the rest. Use the universal coefficient theorem. This says that

$$H^n(K(A, n), B) \cong Hom(H_n(K(A, n); \mathbb{Z}), B) \oplus Ext(H_{n-1}(K(A, n); \mathbb{Z}), B)$$

So we throw away the Ext term, since $H_{n-1}K(A, n) = 0$. This gives: $H^n(K(A, n), B) \cong Hom(A, B)$.

If $A = B$, then we have the identity map $id \in Hom(A, B) \cong H^n(K(A, n), A)$. Let η be an element of this last group. We will show that Eilenberg-Maclane spaces are unique by characterizing them by a universal property. The claim is that η is universal among cohomology classes with coefficients in A. For any CW complex X the construction which takes a map $f : X \to K(A, n)$ and gives it $f * \eta \in H^n(X; A)$ determines a map $[X, K(A, n)] \to H^n(X; A)$. The claim is that this is always a bijection. That is, Eilenberg-Maclane spaces are representing spaces for cohomology. You could have used this to give a definition of cohomology. If you want to understand cohomology, you want to understand what $K(A, n)$ look like.

For X a CW complex, let $Map(X, Y)$ denote the space of continuous maps from X to Y. Assume Y has a base point, which we will not talk about. So $\Omega Map(X, Y) \cong Map(X, \Omega Y)$. Eilenberg-Maclane spaces are connected. When you take the loop space, the homotopy groups of the loop space are the same as the original, only shifted. So there is still just one homotopy group, now in degree $n-1$ rather than n. Write $\Omega K(A, n) = K(A, n-1)$.

What is $\pi_n(Map(X, Y))$? This is $\pi_0(\Omega^n X, Y) \cong \pi_0 Map(X, \Omega^n Y) \cong [X, \Omega^n Y]$. An (apparently) more general claim: assume that X has a base point. We have maps

$$\pi_m(Map_*(X, K(A, n))) \cong [X, \Omega^m K(A, n)] = [X, K(A, n-m)]$$

What happens when m gets large? This should be something with only π_0, and that looks like A. When $m \geq n$ this will by convention just be a contractible space. There is a map from this to $H_{red}^{n-m}(X; A)$. We will claim that all of these maps are bijective.

Proof. By unraveling X. For simplicity assume that X is finite. If not, the same argument works, just requires more notation. Induct on the number of cells of X. If

X is empty, then we're done. (Both sides have a single element.) Otherwise, we can choose some cell of X having maximal dimension. This means that X is obtained by starting with some space and attaching an n-cell

$$
\begin{array}{ccc}
\partial D^p & \longrightarrow & X_0 \\
\downarrow & & \downarrow \\
D^n & \longrightarrow & X
\end{array}
$$

Take the base point to be D^n (or some point inside it, etc.). To get a map into $K(A, n)$ we need a map from X_0 into $K(A, n)$ that restricts to a map from ∂D^n, which in turn extends appropriately.

$$Map(X, K(A, n)) \to Map(X_0, K(A, n)) \to Map(\partial D^n, K(A, n))$$

Taking long exact sequences in homotopy and cohomology, we get a commutative diagram

$$
\begin{array}{ccccccccc}
\pi_{m+1}Map(X_0, K(A, n)) & \longrightarrow & \pi_{m+1}Map(S^{p-1}, K(A, n)) & \longrightarrow & \pi_m Map(X, K(A, n)) & \longrightarrow & \pi_m(Map(X_0, K(A, n))) & \longrightarrow & \pi_m Map(S^{p-1}, K(A, n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{n-m-1}(X_0, A) & \longrightarrow & H^{n-m-1}(S^{p-1}, A) & \longrightarrow & H^{n-m}(X, A) & \longrightarrow & H^{n-m}(X_0, A) & \longrightarrow & H^{n-m}(S^{p-1}; A)
\end{array}
$$

We want to prove that the middle map is an isomorphism. But by the five lemma, it suffices to prove that the other maps are isomorphisms. That is, it suffices to prove this for X_0, and in the case of a sphere. But we know X_0 by induction. So it suffices to assume $X = S^k$. We are trying to compare $H^*(S^k; A)$ with $\pi_* Map(S^k, K(A, n))$.

By shifting, we need only prove this in the original formulation:

$$[S^k, K(A, n)] \cong H^n_{red}(S^k, A)$$

$$[S^k, K(A, n)] \cong \pi_k K(A, n) = \begin{cases} A & \text{for k = n} \\ 0 & \text{otherwise} \end{cases}$$

But this *is* the cohomology of the sphere, with coefficients in A. $\mathcal{Q.E.D.}$

So, $K(A, n)$ is unique up to weak homotopy equivalence. If you have any $K(A, n)$ you can always choose a map to a CW complex that is an isomorphism on all homotopy groups. So $K(A, n)$ might as well have been a CW complex. In this category, $K(A, n)$ represents a specific functor that we wrote earlier. If two objects represent the same functor, then they are canonically isomorphic. So $K(A, n)$ is uniquely determined in the category of CW complexes; otherwise, it is uniquely determined up to weak homotopy.

Why do we like Eilenberg-Maclane spaces? They represent cohomology. But there's more. Let X be a simply connected space. Use the notation $\tau_{\leq n}(X)$ meaning we have killed all homotopy groups above n. So $\tau_{\leq 1}$ is just a point. The fiber of the fourth truncation mapping to the third is $K(\pi_4 X, 4)$. What is the fibration $\tau_{\geq n}(X) \xrightarrow{p} \tau_{n-1}(X)$?

Look at the cone of p. (This is the space formed by taking $\tau_{\geq n}(X)$, and attaching $\tau_{\geq n-1}(X)$ on one end and a point on the other.)

$$
\begin{array}{ccc}
\tau_{\geq n}(X) & \longrightarrow & Cone(\tau_{\geq n}(X)) \\
\downarrow p & & \downarrow \\
\tau_{\geq n-1}(X) & \longrightarrow & Cone(p)
\end{array}
$$

This is excisive. $H_*^{red} \simeq H_*(\tau_{\geq n-1}(X), \tau_{\geq n}(X); Z)$. We know about the relative homotopy groups of this. We know

$$
\pi_*(\tau_{\geq n-1}(X), \tau_{\geq n}(X)) = \begin{cases} \pi_n(X) & \text{if } * = n+1 \\ 0 & \text{otherwise} \end{cases}
$$

The relative homotopy groups are concentrated in one point. The Hurewicz theorem says something about relative homology:

$$
H_*(\pi_*(\tau_{\geq n-1}(X), \tau_{\geq n}(X))) = \begin{cases} 0 & \text{for } * < n+1 \\ \pi_n(X) & \text{for } x = n+1 \end{cases}
$$

This space has no homology until degree $n+1$. We can also compute the fundamental group (van Kampen) and we see it is trivial. By the Hurewicz theorem, the first nonvanishing homotopy group is in degree $n+1$ and it is $\pi_n(X)$. We claim that the following commutative diagram is a fiber square:

$$
\begin{array}{ccc}
\tau_{\geq n}(X) & \longrightarrow & Cone(\tau_{\geq n}(X)) \\
\downarrow p & & \downarrow \\
\tau_{\geq n-1}(X) & \xrightarrow{f} & K(\pi_n(X), n+1)
\end{array}
$$

Because $Cone(\tau_{\geq n}(X))$ is contractible we have a map $\tau_{\geq n}(X) \to fiber(f)$. What is going on in homotopy groups? There is a fiber sequence

$$
F \to \tau_{\geq n-1}(X) \to K(\pi_n X, n+1)
$$

This gives a long exact sequence of homotopy groups:

$$
\ldots \to \pi_{k+1} K(\pi_n(X), n+1) \to \pi_k F \to \pi_k(\tau_{\geq n-1}(X)) \to \ldots
$$

where

$$
\pi_k(\tau_{\geq n-1}(X)) = \begin{cases} \pi_k(X) & \text{for } k < n \\ * & \text{otherwise} \end{cases}
$$

$$
\pi_{k+1} K(\pi_n(X), n+1) = \begin{cases} \pi_n(X) & \text{for } k = n \\ 0 & \text{otherwise} \end{cases}
$$

The upshot is that the long exact sequence tells us exactly what the homotopy groups of F are:
$$\pi_*(F) = \begin{cases} \pi_*(X) & \text{for } * \leq n \\ 0 & \text{otherwise} \end{cases}$$

Check that $\tau_{\geq n}(X) \to F$ is an isomorphism. (We need to show that the map between then "is" the identity.) Both of these map to $\tau_{\geq n-1}(X)$. Because everything is compatible, that says that when $* \neq n$ then this is the identity. So we only have to worry about $* = n$; check at the level of relative homology, and use the details of how $K(\pi_n X, n+1)$ was constructed.

The Postnikov tower is a tower of principal fibrations.

$$
\begin{array}{ccc}
\tau_{\leq 3}(X) & \longrightarrow & \ldots \\
\downarrow & & \\
\tau_{\leq 2}(X) & \longrightarrow & K(\pi_3(X), 4) \\
\downarrow & & \\
\tau_{\leq 1}(X) = * & &
\end{array}
$$

You can build things by an infinite procedure of screwing on one homotopy group at a time.

Say we know $\tau_{\geq n}(X)$ and we also know $\pi_{n+1}(X)$. What are all possibilities $\tau_{\geq n+1}(X)$? This can always be identified with the fiber of a map $\tau_{\geq n}(X) \xrightarrow{f} K(\pi_{n+1}(X), n+2)$. This means that $\tau_{\geq n+1}(X)$ is determined up to homotopy by f. So

$$[f] \in [\tau_{\geq n}(X), K(\pi_{n+1}(X), n+2)] \cong H^{n+2}$$

A space is an n-type if there are no homotopy groups beyond n. This cohomology class is called the $(n+1)^{st}$ K-invariant of X. You know how to build X from its Postnikov tower. If all of these maps are zero, then all of the P maps are nullhomotopic. So all K-invariants of X vanish iff $X = \prod K(\pi_n(X), n)$. Usually, though, spaces are given by a twisted product of K-spaces; this is controlled by these cohomology classes.

Suppose we want to classify all homotopy types. Then we need to compute all of the cohomology groups: $H^*(\tau_{\geq n}(X), A)$. But we can use the Postnikov tower for $\tau \leq n$. This is at the top of a tower of fibrations, where at every step the fiber is an Eilenberg-Maclane space. If you know $\tau_{\geq n-1}(X)$ then you can (in theory) understand $\tau_{\geq n}(X)$ by the Serre Spectral Sequence. The basic calculations are of the following form:

Compute $H^*(K(n, B), A)$.

This is dual to the question of the homotopy groups of spheres. Because Eilenberg-Maclane spaces represent cohomology, this is dual to asking what are all the maps between Eilenberg-Maclane spaces?

We have $H^n(K(B,m)A) \cong [K(B,m), K(A,n)]$ which is the set of all natural transformations of functors $X \mapsto H^m(X;B)$ to $X \mapsto H^n(X;A)$. These are cohomology operations, which takes a cohomology class and produces another one, perhaps of different degree, etc.

What is amazing, is that these calculations are tractable.

9.2 Example. A first example is $K(\mathbb{Z}, 1) = S^1$. We have $K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty = \varinjlim_{n} \mathbb{R}P^n \cong \varinjlim_{n} S^n/antipodal$. In the direct limit, the spheres become contractible. A related statement: $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty = \varinjlim_{n} \mathbb{C}P^n \cong S^{2n+1}/U(1)$. If you take the limit you get something contractible, so the end result is a quotient of a contractible space by the free action of a circle; so the homotopy groups are the same as those of a sphere, only shifted by one.

Lecture 10
2/14

OK. So, I think that there's a lot we're going to build on from the Serre spectral sequence. We got the Hurewicz theorem. But I want to spend a few lectures on how you actually get the Serre spectral sequence.

Today, we will set up some preliminary notions. The Serre spectral sequence doesn't really go from the cohomology of the base with coefficients in the fiber. There is something more. Canonically, it has to do with homology of the base with coefficients in a local system.

§1 Local systems

We want to talk about local systems of abelian groups. Let X be a space. A **local system** on X, which we call $\underline{A}$, consists of

1. An abelian group A_x for each $x \in X$.

2. For every path $\gamma : \Delta[1] \to X$ (joining two vertices $x, y \in X$) an isomorphism $\tau_\gamma : A_x \simeq A_y$. Sometimes it is more convenient to think of τ_γ going in the opposite direction.

This data satisfies:

1. For every $h : \Delta[2] \to X$, then the map $h(v_0) \to h(v_2)$ (τ on the two-face) is the composite $h(v_0) \to h(v_1) \to h(v_2)$ (from the zero- and one-faces). This is kind of like the cocycle condition.

An alternate but equivalent formulation is:

> Consider the **fundamental groupoid** of X, denoted $\pi_{\leq 1}(X)$; this is a category where all the objects are the points of X, the maps $x \to y$ are homotopy classes of paths relative to the boundary, and composition comes from catenation of paths. Then a **local system** is a functor from $\pi_{\leq 1}(X)$ to the category of abelian groups.

It's easy to say what the fundamental groupoid is in the language of categories. However, the language of categories actually postdates fundamental groupoids.

10.1 Example. Suppose $p : E \to B$ is a Hurewicz fibration, and $q \in \mathbb{Z}$. Then the system that associates
$$b \mapsto H_q(p^{-1}b)$$
is a local system. This is because the map $b \mapsto p^{-1}(b)$ is a functor from the fundamental groupoid of B to the homotopy category of topological spaces (this was on the HW). From that the assertion is clear.

Of course, if b, b' are not in the same path-component, then the fibers over b, b' may not have identical homology.

10.2 Example. Consider $S^1 \to S^1, z \mapsto z^2$. Over 1, the abelian group H_0 is $\mathbb{Z} \oplus \mathbb{Z}$. The path γ that goes around the circle (the base S^1) induces the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ by the permutation action flipping the two copies of $\mathbb{Z}$. So the choice of path really does make a difference.

10.3 Example. Let M be a topological manifold. Then $x \mapsto H_n(M, M - x)$ (non-canonically isomorphic to the integers $\mathbb{Z}$) is a local system.

10.4 Definition. There is an obvious notion of an **isomorphism** of local systems, say a natural isomorphism.

An **orientation** of a manifold was an isomorphism of the local system $x \mapsto H_n(M, M - x)$ with the trivial one.

We now need to do a little more. First:

10.5 Definition. A map $f : X \to Y$ is a **weak equivalence** if for all $x \in X$, $n \geq 0$,

$$f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism.

(Taking $n = 0$ means that the path components are in bijection.)

10.6 Theorem. *A weak equivalence induces an isomorphism in homology.*

We'll prove this later, especially after we talk about simplicial sets.

This is interesting. A weak equivalence says that maps of spheres into the two spaces are basically the same. But homology is about maps of simplexes into a space.

10.7 Example. Suppose $p : E \to B$ is a **Serre** fibration. Then the map $b \mapsto H_q(F_b)$ (for $F_b = p^{-1}(b)$) is a local system. Indeed, for $\gamma : I \to B$, consider the diagram

$$
\begin{array}{ccc}
E \times_B I & \longrightarrow & E \\
\downarrow & & \downarrow \\
I & \longrightarrow & B
\end{array}
$$

and the long exact sequence in homotopy groups shows that the maps $F_{\gamma(0)} \to E \times_B I, F_{\gamma(1)} \to E \times_B I$ are weak equivalences. From this the result is clear.

A lot of what you can do for Hurewicz fibrations, you can do for Serre fibrations, because of this example.

§2 Homology in local systems

Let X be a space, $\underline{A}$ a local system. We now want to make a chain complex of singular chains with coefficients in $\underline{A}$. That is, we want to define

$$C_*(X, \underline{A}).$$

The ordinary chain complex of singular chains with coefficients in a group M are

$$C_p(X, M) = \bigoplus_{\Delta[p] \to X} M = C_*(X) \otimes M.$$

The boundary map

$$C_p(X, M) \to C_{p-1}(X, M), \quad \bigoplus_{\Delta[p] \to X} M \to \bigoplus_{\Delta[p-1] \to X} M$$

can be stated simply. Namely, we just use the various restriction of any map $\Delta[p] \to X$ to the various $p - 1$ faces.

Now let's define the analog for local systems:

$$C_p(X, \underline{A}) = \bigoplus_{f : \Delta[p] \to X} A_{f(0)}.$$

Now we want a boundary map. When we take the faces of a standard simplex that are not the zero-face, then the ith face doesn't change. So we can define most of the boundary maps as before, but we have to define how to restrict to the zeroth face. For this we use the identifications given by the local system.

Namely we define restriction maps $\partial_i : C_p(X, \underline{A}) \to C_{p-1}(X, \underline{A})$ by taking the boundary (and possibly using some identification) and then defining the boundary map

$$C_p \to C_{p-1}$$

via $\sum (-1)^i \partial_i$.

10.8 Definition. The **homology of X with coefficients in $\underline{A}$** is the homology of the chain complex $C_*(X, \underline{A})$.

Note that we can make local systems functorial. If $X \to Y$ is a map, we get a functor $\pi_{\leq 1}(X) \to \pi_{\leq 1}(Y)$, so a local system on Y leads to a local system on X.

Homology with coefficients in a local system satisfies:

1. Functoriality (if $f : X \to Y$, $\underline{A}$ a local system on Y, then there is a map $H_*(X, f^*(\underline{A})) \to H_*(Y, \underline{A})$.

2. Homotopy invariance

3. Mayer-Vietoris

4. Excision

5. . . . .

Remark. Given a short exact sequence of local systems, we get a long exact sequence in homology.

For any n-dimensional topological manifold M, the Poincaré duality theorem states that
$$H_*(M, \underline{\mathbb{Z}}) = H_c^{n-*}(M, \mathbb{Z})$$
where $\underline{\mathbb{Z}}$ is the local system discussed earlier.

10.9 Example. Consider the map

$$S^n \to \mathbb{RP}^n$$

and we have the local system which is the cohomology of the fiber. The fiber is always just two points (antipodal). Exercise: show that the homology groups of $\mathbb{RP}^n$ with coefficients in this local system is just the cohomology of the sphere. In fact, we get an isomorphism on chain complexes. This is kind of like the Serre spectral sequence.

10.10 Example. If M is an abelian group, then we have an isomorphism

$$H^n(X, M) = [X, K(M, n)]$$

if X is a CW complex. What if M is replaced by a local system $\underline{A}$?

You can still build a fibration $K(\underline{A}, n) \to X$ where the fiber is $K(A, n)$. Then it turns out that the cohomology $H^n(X, \underline{A})$ is the set of vertical homotopy classes of sections $X \to K(\underline{A}, n)$—that is, sections up to homotopies of sections.

Lecture 11
February 16, 2010

Last time we talked about local systems. That was to set up today's talk in which we will construct Serre spectal sequences. Suppose we have a Serre fibration $f : E \to B$. We want to associate to this a double chain complex.

Let $C_{p,q}$ be the free abelian group on $\Delta[p] \times \Delta[q]$ satisfying

$$
\begin{array}{ccc}
\Delta[p] \times \Delta[q] & \longrightarrow & E \\
\downarrow & & \downarrow \\
\Delta[p] & \longrightarrow & B
\end{array}
$$

This is a double chain complex where $d_{hor} = \sum (-1)^? \partial_i$, the sum of alternating face-maps in the p-direction. The vertical maps are $d_{ver} = \sum (-1)^? \partial_i$, the analogous thing.

Let's first do the non-Serre direction (the q-direction). So in the first spectral sequence we do homology in the p-direction first (fixing q), then in the q-direction. For fixed q, we want to think of this as a map from $\Delta[p]$ to somewhere.

$$
\begin{array}{ccc}
E' & \xrightarrow{\varphi} & E^{\Delta[q]} \\
\downarrow & & \downarrow \\
B & \xrightarrow{const.} & B^{\Delta[q]}
\end{array}
$$

Since E' is a pullback, the map on fibers is a homeomorphism. By the long exact sequence of homotopy groups, the map $E' \to E^{\Delta[q]}$ on fibers is a weak equivalence, and therefore an isomorphism on homology. We are using this fact:

11.1 Theorem. *A weak equivalence is a homology isomorphism.*

If you take homology in the horizontal direction, you just get homology of E:

$$
\begin{array}{ccc}
H_0 E & H_1 E & H_2 E \\
\Downarrow\Downarrow\Downarrow & \Downarrow\Downarrow\Downarrow & \Downarrow\Downarrow\Downarrow \\
H_0 E & H_1 E & H_2 E \\
\Downarrow\Downarrow & \Downarrow\Downarrow & \Downarrow\Downarrow \\
H_0 E & H_1 E & H_2 E
\end{array}
$$

All of the vertical maps are isomorphisms. The vertical maps should subtract to zero, and the second row are the identity. So the E^2 page is just $H_0, H_1 E, H_2 E$, with zero elsewhere.

The conclusion is that the homology of the total complex of $C_{p,q}$ is just $H_*(E)$.

$$
\begin{array}{ccc}
\Delta[p] \times \Delta[q] & \longrightarrow & E \\
\downarrow & & \downarrow \\
\Delta[p] & \longrightarrow & B
\end{array}
$$

We want to fix p and say what it means to give a map of $\Delta[q]$ into $E^{\Delta[p]}$:

$$
\begin{array}{ccc}
F_c & \longrightarrow & E^{\Delta[p]} \\
\downarrow & & \downarrow \\
pt & \xrightarrow{\;c\;} & B^{\Delta[p]}
\end{array}
$$

For fixed p, we have $C_{p,*} = C_*^{Sing} F_c$. We have some simplex c. Let's restrict to the zero vertex:

$$
\begin{array}{ccccc}
F_c & \longrightarrow & F_{c(v_0)} & & \\
\downarrow & & \downarrow & & \\
F_c & \longrightarrow & E^{\Delta[p]} & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
pt & \xrightarrow{\;c\;} & B^{\Delta[p]} & \longrightarrow & B
\end{array}
$$

(So $c \in B^{\Delta[p]}$) This is not a pullback square... This is a commutative diagram of Serre fibrations. The point c goes to the value of the map $c(v_0)$ at zero. This carries the fiber F_c to the fiber $F_{c(v_0)}$.

On the top square we have a weak equivalence between the fibers. So the homology of $C_{p,}$ is canonically $H_*(F_{c(v_0)})$. In a given degree p, the (p, q) spot is $\bigoplus_{c:\Delta[p]\to B} H_q(F_{c(v_0)})$ which maps to $\bigoplus(F_{c(v_0)})$. This is the alternating sum of the face maps. Check that this is exactly the boundary map, and the top complex is exactly the p-chains

$C_p(B; H_q(F))$. Check that this differential *is* the differential in that chain complex, mapping to $C_{p-1}(B; H_q(F))$. All this is happening on the E^2 page. First you take horizontal homology to get E^1, and then you take the vertical homology. The maps here are the d_1 differentials.

$$E^2_{p,q} = H_p(B; H_q(F))$$

This gives us a spectral sequence converging to $H_{p+q}(E)$.

No more details. There is an obvious version using cochains and cohomology. In this picture the multiplicative structure gets dealt with rather nicely. Read about it.

§1 Applications of the Serre spectral sequence

The Hurewicz theorem says that $\pi_n S^n = H_n S^n = \mathbb{Z}$. It doesn't say anything about any other homotopy groups of spheres. There is a great procedure called Killing Homotopy Groups. (If you don't like this imagine that you just snuck up behind them and put a choke-hold on them for the forseeable future.) We can make an Eilenberg-Maclane space so $S^n \to K(\mathbb{Z}, n)$ is an isomorphism on π_n. We can convert any map into a fibration; do this here. Call the fiber X:

$$
\begin{array}{c}
X \\
\downarrow \\
S^n \longrightarrow K(\mathbb{Z}, n)
\end{array}
$$

We get a long exact sequence

$$\pi_k(X) \to \pi_k(S^n) \to \pi_k(K(\mathbb{Z}, n))$$

Look at this when $k = n+1$. $\pi_n S^n \to \pi_n K$ is an isomorphism, as is $\pi_{n+1} X \to \pi_{n+1} S^n$. So $\pi_{n+1}(K) = 0 = \pi_n(X)$. So $\pi_k X = \pi_k S^n$ for $k > n$ and zero otherwise. We have killed the homotopy group that we knew about. If we could calculate $H_*(X)$ we would have the next homotopy group of spheres, because

$$\pi_{n+1} S^n = \pi_{n+1}(X) = H_{n+1}(X)$$

Or if we knew cohomology we could use the universal coefficient theorem. The whole point of all this is to find the homology of X. We don't know the homology of $K(\mathbb{Z}, n)$. But this gain would continue: we could make a map

$$X \to K(\pi_{n+1}(X))$$

take the fiber above X and call it X_2, and learn

$$\pi_{n+2}(S^n) = H_{n+2}(X_2)$$

and we could keep doing this. There's a missing ingredient. We don't know the homology of Eilenberg-Maclane spaces. In order to do this we need to know $H_*(K(\pi, n))$.

Eilenberg-Maclane spaces tell us about cohomology, not homology. *But* the miracle is that you can calculate this! Today we will extract some relatively easy things.

It isn't true that we know nothing about these. We know the homology groups of $S^1 = K(\mathbb{Z}, 1)$, and of $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$, and of $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$. More generally, we know $H_*(K(A, 1))$ when A is abelian. There is a Serre fibration $S^1 \to S^\infty = \varinjlim S^{2n+1} \to \mathbb{C}P^\infty$. But S^∞ is contractible. Homotopy groups commute with taking limits. In this system every map is the zero map, so this holds in the limit too.

We know $\Omega K(\pi, n + 1) = K(\pi, n)$. Suppose we have a fibration $F \to E \to B$, and we factor this as

$$
\begin{array}{ccc}
 & & \widetilde{F} \\
 & \nearrow & \downarrow \scriptstyle{fibration} \\
F & \longrightarrow & E
\end{array}
$$
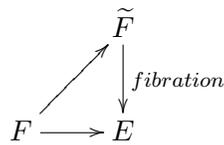
So the fiber of $\widetilde{F} \to E$ is ΩB. (In Hatcher take a look at the Barratt-Puppe sequence.)

$$
\begin{array}{ccc}
K(\mathbb{Z}, 2) & \longrightarrow & X \\
 & & \downarrow \\
 & & S^3 \longrightarrow K(\mathbb{Z}, 3)
\end{array}
$$
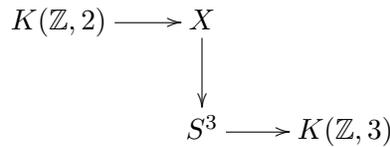
The homology groups of $K(\mathbb{Z}, 2)$ are the same as for $\mathbb{C}P^\infty$. Let's look at the Serre spectral sequence:
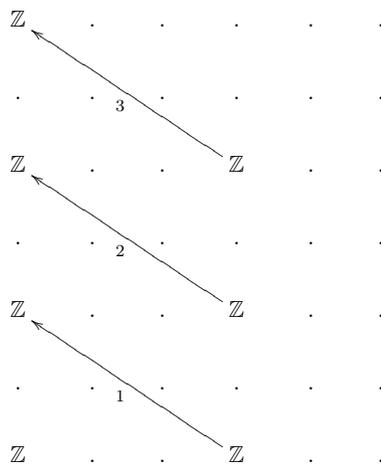
We know that the differential $\mathbb{Z} \to \mathbb{Z}$ in $(3, 0) \to (0, 3)$ is an isomorphism; we are trying to get the homology of the total space. [I am labeling the entries in the lattice as (x,y); the bottom row is $H_*(S^3)$, and the left column is $H_*(\mathbb{C}P^\infty)$.] Here it is in cohomology:
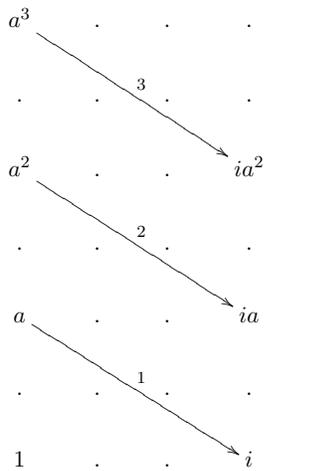
a hits i, so $d_3(a) = i$, $d_3(a^2) = 2ai$, etc. So the downward maps are $3, 2, 1 \ldots$ and in homology it is the same. So we have $H_{odd}(X) = 0$, $H_{2n}(X) = \mathbb{Z}/n$, $H_4(X) = \mathbb{Z}/2$ and so $\pi_4(S^3) = \mathbb{Z}/2$.

Lecture 12
2/18

In the last class, we proved $\pi_4(S^3) = \mathbb{Z}/2$. We did it with this interesting method of killing homotopy groups. Now we want to push that a little further. We did that by looking at the fibration

$$K(\mathbb{Z}, 2) \to X_1 \to S^3$$

where X_1 was obtained from S^3 by killing homotopy groups. We could do this for an arbitrary sphere if we knew the cohomology (or homology) of $K(\mathbb{Z}, m)$.

For instance, we could construct a fibration

$$K(\mathbb{Z}, 3) \to X_1' \to S^4,$$

where X_1' is obtained from S^4 by killing homotopy groups. So we want to know the cohomology of $K(\mathbb{Z}, 3)$. But there is a fibration

$$K(\mathbb{Z}, 2) \to PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3).$$

Let's consider the cohomology spectral sequence for that:

$$\mathbb{Z}\alpha^3$$

$$\mathbb{Z}\alpha^2 \qquad\qquad\qquad \mathbb{Z}\iota\alpha$$

$$\mathbb{Z}\alpha \qquad\qquad\qquad \mathbb{Z}\iota\alpha$$

$$1 \qquad 0 \qquad 0 \qquad \mathbb{Z}\iota$$

we know when the spectral sequence ends, we are supposed to get a bunch of zeros except at the origin because the path space is contractible. In particular, the differential

$$\mathbb{Z}\iota \to \mathbb{Z}\alpha$$

is an iso. So $d^3(\alpha) = \iota$. Thus we get similarly $d^3(\alpha^2) = 2\iota\alpha$. There must be something that $\iota\alpha$ gets sent to. In particular, we find that $H^6(K(\mathbb{Z},3)) = \mathbb{Z}/2$. The universal coefficient theorem now implies that

$$H_3(K(\mathbb{Z},3)) = \mathbb{Z}, \quad H_5(K(\mathbb{Z},3)) = \mathbb{Z}/2.$$

(This is a little counterintuitive, but you can think fast about it by thinking about the cochain complex fast.)

Cool. So let's now try $K(\mathbb{Z},4)$. Let's do homology this time. Use the spectral sequence for

$$K(\mathbb{Z},3) \to * \to K(\mathbb{Z},4).$$

We draw

$$\mathbb{Z}/2$$

$$\mathbb{Z} \qquad\qquad\qquad\qquad \mathbb{Z}$$

$$\mathbb{Z} \qquad\qquad\qquad\qquad \mathbb{Z}$$

where the only thing that can kill $\mathbb{Z}/2$ is a $\mathbb{Z}/2$ in dimension six. In particular, we get the following conclusion:

1. $H_m(K(\mathbb{Z}, n) = \mathbb{Z}$ if $m = 0$.

2. $H_m(K(\mathbb{Z}, n)) = 0$ if $0 < m < n$.

3. $H_m(K(\mathbb{Z}, n)) = \mathbb{Z}$ if $m = n$.

4. $H_m(K(\mathbb{Z}, n)) = 0$ if $n < m < n + 2$.

5. $H_m(K(\mathbb{Z}, n)) = \mathbb{Z}/2$ if $m = n + 2$.

This can be proved by induction by using the Serre spectral sequence. Then one can show:

12.1 Proposition. *For all $n \geq 3$, $\pi_{n+1}(S^n) = \mathbb{Z}/2$.*

One can calculate all the way up to $\pi_{n+3}(S^n)$ for all n. Actually, you can go up to 14 or so, but then you run into problems you can't solve with these techniques (though the answer is known up to 16).

There's a lot of great ways of thinking about this $\mathbb{Z}/2$. We'll come back to that, I hope, towards the end of the course. For now we want to focus on this $\mathbb{Z}/2$.

12.2 Example (Never mind; bridge to nowhere)**.** Consider the homotopy fiber of $S^n \to K(\mathbb{Z}, n)$. The fiber X is more highly connected ($\pi_i(X) = 0$ for $i \leq n + 1$). So if A is a CW complex of dimension n, there is a short exact sequence

$$[A, X_1] \to [A, S^n] \to [A, K(\mathbb{Z}, n)] = H^n(A, \mathbb{Z}).$$

This implies that the homotopy classes of A into S^n is the same thing as a cohomology class of A in n dimensions (since the $n + 1$-skeleton of $K(\mathbb{Z}, n)$ is zero **wait is this true?**. No. Well, $H_*(K(\mathbb{Z}, n), S^n) = 0$ for $* \leq n + 1$). This is a famous result.

What is this when $\dim A \leq n+1$? This is a question that Steenrod asked, and Steenrod answered. This led to the discovery that led to the calculation of the Eilenberg-Maclane spaces.

We had a morphism $S^n \to K(\mathbb{Z}, n)$ which was an iso for $* \leq n$ and an epi for $* = n+1$. The same is thus true for CW complexes. If A is a CW complex of dimension $\leq n+1$, then $[A, S^n] \to [A, K(\mathbb{Z}, n+1)]$ is an epimorphism.

Now we have a map $S^n \to K(\mathbb{Z}, n)$. Choose a map $K(\mathbb{Z}, n) \to K(\mathbb{Z}/2, n+2)$ that gives an isomorphism in homology of degree $n + 2$. Steenrod called this map square 2.

So we have this map
$$Y \to K(\mathbb{Z}, n) \to K(\mathbb{Z}/2, n+2)$$
where Y is the homotopy fiber. The map from $S^n \to K(\mathbb{Z}, n)$ lifts to Y_n because of homotopy properties. Now one can show that the next two groups of Y_1 are zero. By relative Hurewicz, we learn that $H_*(Y_1, S^n) = 0$ is zero in $* \leq n+2$. So is $\pi_*(Y_1, S^n) = 0$ for $* \leq n+2$. Thus if the dimension of A is at most $n+1$,

$$[A, S^n] = [A, Y_1].$$

But Y_1 was something we could control. It was the homotopy fiber of $K(\mathbb{Z}, n) \to K(\mathbb{Z}/2, n+2)$. In fact, you can get an exact sequence

$$H^{n-1}(A, \mathbb{Z}) \to H^{n+1}(A, \mathbb{Z}/2) \to [A, S^n] \to H^n(A, \mathbb{Z}) \to H^{n+2}(A, \mathbb{Z}) = 0.$$

(from the Barratt-Puppe sequence)

There is more in cohomology than just the ring structure. There is an interesting natural transformation
$$H^m(A, \mathbb{Z}) \to H^{m+2}(A, \mathbb{Z}/2).$$

This natural transformation holds the key to figuirng out maps of an $n+1$-dimensional complex into a sphere. Steenrod figured out what all the cohomology operations are, and that is called the Streenrod algebra. We could come back to talk about the Steenrod algebra in the future.

What is a good mathematical question is not obvious. Why you would want to consider maps of an $n + 1$ complex into S^n doesn't seem obvious. But it led to really extraordinary computational techniques in algebraic topology.

§1 Serre classes

Okay. I know this was kind of quick, but it gives you some sense of the computational devices. I want to start a new topic now. I'm just going to begin talking about it though. A lot of these arguments in this fashion are due to Serre. He brought to this one really beautiful thing though. (Serre.) Recall that we were looking at the fiber

$$\mathbb{CP}^\infty \to X_1 \to S^3$$

and we computed the cohomology of X_1 last time. We figured it out completely by using the Serre spectral sequence.

Now we will start to motivate philosophically something Serre introduced. If we want to figure out the next homotopy group of spheres, we would do something like

$$X_2 \to X_1 \to K(\mathbb{Z}/2, 4).$$

We want to find the homology of X_2. We can draw the homology spectral sequence where we know that of X_1 completely. Philosophically, let's think about $K(\mathbb{Z}/2, 4)$. We would expect that only 2-torsion in the homology of $K(\mathbb{Z}/2, 4)$. How would it know anything about $\mathbb{Z}/3$? They can't talk to or map to each other. So there should be only 2-torsion in the spectral sequence in the vertical statement. So if we were working only mod 3 or mod 5, the spectral sequence would be very easy, and the only 3-torsion or five-torsion would have to come up from X_1.

This is what Serre observed, and he came up with a beautiful way of ignoring part of the calculation. This is the notion of a Serre class.

12.3 Definition. A **Serre class** of abelian groups is a class $\mathcal{C}$ with the following properties:

1. $\mathcal{C}$ is closed under isomorphisms.

2. If $0 \to A \to B \to C \to 0$ is a short exact sequence, then $B \in \mathcal{C}$ iff $A, C \in \mathcal{C}$.

12.4 Example.   1. Fix a prime p. $\mathcal{C}$ is the class of A such that every element of A is torsion of order prime to p. (Exercise.)

2. $\mathcal{C}$ could be all torsion abelian groups.

3. $\mathcal{C}$ could be the class of all finitely generated abelian groups.

The point is that if you have a spectral sequence, and one of them is in a Serre class at the E_2 page, then it stays in that class at the E_∞ page, because you are just systematically taking subquotients.

So fix a Serre class $\mathcal{C}$. A morphism $M \to N$ is a epi mod $\mathcal{C}$ if the cokernel is in $\mathcal{C}$. It is a mono mod $\mathcal{C}$ if the kernel is in $\mathcal{C}$. It is an iso mod $\mathcal{C}$ if both the kernel and cokernel are in $\mathcal{C}$.

(Presumably it is true that composites of monos mod $\mathcal{C}$ is a mono, etc.)

12.5 Definition. $A \xrightarrow{i} B \xrightarrow{j} C$ is **exact at B mod $\mathcal{C}$** if im$i$ and ker$j$ are isomorphic mod $\mathcal{C}$ in a natural way. We'll spell this out in detail next time.

Now we want to start studying the Postnikov tower of the sphere. We need to use a fact that we can't justify today.

12.6 Proposition. *If X is a CW complex, then $[X, K(A,n)]$ is naturally isomorphic to $H^n(X, A)$.*

The sphere S^n maps to $K(\mathbb{Z}, n)$. That's an equivalence in homology up to n.

Lecture 13
February 23, 2011

We were talking about Serre classes. This is a category $C \subset AbelGp$. We defined the notions of epi, mono, and iso mod C; this just means $coker \in C$, $ker \in C$, and both, respectively. We were talking about an exact sequence mod C. Suppose we have a sequence

$$L \xrightarrow{f} M \xrightarrow{g} N$$

This is exact mod C if the image $g \circ f \in C$, and if

$$L \xrightarrow{f} M \xrightarrow{\overline{g}} \overline{N} = N/Im(g \circ f)$$

we want $Im(f) \cong_C ker(\overline{g})$.

$$\begin{array}{ccc}
 & & N \\
 & \nearrow^{g} & \downarrow \cong (mod\ C) \\
M & \xrightarrow{\overline{g}} & \overline{N}
\end{array}$$

You can do all of homological algebra mod C. Basically it means that you have a functor to some other abelian category, and you're just doing homological algebra there.

13.1 Example. If C is all torsion abelian groups, then epi, mono, and iso mod C is the same thing as epi, mono, and iso after tensoring with $\mathbb{Q}$. Similarly, if C is all ℓ-torsion abelian groups where $(\ell, p) = 1$ for some prime p [i.e. where all elements have order prime to p], then epi, mono, and iso mod C are the same things as epi, mono, and iso after tensoring with $\mathbb{Z}_{(p)}$. Another example is the class C of finitely generated abelian groups.

Consider just these three examples of Serre classes:

1. all torsion abelian groups;

2. all ℓ-torsion abelian groups (for ℓ prime to some fixed prime p);

3. all finitely generated abelian groups.

We want to talk about Serre's generalization of the Hurewicz theorem mod C:

13.2 Theorem. *If X is simply connected, then $\pi_i(X) = 0$ mod C for $i < n$ implies $H_i(X) = 0$ mod C for $i < n$ and $\pi_n(X) \to H_n(X)$ is an isomorphism mod C.*

We will give a slightly different argument from Serre's.

13.3 Corollary. $\pi_k(S^n)$ *is finitely generated.*

They are simply connected, the homology groups are finitely generated (zero mod C), so the first homotopy group that is nonzero is isomorphic to the first homology group that is nonzero (mod C) ... but *all* of these are zero mod C. This is great, because geometric methods can only tell you that these groups are countable; you won't get anywhere with the group structure.

13.4 Theorem.
$$\pi_* S^{2m+1} \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & for * = 2m + 1 \\ 0 & otherwise \end{cases}$$

$$\pi_* S^{2m} \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & for * = 2m \\ \mathbb{Q} & for * = 4m - 1 \\ 0 & otherwise \end{cases}$$

13.5 Corollary. $\pi_k S^n$ *is finite if* $k \neq n$ *and* $k \neq 2n - 1$ *for n even.*

To do this we will be systematically exploiting Eilenberg-Maclane spaces. These are what I call "designer" homotopy types; they are constructed algebraically, and usually don't appear "in nature." Let's first talk about the Hurewicz theorem mod C. We will do this by reducing it to the usual Hurewicz theorem. But first we need a theorem about Eilenberg-Maclane spaces.

13.6 Theorem. *Suppose C is one of the Serre classes mentioned above, and $A \in C$. Then for all q, and all $n \geq 1$, $\widetilde{H}_q(K(A, n); \mathbb{Z}) \in C$.*

Does this hold for all Serre classes? Well no... Here's a lemma about Serre classes:

13.7 Lemma. *Suppose C is a Serre class. If M is a finitely-generated abelian group, and $A \in C$ then both $A \otimes M$ and $Tor(A, M)$ are in C. If C is closed under arbitrary direct sums then you can drop the assumption on M being finitely generated.*

Proof of the lemma. If M is finitely generated, we can give a presentation $\mathbb{Z}^{r_1} \hookrightarrow \mathbb{Z}^{r_2} \twoheadrightarrow M$. Then we get part of an exact sequence:

$$Tor(A, M) \to \overset{r_1}{\bigoplus} A \to \overset{r_2}{\bigoplus} A$$

Then if $A \in C$ then $\bigoplus^r A \in C$ so we have

$$A \hookrightarrow \overset{r}{\bigoplus} A \twoheadrightarrow \overset{r-1}{\bigoplus} A$$

This shows that each of the above things are in C. If we drop the finitely generated assumption, all we need is for all of these direct sums to be in C. $\mathscr{Q.E.D.}$

13.8 Corollary. *Suppose C is closed under infinite sums, and X is any space. Then $A \in C$ implies that $H_q(X; A) \in C$.*

This is just because the homology groups sit in an exact sequence with tor, and things tensored with things in C. If C is arbitrary, we need to know something about X. If $H_q(X)$ and $H_{q-1}(X)$ is finitely generated, then $A \in C$ implies $H_1(X; A) \in C$. (Use the exact sequence with this, Tor, and tensors.)

Now let's prove the Hurewicz theorem mod C, assuming everything else we've asserted. Assumptions:

- X is simply connected

- $\pi_i X = 0 \mod C$ for $i < n$

Suppose that $\pi_i(X) = 0$ for $i < k < n$. Then we can map $X \to K(\pi_k(X), k)$ and take the homotopy fiber X'. As usual we want to study the fibration

$$
\begin{array}{ccc}
K(\pi_k, k-1) & \longrightarrow & X' \\
& & \downarrow \\
& & X \longrightarrow K(\pi_k(X), k)
\end{array}
$$

We will show that $H_*(X') \to H_*(X)$ is an isomorphism mod C, and then we can replace X by X' without changing anything. By construction, this is an isomorphism in homotopy mod C.

$$
\begin{array}{ccc}
\pi_i(X') & \longrightarrow & H_i(X') \\
\downarrow \cong C & & \downarrow \cong (C) \\
\pi_i(X) & \longrightarrow & H_i(X)
\end{array}
$$

We want to show that the bottom map is an isomorphism mod C for $i \leq n$, so it suffices to prove that the top map is an isomorphism mod C. But we've improved the connectivity of X. Once we do this, we can replace X' by X and continue. This argument is going to quit as soon as $k = n$. So this reduces to the case where $\pi_i(X) = 0$ for $i < n$. This is covered by the Hurewicz theorem. (We keep taking higher and higher connected covers; as long as we keep killing homotopy groups that are in the Serre class, we are done.)

CLAIM: Suppose we have a sequence $F \to E \to B$ where B is simply connected. $H_*(F) = 0 \mod C$ for $* > 0$. Then $H_*(E) \to H_*(B)$ is an isomorphism mod C.

$$0_C \qquad . \qquad . \qquad . \qquad . \qquad .$$

$$0_C \qquad . \qquad . \qquad . \qquad . \qquad .$$

$$0_C \qquad . \qquad . \qquad . \qquad . \qquad .$$

$$0_C \qquad . \qquad . \qquad . \qquad . \qquad .$$

$$. \qquad H_2(B) \qquad . \qquad . \qquad .$$

When we write 0_C, we mean 0 mod the Serre class. When we get to the infinity page, we get to a quotient of a group that was zero. We have a short exact sequence

$$0_c \hookrightarrow H_2 E \twoheadrightarrow ker(H_2 B \to 0_C)$$

There is an isomorphism $ker(H_2 B \to 0_C) \to H_2 E$, and so there is an isomorphism $H_2 E \to H_2 B$ is an isomorphism. If C is closed under infinite sums, then the differentials don't change anything mod the Serre class, and assembling them doesn't change anything mod the Serre class. So really, all of the groups are zero mod C.

The claim also holds if, in addition, we know that the homology groups of B are finitely generated.

Missing something here; oops. This is good in cases 1 and 2. Need a different argument in the third case.

We don't have time to coherently start something new. Ah, oh well.

Lecture 14
2/25

So let's just focus on that case. Namely, we will take $\mathcal{C}$ as the Serre class of finitely generated abelian groups. We want to prove the Hurewicz theorem mod $\mathcal{C}$. Namely, if X is simply connected, and

$$\pi_i(X) \in \mathcal{C}, \quad i < n,$$

then we have that

$$H_i(X) \in \mathcal{C}, \quad i < n$$

and the Hurewicz map

$$\pi_n(X) \to H_n(X)$$

is an isomorphism mod $\mathcal{C}$. We are going to have to go through this more carefully.

Remark. Take $K(\mathbb{Z}/2, 2)$ and consider

$$\bigvee S^n \to K(\mathbb{Z}/2, 1) \times \bigvee S^n to K(\mathbb{Z}/2, 1);$$

then the base space has homology in $\mathcal{C}$ (i.e. is finitely generated), *but* the morphism

$$\bigvee S^n \to K(\mathbb{Z}/2, 1) \times \bigvee S^n$$

is not an isomorphism in homology (by the Künneth formula).

Here's the plan to prove the Hurewicz theorem. We are going to kill a bunch of homotopy groups to reduce to the case of X highly connected, and then use the usual Hurewicz theorem. Suppose X is $j - 1$-connected for $2 \leq j \leq n$, and we want to bump that up one more to j-connected. As we keep doing that, we can make X more connected.

By the usual Hurewicz theorem, and by assumption, $H_j(X) = \pi_j(X) \in \mathcal{C}$. By the universal coefficient theorem (or whatever), we know that $H_j(X, A) \in \mathcal{C}$ for any finitely generated abelian group. Now consider a fibration

$$E \to X \to K(\pi_j(X), j)$$

and convert $E \to X$ to a fibration, letting F to be the homotopy fiber (so $F = K(\pi_j(X), j - 1)$). We have a fibration

$$F \to E \to X.$$

Let us consider the Serre spectral sequence. Then in the first j columns, everything is zero mod $\mathcal{C}$.

NO NEVER MIND

We are going to prove this by *decreasing* induction on j. We are going to suppose X $(j - 1)$ connected for $j \leq n$. We start with $j = n$. Then the usual Hurewicz theorem does it. OK. Suppose we know for j-connected spaces. We want to prove it for $j - 1$-connected spaces. We look at the same fibration

$$X' \to X \to K(\pi_j(X), j)$$

and we look at the spectral sequence of this fibration. X' is more highly connected than X, so we know the mod $\mathcal{C}$ theorem for X'. We use the fact that $H_*(K(\pi_j(X)), j)$ is finitely generated. In particular, we find that in this spectral sequence, everything below n is 0 mod $\mathcal{C}$. So the homology of X' in dimension n maps isomorphically mod $\mathcal{C}$ to $H_*(X')$. From this we can do the inductive step.

Let us now go back to a baisc fact:

14.1 Theorem. *If $A \in \mathcal{C}$, then $H_*(K(A, n)) \in \mathcal{C}$ for $n \geq 1$.*

(We also need to do the rational homotopy groups of spheres.)

Let us start with the rational homotopy groups of spheres. We want to understand $\pi_*(S^n) \otimes \mathbb{Q}$. This is equivalent to working mod $\mathcal{C}$ where $\mathcal{C}$ is the Serre class of torsion abelian groups.

The best method of computing homotopy groups is to work your way up the Postinkov tower, computing homologies of Eilenberg-Maclane spaces each time. So we need

$$H_*(K(A, n), \mathbb{Q})$$

for A f.g. abelian. As part of this, we'll prove later

1. If A is torsion, then $H_*(K(A, n), \mathbb{Q}) = 0$.

2. $K(\mathbb{Z}, n) \to K(\mathbb{Q}, n)$ is an isomorphism in rational homology (by relative Hurewicz theorem mod $\mathcal{C}$, or by taking the homotopy fiber $K(\mathbb{Q}/\mathbb{Z}, n-1)$ and looking at its homology via the Serre spectral sequence).

So we want to figure out the homology groups of the Eilenberg-Maclane spaces. That is, when doing rational stuff, we can not distinguish between $K(\mathbb{Z}, n)$ and $K(\mathbb{Q}, n)$. So we will try to compute

$$H^*(K(\mathbb{Z}, n), \mathbb{Q}).$$

Start with $n = 1$. Then we have an exterior algebra on a class in degree one, that is $\mathbb{Q}[x]/x^2$ where $\deg x = 1$ (because we can take a circle). For $K(\mathbb{Z}, 2)$, we consider a spectral sequence from the fibration $K(\mathbb{Z}, 1) \to * \to K(\mathbb{Z}, 2)$

$$K(\mathbb{Z}, 1)$$

(OK, can't livetex this fast). Anyway, we find that $H^*(K(\mathbb{Z}, 2), \mathbb{Q})$ is a polynomial ring.

Note that if we are working with graded commutative rings, then the exterior algebra on odd degree generators is still a *free* algebra, because the square is zero. So in a sense, these are kind of analogous.

This needs to be written up slowly. But using the Serre spectral sequence, we find that $H^*(K(\mathbb{Z}, 3), \mathbb{Q})$ is an exterior algebra on a class. Now we are back in the same situation. The conclusion is that

14.2 Theorem. *$H^*(K(\mathbb{Z}, n), \mathbb{Q})$ is an exterior algebra on a class of degree n if n is odd, and a polynomial algebra on a class of degree n if n is even.*

Consider a map

$$S^{2n+1} \to K(\mathbb{Z}, 2n+1);$$

this is an iso on homology for rational coefficients; so by the relative mod $\mathcal{C}$ Hurewicz, it is an iso on homotopy. Thus

$$\pi_*(S^{2n+1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } * = 2n+1 \\ 0 & \text{else} \end{cases}$$

That takes care of the odd spheres. By the finite generation result, we know that $\pi_k(S^{2n+1})$ is finite if $k > 2n+1$. There is no way of seeing this with geometric techniques. You need the Eilenberg-MacLane spaces.

What about the even spheres? We could try the same thing

$$S^{2n} \to K(\mathbb{Z}, 2n)$$

but in rational cohomology this is not an isomorphism. Let us take the first class in the kernel and get rid of it. For instance, consider the squaring map

$$K(\mathbb{Z}, 2n) \to K(\mathbb{Z}, 4n).$$

The homotopy fiber of this map is X, so we get a map

$$S^{2n} \to X \to K(\mathbb{Z}, 2n)$$

and we can use the Serre spectral sequence to get that the cohomology of X is $\mathbb{Q}[x_{2n}]/(x_{2n}^2)$. Thus we can get the rational homotopy groups of X and so those of S^{2n}.

Alternatively, map

$$F \to S^{2n} \to K(\mathbb{Z}, 2n)$$

where F is the homotopy fiber. We could calculate the homotopy fiber F's rational cohomology. Namely use the fibration

$$K(\mathbb{Z}, 2n-1) \to F \to S^{2n}.$$

With the Serre SS, we learn that $H^*(F, \mathbb{Q}) = \mathbb{Q}$ if $* = 4n-1$ and 0 otherwise. Thus F is the Eilenberg-MacLane space $K(\mathbb{Q}, 4m)$ which is kind of hard to see. Anyway this leads to something...

So there is another $\mathbb{Z}$ in the homotopy groups of spheres, namely in dimension $4n-1$. We know that

$$\pi_{4n-1}(S^{2n}) = \mathbb{Z} + \text{finite}.$$

What is the generator of that $\mathbb{Z}$? There is a class, the Whitehead product, that goes

$$[\iota, \iota] \in \pi_{2m-1}(S^m).$$

To make it, note that $S^m \times S^m$ has a cell decomposition where the $S^m \vee S^m \cup e^{2m}$. There is an attaching map

$$S^{2m-1} \to S^m \vee S^m$$

for which we can get an explicit formula. This followed with the crushing map gives a map

$$S^{2m-1} \to S^m$$

which is the **Whitehead square.** This has infinite product. It doesn't always generate the $\mathbb{Z}$, but it does most of the time. The interesting question is when when $[\iota, \iota]$ is divisible by two on any sphere. There are a bunch of questions here that were really important. When m is even, this has to do with linearly independent vector fields on the sphere, and the answer is only when $m = 2$. All the rest of the time, it is not divisible by two. (This is related to the Hopf invariant one problem.) When m is odd, this was recently solved by Hopkins and a few others. Only at most six values of m.

It's time to move on to another topic. But, before we leave this, we let $\mathcal{C}$ be the Serre class of torsion abelian groups with torsion of order prime to p. In this case, an iso (resp. epi, mono) mod $\mathcal{C}$ is equivalent to the same thing after tensoring with $\mathbb{Z}_p$. Let's go through and look at these Eilenberg-Maclane spaces. We work over $\mathbb{Z}_p$.

For instance, consider

$$K(\mathbb{Z}, 2) \to X \to S^3 \to K(\mathbb{Z}, 3)$$

and imagine we are working over $\mathbb{Z}_p$. We want the homology. Draw the spectral sequence. When you've localized at p, all these differentials are isomorphisms until you get multiplication by p. That's in degree $2p$. We find that $H_*(X, \mathbb{Z}_p) = 0$ for $* < 2p$ and is $\mathbb{Z}/p$ when $* = 2p$. We conclude that the p-torsion part of $\pi_{2p}(S^3)$ is $\mathbb{Z}/p$ $(p > n)$.

Lecture 15
February 28, 2011

The goal is to describe an abstract place where we can do homotopy theory, and say where two such places give us equivalent homotopy theories. We will be doing this in an abstract category, so we can't talk about the unit interval: how do you define homotopy theory without homotopies?. He viewed it as a nonabelian view of homological algebra. The theory of model categories reduces to homological algebras.

15.1 Definition. A *model category* is a category C equipped with three classes of maps called *cofibrations*, *fibrations*, and *weak equivalences*. They have to satisfy five axioms $M1 - M5$. Denote cofibrations as $\hookrightarrow$, fibrations as $\twoheadrightarrow$, and weak equivalences as $\xrightarrow{\sim}$.

(M1) C is closed under all (finite) limits and colimits. (Some people want it to be

closed under *all* limits. Many of our arguments will involve infinite colimits. The original formulation required only finite such, but most people assume infinite.)

(M2) Each of the three classes is *closed under retracts*:

$$
\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
X & \longrightarrow & Y & \longrightarrow & X
\end{array}
$$

with Id over the top and Id below.
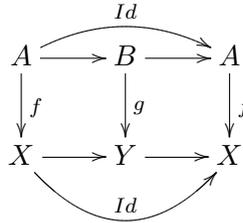
If g is one, then f is one too.

(M3) If *two of three* in a composition are weak equivalences, so is the third.

$$
\begin{array}{cc}
 & \xrightarrow{\;f\;} \\
{\scriptstyle h}\big\downarrow & \swarrow{\scriptstyle g} \\
\end{array}
$$
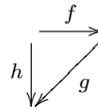
(M4) (*Lifts*) Suppose we have

$$
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
{\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\
\cdot & \longrightarrow & \cdot
\end{array}
$$
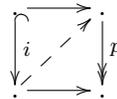
Then a lift exists if i or p is a weak equivalence.

(M5) (*Factorization*) Every map can be factored in two ways:

$$
\begin{array}{ccc}
 & \cdot & \\
\nearrow & & \searrow^{\sim} \\
\cdot & \xrightarrow{\;f\;} & \cdot \\
\searrow_{\sim} & & \nearrow \\
 & \cdot &
\end{array}
$$
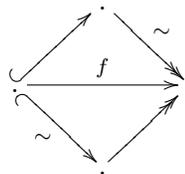
Remark. If C is a model category, then C^{op} is a model category, with the notions of fibrations and cofibrations reversed. So if we prove something about fibrations, we automatically know something about cofibrations.

Suppose that P is a class of maps. A map f has the left lifting property with respect to P iff: for all $p \in P$ and all diagrams

$$
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
{\scriptstyle f}\big\downarrow & {\scriptstyle \exists!}\nearrow & \big\downarrow{\scriptstyle p} \\
\cdot & \longrightarrow & \cdot
\end{array}
$$
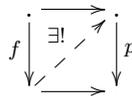
a lift exists. We call this property LLP. There is also a notion of a right lifting property, where f is on the right. (Just look at the opposite category, and the definition will be clear.)

A map which is a weak equivalence and a fibration will be called an acyclic fibration. Denote this by $\xrightarrow{\sim}$. A map which is both a weak equivalence and a cofibration will be called an acyclic cofibration, denoted $\xhookrightarrow{\sim}$. (Etymology: think of a chain complex with zero homology. There is actually a connection here.)

15.2 Theorem. *Suppose C is a model category. Then:*

(1) *A map f is a cofibration iff it has the left lifting property with respect to the class of acyclic fibrations.*

(2) *A map is a fibration iff it has the right lifting property w.r.t. the class of acyclic cofibrations.*

Proof. Suppose you have a map f, that has LLP w.r.t. all acyclic fibrations and you want it to be a cofibration. (The other direction is an axiom.) Somehow we're going to have to get it to be a retract of a cofibration. Somehow you have to use factorization. Factor f:

$$
\begin{array}{ccc}
A & & \\
\downarrow{\scriptstyle f} & \searrow & \\
X & \xleftarrow[\sim]{} & X'
\end{array}
$$
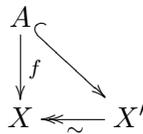
We had assumed that f has LLP. There is a lift:

$$
\begin{array}{ccc}
A & \xhookrightarrow{\;i\;} & X' \\
\downarrow{\scriptstyle f} & \nearrow & \downarrow{\scriptstyle \sim} \\
X & \xrightarrow{Id} & X
\end{array}
$$
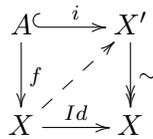
This implies that f is a retract of i.

$$
\begin{array}{ccccc}
A & \longrightarrow & A & \longrightarrow & A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\
X & \overset{\exists}{\dashrightarrow} & X' & \longrightarrow & X
\end{array}
$$
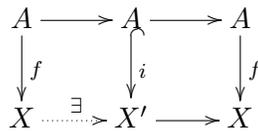
$\mathcal{Q.E.D.}$

15.3 Theorem. (1) *A map p is an acyclic fibration iff it has RLP w.r.t. cofibrations*

(2) *A map is an acyclic cofibration iff it has LLP w.r.t. all fibrations.*

Suppose we know the cofibrations. Then we don't know the weak equivalences, or the fibrations, but we know the maps that are both. If we know the fibrations, we know the maps that are both weak equivalences and cofibrations. This is basically the same argument. One direction is easy: if a map is an acyclic fibration, it has the lifting property by the definitions. Conversely, suppose f has RLP w.r.t. cofibrations. Factor this as a cofibration followed by an acyclic fibration.

$$
\begin{array}{ccc}
X & \xrightarrow{\ Id\ } & X \\
\downarrow & \nearrow & \downarrow f \\
Y' & \xrightarrow[\sim]{\ p\ } & Y
\end{array}
$$

f is a retract of p; it is a weak equivalence because p is a weak equivalence. It is a fibration by the previous theorem.

15.4 Corollary. *A map is a weak equivalence iff it can be written as the product of an acyclic fibration and an acyclic cofibration.*

We can always write

$$
\begin{array}{ccc}
& \overset{\cdot}{\underset{f}{\nearrow}} & \searrow^{p} \\
\cdot & \xrightarrow{\quad f \quad} & \cdot
\end{array}
$$

By two out of three f is a weak equivalence iff p is. The class of weak equivalences is determined by the fibrations and cofibrations.

15.5 Example (Topological spaces)**.** The construction here is called the Serre model structure (although it was defined by Quillen). We have to define some maps.

(1) The fibrations will be Serre fibrations.

(2) The weak equivalences will be weak homotopy equivalences

(3) The cofibrations are determined by the above classes of maps.

15.6 Theorem. *A space equipped with these classes of maps is a model category.*

Proof. More work than you realize. M1 is not a problem. The retract axiom is also obvious. (Any class that has the lifting property also has retracts.) The third property is also obvious: *something is a weak equivalence iff when you apply some functor (homotopy), it becomes an isomorphism.* (This is important.) So we need lifting and factorization. One of the lifting axioms is also automatic, by the definition of a cofibration. Let's start with the factorizations. Introduce two classes of maps:

$$ A = \{ D^n \times \{0\} \to D^n \times [0,1] \ : \ n \geq 0 \} $$

$$B = A \cup \{S^{n-1} \to D^n \; : \; n \geq 0, S^{-1} = \emptyset\}$$

These are compact, in a category-theory sense. By definition of Serre fibrations, a map is a fibration iff it has the right lifting property with respect to A. A map is an acyclic fibration iff it has the RLP w.r.t. B. (This was on the homework.) I need another general fact:

15.7 Proposition. *The class of maps having the left lifting property w.r.t. one of these classes is closed under arbitrary coproducts, co-base change, and countable (or even transfinite) composition. By countable composition*

$$A_0 \hookrightarrow A_1 \to A_2 \to \cdots$$

we mean the map $A \to colim_n A_n$.

Suppose I have a map $f_0 : X_0 \to Y_0$. We want to produce a diagram:

We have $\sqcup V \to \sqcup D$ where the disjoint union is taken over commutative diagrams

where $V \to D$ is in A. Sometimes we call these lifting problems. For every lifting problem, we formally create a solution by defining X_1 as the pushout:

Since Y already made this diagram commute, there is a map f_1. By construction, every lifting problem in X_0 can be solved in X_1.

We know that every map in A is a cofibration. Also, $\sqcup V \to \sqcup D$ is a homotopy

equivalence. k is an acylic cofibration because it is a weak equivalence (recall that it is a homotopy equivalence) and a cofibration.

Now we make a cone of $X_0 \to X_1 \to \cdots X_\infty$ into Y. The claim is that f is a fibration:

$$
\begin{array}{ccc}
X & \overset{\sim}{\hookrightarrow} & X_\infty \\
 & \searrow & \downarrow f \\
 & & Y
\end{array}
$$

by which we mean

$$
\begin{array}{ccccc}
V & \longrightarrow & X_n & \longrightarrow X_{n+1} & \longrightarrow X_\infty \\
\ell \downarrow & \nearrow & \downarrow & \downarrow & \downarrow \\
D & \longrightarrow & Y & \longrightarrow Y & \longrightarrow Y
\end{array}
$$

where $\ell \in A$. V is compact Hausdorff. X_∞ was a colimit along closed inclusions.

$$\mathscr{Q.E.D.}$$

So I owe you one lifting property, and the other factorization.

<div align="center">

Lecture 16
3/2/2011

</div>

We were in the middle of proving that spaces forms a (Quillen) model category with the Serre fibrations and weak equivalences. Last time, we came pretty close. Let's remember where we were. We were going through the axioms, which I'll put up again.

A category $\mathcal{C}$ with classes of morphisms called *fibrations, cofibrations, and cofibrations* is a *model category* if

1. Cocomplete and complete.

2. All classes closed under retracts

3. 2-out-of-three

4. In a diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
$$

with $A \hookrightarrow B$ a cofibration, $X \twoheadrightarrow Y$ a fibration and one a weak equivalence, a lift exists.

5. Factorization.

We proved factorization, i.e. that each map can be factored as an acyclic cofibration followed by a fibration. This we did last time. We also have to show that any map can be factored as a cofibration and an acyclic fibration. This is analogous.

Recall that *fibrations* in the topological space category were Serre fibrations, and weak equivalences were weak homotopy equivalences. Cofibrations were anything with the lifting property w.r.t. trivial fibrations.

We introduced two classes of maps

$$\mathcal{A} = \{D^n \hookrightarrow D^n \times I\}$$

and

$$\mathcal{B} = \mathcal{A} \cup \left\{S^{n-1} \hookrightarrow D^n\right\}.$$

This situation is quite common, when you have two classes of maps like this. We checked that *fibration* was equivalent to having RLP w.r.t. $\mathcal{A}$ and *acyclic fibration* was equivalent to having RLP w.r.t. $\mathcal{B}$. This is a common situation. You have two small collections that, we say, "generate" the model structure.

Let's continue the proof. We need to understand the trivial cofibrations better. We do know that $D^n \hookrightarrow D^n \times [0, 1]$ is a trivial cofibration.

Let's do the other factorization of a morphism. Given $f : X \to Y$, we will produce a cofibration $X \hookrightarrow X_1$ and a map $X_1 \to Y$. Given this, we will produce

$$X \longrightarrow X_1$$
$$\searrow \quad \downarrow$$
$$Y$$

such that for every lifting problem

$$
\begin{array}{ccc}
S & \longrightarrow X \longrightarrow X_1 \\
\downarrow & \downarrow \nearrow \\
D & \longrightarrow Y
\end{array}
$$

there is a solution in X_1. To get X_1, we take the coproduct over all lifting problems

$$
\begin{array}{ccc}
S & \longrightarrow X \\
\downarrow & \nearrow \downarrow \\
D & \longrightarrow Y
\end{array}
$$

and X_1 will be defined to be the pushout. It is easy to see that $X \to X_1$ is a cofibration because it is a pushout of cofibrations. Then, just as before, we do the same for $X_1 \to Y$ to get $X_1 \hookrightarrow X_2$, $X_2 \to Y$; we eventually repeat this and let $X_\infty = \varinjlim X_n$. The claim is that the map

$$\varinjlim X_n \to Y$$

is an acyclic fibration. This proof is analogous. If we have a lifting problem

$$\begin{array}{ccc} S & \longrightarrow & X_\infty \\ \downarrow & \nearrow & \downarrow \\ D & \longrightarrow & Y \end{array}$$

then we can factor this through some X_n by compactness. Then we get a diagram

$$\begin{array}{ccc} S & \longrightarrow & X_n \\ \downarrow & \nearrow & \downarrow \\ D & \longrightarrow & Y \end{array}$$

and we can lift out of D at least to X_{n+1}, thus to X_∞. We are using the fact that

$$\mathrm{Hom}(S, \varinjlim X_n) = \varinjlim \mathrm{Hom}(S, X_n).$$

This is a general categorical statement.

So finally we have to do the last lifting property. If we have a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y \end{array}$$

where $A \to B$ is an acyclic cofibration and $X \to Y$ a fibration, we want a lift. The first thing is to factor $A \to B$ as an acyclic cofibration followed by an acyclic fibration, $A \to A' \to B$. We can use the construction done yesterday. Namely, we can choose $A \to A'$ such that it has the LLP with respect to all fibrations because it can be chosen by the usual argument as a pushout, countable composition, and cobase change of maps in $\mathcal{A}$. We can draw a diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & B \end{array}$$

such that $A \to B$ is a retract of $A \to A'$. But $A \to A'$ has the LLP with respect to all fibrations, so $A \to B$ does too.

There's a whole bunch of things to collect about the experience we've just shared. First of all, when we made these constructions of attaching things, we were attaching cells. So we can call a map $A \to B$ constructed as a colimit of cobasechanges of coproducts

of $\mathcal{B}$ **cellular.** That's what it is. We showed that the cofibrations are *the retracts of the cellular maps.*

This entire year, we've been secretly making a distinction between cellular complexes and CW complexes. Here we are working with cell complexes. Cell complexes don't have to have their cells attached in order of dimension.

16.1 Example. Let X be a space. Then the map $* \to X$ allows for a factorization

$$* \to X' \to X$$

where X' is a cell complex and $X' \to X$ is an acyclic fibration.

In fact, all these theorems like the Whitehead theorem, etc. work for cellular complexes as well. The arguments can be cleaner.

We also want to note that in the above situation, there were two sets of maps $\mathcal{A}, \mathcal{B}$ that detected the fibrations and the cofibrations; these two sets of maps had the property that homming out of them commuted with certain colimits.

16.2 Definition. Fix a model category. A pair of sets of maps $\mathcal{A} = \{V \hookrightarrow \Delta\}, B = \{S \hookrightarrow D\}$ **generate** the model category structure if being a fibration is equivalent to having RLP with respect to $\mathcal{A}$ and being an acyclic fibration is equivalent to having RLP with respect to $\mathcal{B}$.

The point is that the above argument of taking transfinite pushouts and colimits generalizes when we have a pair $\mathcal{A}, \mathcal{B}$ as above. Given $X \to Y$, we can construct $X \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots$.

16.3 Definition. An object S in a category is called **compact** if $\mathrm{Hom}(S, -)$ commutes with filtered colimits.

16.4 Example. For sets, compact means finite.

16.5 Example. In R-modules, compact means finitely presented.

In spaces, *no* infinite object is compact. In the category of compactly generated weak Hausdorff spaces, even a singleton is not compact.

The thing is, often we will want compactness to hold with respect to special types of morphisms. Like, we can restrict the transition maps in that colimit system belong to a restricted set. For instance, in the above argument, we needed compactness with respect to closed inclusions.

There is a general language that people use, but people use it differently in different situations.

If $\mathcal{A}, \mathcal{B}$ generate a model structure and the domains of the maps of $\mathcal{A}, \mathcal{B}$ are compact, then we say that the model structure is **compactly generated.** Some people say **combinatorial.**

When the model category is compactly generated, then we can make the above argument. We can form the factorizations as we did. There is a name of the construction above, called the **small object argument.**

Let us do some more examples of model categories. Well, at least one.

16.6 Example. There is a model structure on spaces where the fibrations are the Hurewicz fibrations, and cofibrations the Hurewicz cofibrations, and the weak equivalences the homotopy equivalences. This is called the **Strom model structure.**

16.7 Example. Let R be a ring. Let $\mathcal{C}$ be the category of chain complexes of left R-modules. Here chain complexes can be infinite in both directions. We want to define a model structure here. We do it as follows:

1. A weak equivalence is an iso in homology

2. Fibrations are surjections

3. Cofibrations—forced

The claim is that this is a model structure. The proof is left to the reader. Let us note that the generators are as follows. Here $\mathcal{A} = \{0 \to (0 \to 0 \to \ldots \to M \to M \to 0 \to 0\}$. Here $\mathcal{B} = \mathcal{A} \cup \{(0 \to M \to 0) \to (0 \to M \to M)\}$. The claim is that $\mathcal{A}, \mathcal{B}$ generate a model structure. It is a fun exercise to work out.

You learn that the cofibrations are the monomorphisms whose cokernel is termwise projective. This recovers most of the story of homological algebra.

16.8 Example. Chain complexes which are bounded below degree zero. We can do something similar. We find that weak equivalences are homology isomorphisms. $M_* \to N_*$ is a fibration iff $M_i \to N_i$ is surjective for all $i > 0$. An acyclic fibration is a surjection that induces a homology isomorphism.

Lecture 17
March 4, 2011

Model categories are supposed to be a place to do homotopy theory. We need something to play the role of a space cross the unit interval. These are called *cylinder objects*.

17.1 Definition. Let A be in a model category, which is closed under finite coproducts, so $A \sqcup A \in \mathcal{C}$. A cylinder object $Cyl(A)$ results from a factorization of the standard map $A \sqcup A \to A$:

$$
\begin{array}{ccc}
A \sqcup A & \hookrightarrow & Cyl(A) \\
 & \searrow & \Big\downarrow f \sim \\
 & & A
\end{array}
$$

Sometimes you want f to be a fibration.

For example, if A is a CW complex, then $Cyl(A) = A \times [0,1]$. This is the example you should have in mind. Now that we've got cylinders, we can talk about homotopies.

17.2 Definition. Two maps $f, g : A \to X$ are (left) homotopic if there is some cylinder object $Cyl(A)$, along with a map h, which extend f and g. The picture is:

$$
\begin{array}{ccc}
A \sqcup A & & \\
\Big\downarrow & \searrow^{f,g} & \\
Cyl(A) & \xrightarrow{\ h\ } & X
\end{array}
$$

There is a dual notion, and that is a path object:

17.3 Definition. A *path object* for X is a factorization of the diagonal map $X \to X \times X$:

$$
\begin{array}{ccc}
 & PX & \\
\nearrow & & \Big\downarrow \\
X & \xrightarrow[\Delta]{} & X \times X
\end{array}
$$

For example, in spaces this is just the space of paths in X:

$$
\begin{array}{ccc}
 & X^I & \\
\nearrow & & \Big\downarrow fib \\
X & \xrightarrow[\Delta]{} & X \times X
\end{array}
$$

[The map $X \to X^I$ takes x to the constant path; $X^I \to X \times X$ takes a path to the pair of its endpoints.] Maps $f, g : A \to X$ are right homotopic if there is a homotopy h making the diagram commute:

$$
\begin{array}{ccc}
 & & PX \\
 & \nearrow^{h} & \downarrow \\
X & \xrightarrow{f,g} & X \times X
\end{array}
$$
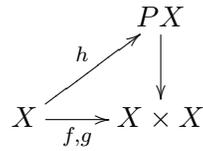
Right now, we don't know that right and left homotopies have anything to do with each other.

Question: Is left homotopy an equivalence relation? Suppose f, g are homotopic. To show symmetry, just compose with the isomorphism that switches these.

$$
\begin{array}{ccc}
A \sqcup A & \xrightarrow{flip} & A \sqcup A \\
& & \downarrow \quad \searrow^{f,g} \\
& Cyl(A) & \longrightarrow X
\end{array}
$$

$$
\begin{array}{c}
A \sqcup A \\
\downarrow \\
Cyl(A) \xrightarrow{p} A \xrightarrow{f} X
\end{array}
$$
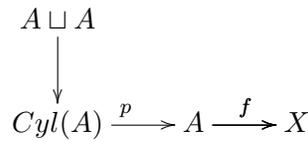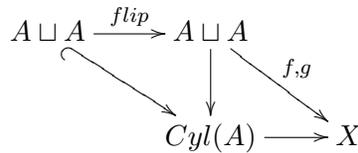
Suppose $f_1 \sim L f_2 \sim L f_3$. We have homotopies

$$
\begin{array}{cc}
A \sqcup A & \\
\downarrow \searrow^{f_1, f_2} & \\
Cyl_{12}(A) \longrightarrow X &
\end{array}
\qquad
\begin{array}{cc}
A \sqcup A & \\
\downarrow \searrow^{f_2, f_3} & \\
Cyl_{23}(A) \longrightarrow X &
\end{array}
\qquad \text{and we want} \qquad
\begin{array}{cc}
A \sqcup A & \\
\downarrow \searrow^{f_1, f_3} & \\
Cyl_{13}(A) \longrightarrow X &
\end{array}
$$
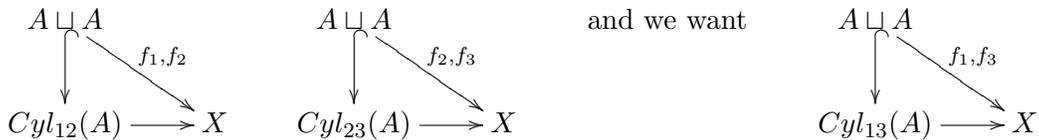
The idea is to make $Cyl_{13}(A)$ a disjoint union. In particular, define $Cyl_{13}(A)$ as the pushout in the following diagram

$$
\begin{array}{ccc}
A & \xrightarrow{i_1} & Cyl_{12}(A) \\
\downarrow^{i_0} & & \downarrow \\
Cyl_{23}(A) & \longrightarrow & Cyl_{13}(A) \\
& & \searrow \\
& & X
\end{array}
$$
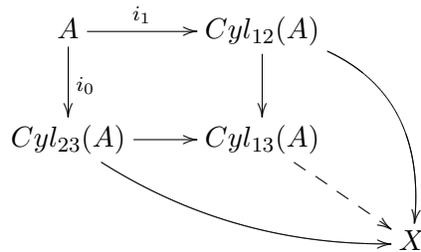
where i_0 is the inclusion of A into the second spot of $A \sqcup A$, followed by the map into $Cyl_{12}(A)$. So we get the diagram

$$A \sqcup A$$
$$\downarrow \qquad \searrow^{f_1, f_3}$$
$$Cyl_{13}(A) \longrightarrow X$$

where the map $A \sqcup A \to Cyl_{13}(A)$ is given by the two maps $A \to Cyl_{13}(A)$ above. We need $Cyl_{13}(A)$ to be a cylinder object, so we need $A \sqcup A \to Cyl_{13}(A)$ to be a cofibration, and $Cyl_{13}(A) \to A$ to be a weak equivalence.

$$A \sqcup A \hookrightarrow Cyl_{13}(A)$$
$$\searrow \qquad \downarrow$$
$$A$$

Do the cofibration first. Actually, it's not a cofibration. But, it is so, when A is cofibrant.

Our category has finite products and coproducts, and so it has the empty colimit, i.e. the initial object. It also has the empty limit, i.e. the terminal object.

17.4 Definition. An object A is *cofibrant* if the map from the initial object to A is a cofibration. An object is *fibrant* if the map from A to the terminal object is a fibration. In spaces, every space is fibrant.

For example, the cofibrant chain complexes are complexes of projective modules. Every cochain complex is cofibrant. This is a good class for mapping out of.

Suppose A is cofibrant. Then $A \to A \sqcup A$ is a cofibration, because it is a pushout:

$$\emptyset \hookrightarrow A$$
$$\downarrow \qquad \downarrow$$
$$A \hookrightarrow A \sqcup A$$

Therefore, $A \sqcup A \hookrightarrow Cyl$ is a cofibration by definition. There is a projection map back to A. By stuff, we can show that $A \to Cyl(A)$ is a weak equivalence:

$$A$$
$$\downarrow \qquad \searrow^{\sim}$$
$$A \sqcup A \hookrightarrow Cyl(A)$$
$$\downarrow^{\sim}$$
$$A$$

So we have $A \hookrightarrow Cyl_{12}(A)$ is an acyclic cofibration, and hence $Cyl_{23}(A) \hookrightarrow Cyl_{13}(A)$ is an acyclic cofibration. This makes $Cyl_{13}(A) \to A$ into a weak equivalence.

Claim: If A is cofibrant, then $A \sqcup A \to Cyl_{13}(A)$ is a cofibration. Glue on one of the ends (downward map). When we take the pushout we get:

$$
\begin{array}{ccc}
A \sqcup A & \hookrightarrow & Cyl_{12} \\
\downarrow & & \\
A \sqcup A \hookrightarrow A \sqcup Cyl_{23}(A) & \hookrightarrow & Cyl_{13}(A)
\end{array}
$$

17.5 Proposition. *If A is cofibrant, then left homotopy is an equivalence relation.*

Question: Can this be calculated with just one cylinder object?

Answer: No, in general... but yes, if X is fibrant.

Oops. If $A \sqcup A \to Cyl_{13}(A)$ wasn't a cofibration, you could just have factored this into a cofibration followed by a weak equivalence, and then used that.

Suppose we have some cylinder object where you have a cofibration followed by a weak equivalence, and then another cylinder object:

$$
\begin{array}{ccc}
A \sqcup A & \hookrightarrow & Cyl'(A) \\
\downarrow & \nearrow & \downarrow \sim \\
Cyl & \longrightarrow & A
\end{array}
$$

So there exists a lift. All you need is a cofibration on the left, and you will have: cyl' homotopic implies Cyl homotopic for any $Cyl(A)$. Suppose X is fibrant, and we want $f, g : A \to X$ $Cyl(A)$-homotopic. I want to factor this map is a cofibration $Cyl''A$ followed by an acyclic fibration. One thing was already a weak equivalence:

$$
\begin{array}{ccc}
Cyl(A) & \longrightarrow & Cyl'(A) \\
\wr \downarrow \sim & \nearrow \sim & \\
Cyl''(A) & &
\end{array}
$$

If I had a $Cyl(A)$ homotopy to X:

$$
\begin{array}{ccc}
Cyl & \longrightarrow & X \\
\wr \downarrow & & \downarrow \\
Cyl''A & \longrightarrow & *
\end{array}
$$

So a lift exists. So if X is fibrant, then f and g are $Cyl''(A)$ homotopic. I need to get a section somewhere. If you have

$$A \sqcup A \longrightarrow Cyl''A \longrightarrow X$$

$$\downarrow \quad\quad\quad \downarrow \sim$$

$$Cyl'A \xrightarrow{\;Id\;} Cyl'A$$

So you have a lift. This shows that $Cyl''(A)$ homotopic implies Cyl' homotopic.

17.6 Proposition. *If A is cofibrant, and X is fibrant, then left homotopy can be calculated using any single cylinder object.*

When you try to set this up in abstract homotopy theory, you want to be going from a cofibrant object to a fibrant one. This is clear in simplicial sets. There aren't enough maps between two random sets. You need a Khan complex (fibrant). In homological algebra, chain homotopy of maps might not work unless you've got projectives. The dual notion holds as well: if A is cofibrant, and X is fibrant, then right homotopy can be calculated using any single path object.

17.7 Proposition. *If A is cofibrant and X is fibrant, then right-homotopic is the same as left-homotopic on the set of maps from $A \to X$.*

Proof: go do it yourself.

The next thing to do is to define the homotopy category, but I'll do that next time. In principle, you'd like to take maps, and mod out by left or right homotopy. But you need to worry about fibrant and cofibrant objects. So it takes a little setting up.

I want to spend some time pointing something out. We've defined $Cyl(A)$, which is supposed to be a model for $A \times [0,1]$, which I've suggested we write as $A \times \Delta[1]$. You can also have $A \times \Delta[2]$.

$$0 \xrightarrow{Cyl_{02}(A)} 2$$
$$\quad\quad Cyl_{12}(A)$$
$$1$$

Call this object K. You can take:

$$K \hookrightarrow A \otimes \Delta[2]$$
$$\downarrow \quad \swarrow$$
$$A$$

We should be able to form $A \times \Delta[n]$ for any n. We can make any simplicial complex this way. We have $A \otimes S$ for S any simplicial complex. We have $C(A \otimes S, X)$, which should sort of be $Spaces(S, C(A, X))$. Even though we didn't start this way, it tells us that somehow, we can associate a space, up to homotopy, of maps $A \to X$.

Suppose we have A in a model category that is cofibrant. We can make ΣA by making the cylinder of A, and collapsing the ends to the terminal object:

$$\begin{array}{ccc} A \sqcup A & \longrightarrow & * \sqcup * \\ \downarrow & & \downarrow \\ Cyl(A) & \longrightarrow & \Sigma A \end{array}$$

If Xx is fibrant, with base point (i.e. a map from $* \to X$), we can consider $[\Sigma A, X]$, pointed maps modulo homotopy. That group should be the fundamental group of some sort of mapping group: $\pi C(A, X)$.

[Imagine that the tetrahedron is associated with $A \otimes \Delta[3]$]

Lecture 18
3/7

So we're continuing our journey into the theory of Quillen model categories. Let $\mathcal{C}$ be a model category. Last time, we defined the notion of *homotopy*. Now we want to define the *homotopy category* of $\mathcal{C}$, written $\text{Ho}(\mathcal{C})$.

§1 Localization

Let's first start with a general construction. Let $\mathcal{C}$ be a category and S a collection of morphisms. We will consider the case in particular where S is the class of weak equivalences of $\mathcal{C}$. What we want to do is to define a new category $S^{-1}\mathcal{C}$ and a functor

$$\mathcal{C} \to S^{-1}\mathcal{C}$$

that is universal for functors taking the maps to S to isomorphisms. The notation is of course borrowed from commutative algebra, and is supposed to remind you of the universal property of localization of a ring, where you invert elements.

18.1 Definition. This category $S^{-1}\mathcal{C}$ will be called the **localization** of $\mathcal{C}$ at S.

The universal property is the following. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(f)$ is an isomorphism for all $f \in S$; then there exists a functor

$$S^{-1}\mathcal{C} \to \mathcal{D}$$

and a diagram

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & S^{-1}\mathcal{C} \\
 & \searrow \quad \swarrow & \\
 & \mathcal{D} &
\end{array}
$$

that commutes up to unique natural isomorphism.

The diagram is *not* required to commute on the nose. If $S^{-1}\mathcal{C}$ exists, it is characterized uniquely up to unique(?) *equivalence.*

We're going to focus on the existence of $S^{-1}\mathcal{C}$ today, and worry about the details of uniqueness later.

Here's the construction that you would like to make. Form a new category out of $\mathcal{C}$ by taking $\mathcal{C}$ and adding formally a map f^{-1} for each $f \in S$. These are required to satisfy that $f \circ f^{-1}, f^{-1} \circ f$ be the identity. This is analogous to how one can build groups out of generators and relations. However, doing so can add a whole bunch of new maps indirectly.

There is, however, a set-theoretic difficulty. If we do this, and formed $S^{-1}\mathcal{C}$, we have no guarantee that the new hom-sets will in fact be sets. S might be large, and $\mathcal{C}$ larger still. In different contexts, there are various ways around this.

18.2 Example. If you want to construct an abelian category mod a Serre class, then you are localizing.

§2 The homotopy category

Now we *want* to define $\mathrm{Ho}(\mathcal{C})$ to be the localization of a model category at the set of weak equivalences. But the set-theoretic difficulties come up since the weak equivalences don't form a set.

Actually, what we are going to do is to construct the homotopy category in such a way that it is a legitimate category.

What you can show is that, for a model category, if A is cofibrant and B fibrant, then the maps in this localized category from $A \to B$ is the same thing as the homotopy classes of maps $A \to B$. If A is an arbitrary object, we can factor $\emptyset \to A$ into a cofibration and a weak equivalence, so A is weakly equivalent to a cofibrant object A_c. (The map $A_c \to A$ is called a *cofibrant approximation* to A, and is analogous to a projective resolution in homological algebra.) If we invert the weak equivalences, then A_c is going to become the same as A. Similarly we can get for any object B a *fibrant approximation* $B \to B_f$ which is a weak equivalence.

Suppose we have two homotopic maps $f, g : A \to X$. Then we can factor them as $A \rightrightarrows \mathrm{Cyl}(A) \to X$. Since the two maps $A \rightrightarrows \mathrm{Cyl}(A) \to A$ are the same and the latter is a weak equivalence, we find that in the localized category the two inclusions $A \rightrightarrows \mathrm{Cyl}(A)$ become the same.

It isn't hard to check that if you are working with a cofibrant object and a fibrant object, then the maps between the two of them already see everything you might get by inverting weak equivalences. By using cofibrant and fibrant replacements, we get a picture of what maps between any two objects A, X look like in $\mathrm{Ho}(\mathcal{C})$.

Given Y, we can first form a cofibrant approximation $Y_c \to Y$, and then get an acyclic cofibration $Y_c \to Y_{cf}$ where $Y_{cf} \to *$ is a fibration. Then Y_{cf} is cofibrant-fibrant and is related to Y by a chain of weak equivalences.

To construct $\mathrm{Ho}(\mathcal{C})$, choose cofibrant-fibrant replacements for each object in $\mathcal{C}$. In practice, we can always do this easily because the two factorizations in the axiom M5 are even *functorial*. This always happens in the compactly generated case (i.e. when we have sets $\mathcal{A}, \mathcal{B}$ with which we do the small object argument). Thus the cofibrant-fibrant approximation becomes a *natural* thing.

From this, we can define the *homotopy category* of $\mathcal{C}$ whose objects are the same as the objects of $\mathcal{C}$, but the maps $A \to X$ are defined to be maps in $\mathcal{C}$ from $A_{cf} \to X_{cf}$ modulo homotopy. Then you have to check a lot of things. You have to check that a composition law is defined. This is much easier to do when you have a functorial factorization. Then you can check the universal property, which is easy to do.

So we get that $\mathrm{Ho}\mathcal{C} = S^{-1}\mathcal{C}$ for S the class of weak equivalences. We won't go through the details, though you might want to do them.

18.3 Example. Suppose $\mathcal{C}$ is the category of spaces with the Quillen model structure (built on Serre fibrations, etc.). To calculate in the homotopy category maps $A \to X$, then we have to find cellular approximations A_{cf}, X_{cf} and consider $[A_{cf}, X_{cf}]$. The homotopy category is thus equivalent to the homotopy category of CW (or cell) complexes.

There is another model structure on topological spaces where the homotopy category is the usual homotopy category.

We are thinking that the homotopy category contains most of the information of the model category. Our goal was to carefully formulate the idea of an algebraic model for rational homotopy theory. Actually, first we wanted to define rational homotopy theory. We still have some things to do in order to do that.

In order to compare two model categories, we need to consider what a map between them might be.

§3 Morphisms between model categories

Let us first say some false starts. We might say that a morphism of model categories from $\mathcal{C} \to \mathcal{D}$ is a functor taking weak equivalences to weak equivalences. If we had such a functor, then we would get a functor

$$\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$$

by the universal properties. This is the *minimum* that we would want.

In practice, though, this is not right. For example, suppose we have a map of rings $R \to S$, and consider the categories $\mathcal{C}_R, \mathcal{C}_S$ of chain complexes of R-modules and chain complexes of S-modules (that stop in degree zero, say). There is a functor from $\mathcal{C}_R \to \mathcal{C}_S$ obtained by tensoring. Unless S is flat over R, though, this *won't* preserve weak equivalences (though it will when we have projectives). But we *want* a map of homotopy categories here.

So our first notion demanded too much. Suppose instead we have a functor $F : \mathcal{C} \to \mathcal{D}$ and F sends weak equivalences *between cofibrant objects* to weak equivalences. Then we can still get a functor

$$\mathrm{Ho}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D}).$$

Suppose $A \in \mathcal{C}$; then $\mathrm{Ho}F$ sends $A \mapsto F(A_c)$.

18.4 Definition. Let $F : \mathcal{C} \to \mathcal{D}$ is a functor. The **left derived functor** of F, if it exists, is a functor $LF : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ together with a natural transformation

$$LF \circ (\mathcal{C} \to \mathrm{Ho}(\mathcal{C})) \to (\mathcal{D} \to \mathrm{Ho}(\mathcal{D})) \circ F$$

which is universal from the left.

This is explained in the notes for more information.

18.5 Definition. A **Quillen morphism** $\mathcal{C} \to \mathcal{D}$ (model categories) is a pair of adjoint functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ (F the left adjoint, G the right adjoint) such that F preserves cofibrations and acyclic cofibrations. (This is equivalent to demanding that G preserve fibrations and acyclic fibrations.)

In this case, the left-derived functor of F exists (as does the right-derived functor of G), and they are adjoint.

Lecture 19
March 11, 2011

Last time we talked about homotopy categories.

19.1 Definition. A Quillen functor (pair) from $\mathcal{D} \to \mathcal{C}$ is a pair of adjoint functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$, such that F preserves cofibrations and acyclic cofibrations. This is equivalent to asking for G to preserve fibrations and acyclic fibrations.

19.2 Definition. Let $F : \mathcal{C} \to \mathcal{D}$. The left derived functor, if it exists, is a functor $LF : ho\mathcal{C} \to ho\mathcal{D}$ such that there is a universal transformation

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{D} \\
{\scriptstyle p_C} \downarrow & \nearrow & \downarrow {\scriptstyle p_D} \\
ho\mathcal{C} & \xrightarrow{\ LF\ } & ho\mathcal{D}
\end{array}
$$

from $LF \circ p_c \to p_D \circ F$ which is universal in the sense that given any other diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{D} \\
{\scriptstyle p_C} \downarrow & & \downarrow {\scriptstyle p_D} \\
ho\mathcal{C} & \xrightarrow{\ G\ } & ho\mathcal{D}
\end{array}
$$

with a natural transformation $G \circ p_C \to p_D \circ F$ there is a unique natural transformation $G \to LF$ making the obvious diagram commute.

$$
\begin{array}{c}
G \circ p_c \\
\downarrow \qquad \searrow \\
LF \circ p_c \longrightarrow p_D \circ F
\end{array}
$$

You can think of the left derived functor as creating something that is "closest on the left," or "closest to making the diagram commute."

Almost always, you calculate $LFX = FX_c$ as the cofibrant approximation:

$$
\begin{array}{c}
X_c \\
\nearrow \quad \downarrow {\scriptstyle \sim} \\
\emptyset \longrightarrow X
\end{array}
$$

Let $R \to S$ be a map of commutative rings. Let Mod_R be chain complexes of R-modules. Similarly, define Mod_S. (We are assuming that the indices are $0, 1, 2, \cdots$) Then there is a pair of adjoint functors $Mod_R \leftrightarrows Mod_S$, where $M \mapsto S \otimes_R M$ and $Hom_S(R, N) \leftarrow N$, that forms a Quillen morphism. Recall if you have a chain complex $0 \to 0 \to M$, you can make a projective resolution $P_0 \to M$; that is $LF(M)$. So $FP_0 = S \otimes_R P$.

19.3 Theorem. *If $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is a Quillen morphism, then the left and right derived functors exist and are adjoint functors $LF : ho\mathcal{C} \leftrightarrows ho\mathcal{D} : RG$.*

Suppose there is a Quillen morphism (F, G). Here you take LFX by taking the cofibrant

approximation. Suppose you have X and you factor it. Then you can also put in:

$$
\begin{array}{ccc}
X_c & \overset{\sim}{\hookrightarrow} & X_{cf} \\
& & \\
\nearrow \quad \downarrow{\scriptstyle\sim} & & \downarrow \\
\hookrightarrow \quad X & & \\
\emptyset \longrightarrow X & & *
\end{array}
$$

since F preserve acyclic cofibrations (and cofibrations).

19.4 Definition. A Quillen morphism is a Quillen equivalence if for every cofibrant $A \in C$ and fibrant $D \in D$ a map $A \to GX$ is a weak equivalence iff $FA \to X$ is a weak equivalence. (So iff having a w.e. on one side gives you one on the other side.)

19.5 Theorem. *If (F, G) is a Quillen equivalence, then $LF : ho\mathcal{C} \to ho\mathcal{D}$ forms an equivalence of categories.*

Anything I want to do in homotopy theory in $\mathcal{C}$ is equivalent to something I can do in homotopy theory in $\mathcal{D}$. This is fairly straight-forward to prove.

We were seeking an algebraic model of $\mathbb{Q}$-homotopy theory. But what does this even mean? Now we know part of this: we can construct a purely algebraically-defined model category, which should be equivalent via a Quillen equivalence, to spaces modulo torsion.

§1 Model Category on Simplicial Sets

Recall we had the standard n-simplex $\Delta[n]$, the linear simplex with vertices $v_0 \cdots v_n$. We also considered linear maps $\Delta[n] \to \Delta[m]$ that are order-preserving on the vertices. Call Δ the category with objects $\Delta[n]$, and morphisms linear maps that are order-preserving on the vertices. But the maps are determined simply by where the vertices go; you could have forgotten all about the rest of the simplex. So an equivalent (isomorphic) category is the category with objects the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and order preserving maps. You could also say that Δ is equivalent to the caegory of finite-ordered sets. Every finite-ordered sets is equivalent, via unique isomorphism, to some $[n]$.

So a simplicial set is a contravariant functor $X_\bullet : \Delta \to Sets$. No one ever writes $X(\Delta[n])$; instead you write $X([n]) = X_n$, where $[n] \mapsto X_n$. X_n is called the set of n-simplices of X. For $0 < 1$ represented as $0 - 1$ and another point which can map into both 0 and 1; that is, you get two maps $X_1 \to X_0$, that we call d_0 and d_1. There are three maps out of X_2, and more generally you can define $d^i : [n] \to [n+1]$ where $0 \mapsto 0$, $1 \mapsto 1$, $\ldots$, $i-1 \mapsto i-1$, $i \mapsto i+1$, $\ldots$ Geometrically, this is the inclusion of the face opposite the nth vertex. You can also map a simplex into a lower one. (You didn't insist that the order-preserving maps were one-to-one.) So $d^i : [n] \to [n+1]$ is a coface

map, and $d_i : X_{n+1} \to X_n$ is a face map. You can also define $s^i : [n+1] \to [n]$ where $0 \mapsto 0, \ldots, i \mapsto i, i+1 \mapsto i, \ldots$. For example, collapsing a tetrahedron onto its base is s^1. But they have to be order-preserving: you can't have just any collapse-map. So you can collapse $2-3$ and $2-1$ but not $2-0$.

So we have maps $d^0, d^1 : \Delta[0] \to \Delta[1]$ and degeneracy map s^0; we have $d^0, d^1, d^2 : \Delta[1] \to \Delta[2]$ and degeneracy maps s^0 and s^1.

19.6 Example. Suppose X is a space. The singular simplicial set $Sing(X)$ consists of all continuous functions $\Delta[n] \to X$. This defines a contravariant functor from $\Delta \to Sets$.

19.7 Example. As another example, suppose you have a simplicial complex. Suppose I have a simplicial complex K, that is maybe a triangle with an extra stick. Order the vertices $a \to b$, *bcd*. We can form a simplicial set $\mathcal{K}_n$ that is the order-preserving maps $[n] \xrightarrow{f} vert(K)$ such that $f([n])$ is a simplex. Note that you have to explicitly choose a total ordering for this to make sense. Taking K the be the standard n-simplex on $\{0 \cdots n\}$ that gives us a simplicial set $\Delta[n]$. where the dot indicates it is a simplicial set, instead of just the standard n-simplex.

There is a more direct way of saying this. $\Delta[n]_\bullet$ is a functor from $\Delta \to Sets$; it is the functor represented by $[n]$. So $\Delta[n]_k = \Delta([k], [n]$; that is, $\Delta[n]_k$ is just $\Delta(-, [n])$.

Next lecture: Yoneda lemma.

Let's map simplicial sets to topological spaces, via a pair of adjoint functors. One of these is $Sing$; the other one is written $|\ |$ and is called geometric realization. The geometric realization of $\Delta[n]_\bullet$ is just $\Delta[n]$. This being the adjoint actually determines this functor. So we can write

$$|X_\bullet| = \sqcup X_n \times \Delta[n] / \sim$$

where this is defined in terms of face maps, &c. We will construct a model category structure on simplicial sets, and show that $|\ | : sSets \to Spaces$ forms a Quillen equivalence.

Lecture 20
3/11

Simplicial sets are good. We can think of them in geometric terms. They are like a generalization of a simplicial complex, and in fact we can do homotopy theory with them. But they are also completely combinatorial. Everything we want to do, we can do hand-by-hand, moving the points around. This is very convenient. But by definition, simplicial sets are a functor category.

The category of simplicial sets is the category of functors

$$\Delta \to \mathbf{Sets}$$

that map the simplex category Δ into sets. This means that you can always write proofs in the language of categories. Today, we will take some time to review (or tell you for the first time) some facts from category theory that make it convenient to understand things about simplicial sets.

§1 The Yoneda embedding

Suppose $\mathcal{C}$ is a category. We consider *small* categories $\mathcal{C}$, but maybe you can make something work in general. We pick an object $X \in \mathcal{C}$. We can make a functor

$$Y_X : \mathcal{C}^{op} \to \mathbf{Sets}$$

that maps

$$C \mapsto \mathrm{Hom}_{\mathcal{C}}(C, X).$$

So Y_X is the functor of maps into X.

20.1 Definition. We say that Y_X is **represented** by the object X. We call Y_X is a **representable functor.**

One of the big theorems we talked about this semester was that if $\mathcal{C}$ was the homotopy category of CW complexes, then the cohomology functors $H^n(-, \pi)$ are representable (by the Eilenberg-Maclane spaces).

So we have the **Yoneda embedding** from $\mathcal{C}$ into the category of functors $\mathcal{C}^{op} \to \mathbf{Sets}$, sending $X \mapsto Y_X$. We use the letter Y because Y stands for "Yoneda."

The first question you might ask is, in what sense is this an embedding? The sort of trivial, but very important fact, called the Yoneda lemma, makes this precise:

20.2 Lemma (Yoneda). *Suppose $F : \mathcal{C}^{op} \to \mathbf{Sets}$ is any contravariant functor and $X \in \mathcal{C}$. Then the natural transformations $Y_X \to F$ are naturally in bijection with elements of $F(X)$.*

We'll indicate the proof (which is completely trivial; there isn't much to play with).

20.3 Example. Natural transformations

$$Y_A \to Y_B$$

correspond to maps in $\mathcal{C}$ from $A \to B$. Thus the Yoneda embedding is fully faithful. This often lets you reduce things in category theory to things in sets. Often you can check things in category theory objectwise.

Proof. Given a natural transformation $T : Y_X \to F$, look at the element X. We get a map

$$\mathrm{Hom}_{\mathcal{C}}(X, X) = Y_X(X) \to F(X)$$

and look at where the identity map $1_X : X \to X$ is sent to in $F(X)$. There is thus a morphism from natural transformations to elements of $F(X)$.

Now the point is that a natural transformation is uniquely determined by where it sends the identity $1_X : X \to X$. For instance, suppose we are given $a \in F(X)$; then we need to make a natural transformation

$$Y_X \to F$$

sending the identity to a. For each B, we need a map $\mathrm{Hom}(B, X) \to F(B)$. For each map $\varphi : B \to X$, we send it to $\varphi^*(a)$ because this is a natural transformation. This is forced by naturality. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}(B, X) & \xrightarrow{\varphi^*} & \mathrm{Hom}(X, X) \\
\downarrow & & \downarrow \\
F(B) & \xrightarrow{\varphi^*} & F(X)
\end{array}
$$

that forces this. $\mathcal{Q.E.D.}$

People don't appreciate how many things this Yoneda lemma tells you.

20.4 Corollary (Uniqueness of the object representing a functor)**.** *An object $A \in \mathcal{C}$ is determined uniquely (up to unique isomorphism) by the functor Y_A.*

§2

We've tried hard not to give a name for the category of contravariant functors $\mathcal{C}^{op} \to$ **Set**. Suppose we have two such functors F, G. What can we say about natural transformations

$$\mathrm{Hom}(F, G)?$$

Here F, G are two general functors. Hom denotes hom in the category of contravariant functors.

We know how to hom out of a representable functor. On the other hand we know that $\mathrm{Hom}(F, -)$ determines F up to unique isomorphism (by the Yoneda lemma applied to this new category, or rather its opposite).

What would a natural transformation $F \to G$ be? To give a natural transformation means to give two pieces of data. One for all objects in $\mathcal{C}$, we must give $F(X) \to G(X)$, and second these morphisms are required to satisfy the naturality condition.

A map $F(X) \to G(X)$, by Yoneda, is the same thing as a map $\sqcup_{x \in F(X)} Y_X \to G$. In other words, it is the same thing as giving a natural transformation

$$F(X) \times Y_X \to G.$$

Here $F(X)$ is the constant functor. So this is equivalent to a natural transformation

$$\sqcup_{a \in \mathcal{C}} F(a) \times Y_a \to G$$

that make certain diagrams commute. Second, for each $a \to b$, the two maps $F(b) \times Y_a \rightrightarrows G$ are to be the same. Since this is true for any G, we know that there is a universal pair of maps

$$\sqcup_{a \to b} F(b) \times Y_a \rightrightarrows \sqcup_a F(a) \times Y_a$$

So to give a natural transformation $F \to G$ is the same thing as saying that we have a natural transformation $\sqcup_a F_a \times Y_a \to G$ such that the two pull-backs to $\sqcup_{a \to b} F_b \times Y_a \to G$ are the same. In particular, we have a coequalizer

$$\sqcup_{a \to b} F_b \times Y_a \rightrightarrows \sqcup_a F_a \times Y_a \to F$$

so that:

20.5 Corollary. *Any F is tautologically a colimit of representable functors.*

Here's an example. Suppose we want to make a functor

$$T : \mathbf{Set}^{\mathcal{C}^{op}} \to \mathcal{D}$$

which we want to commute with colimits. Well, if it commutes with colimits, it will commute with the above coequalizer. Thus such a functor is determined by its values on the representable functors. In fact, we can actually construct the functor as long as we have $T(Y_a)$ for all a. Thus we only need to have a covariant functor

$$\mathcal{C} \to \mathcal{D}.$$

Thus the colimit-preserving functors from $\mathbf{Set}^{\mathcal{C}^{op}} \to \mathcal{D}$ are the same thing as covariant functors $\mathcal{C} \to \mathcal{D}$.

Pretty much anything you can say about contravariant functors into sets can be thought of using the Yoneda's lemma.

§3 Simplicial sets

We got into this to talk about simplicial sets. Suppose $\mathcal{C} = \Delta$. Then the contravariant functors

$$\mathbf{Sets}^{\mathcal{C}^{op}} = \mathbf{SSet}$$

are the simplicial sets. The representable functors are the following. The objects in Δ are the finite ordered sets $[n] = \{0, 1, \ldots, n\}$ and the functor represented by $[n]$ is, by definition, the **standard n-simplex $\Delta[n]$**.

Let X now be any simplicial set. Then we know by Yoneda that simplicial set maps

$$\Delta[n] \to X$$

are the same thing as n-simplices, i.e. elements of X_n. The tautological presentation tells us now that any simplicial set is a colimit of things of the form $\Delta[n]$. It looks like

$$\sqcup_{[n]\to[m]}\Delta[n] \times X_m \rightrightarrows \sqcup_n\Delta[n] \times X_n \to X.$$

The last point to make today is simple and can be done many other ways. We want to make a functor
$$\textbf{SSet} \to \textbf{Top}.$$

We want to make one that is *left adjoint* to the singular simplicial set functor $\textbf{Top} \to \textbf{SSet}$. This is entirely determined by where $\Delta[n]$ goes. We know that a left adjoint automatically commute with colimits and so are determined by the $\Delta[n]$. Thus it suffices to give a covariant functor

$$\Delta \to \textbf{Spaces},$$

or a *cosimplicial* space.

We just need to figure out where the standard n-simplex goes. And now we can demand that

$$\mathrm{Hom}_{\textbf{Top}}(\Delta_n, E) = \mathrm{Sing}E_n = \mathrm{Hom}_{\textbf{SSet}}(\Delta[n], \mathrm{Sing}E) = \mathrm{Hom}_\top(|\Delta[n]|, E).$$

It follows that we must demand $|\Delta[n]| = \Delta_n$ the standard n-simplex.

But $n \mapsto \Delta_n$ is a covariant functor from Δ into spaces, so that extends uniquely to the geometric realization functor. Thus there is a tautological presentation of $|X|$ for any X as a coequalizer diagram. So to get $|X|$, we take the disjoint union $\sqcup X_n \times \Delta_n$ and mod out by the relation that for every order-preserving map $n \to m$, corresponding faces are identified.

Lecture 21
March 21, 2011

We had two ways of thinking of simplicial sets. On the one hand, they are contravariant functors $\Delta \to Sets$. We can talk about them in category-theoretic ways, which we

talked about last time. We can also think about them geometrically, as a generalization of simplicial complexes. If I have a simplicial set X, we don't write $X[n]$; we write X_n, and call it the set of n-simplices. An important example is the functor represented by $[n]$, where $\Delta[n]_k = \Delta[n]_\bullet = \Delta([k], [n])$. We think of this as *the* n-simplex. We defined the geometric realization of this as $|\Delta[n]_\bullet| = \Delta[n]$, the standard n-simplex. By the Yoneda lemma, $sSet(\Delta[m]_\bullet, X_\bullet) \cong S_n$. This at least agrees with our standard terminology: the set of n-simplices in X is the set of maps from the standard n-simplex to X.

§1 Products

Suppose $X_\bullet$ and $Y_\bullet$ are simplicial sets. The product is a categorically defined thing: it is characterized by the property that $sSet(K_\bullet, X_\bullet \times Y_\bullet) \cong sSet(K_\bullet, X_\bullet) \times sSet(K_\bullet, Y_\bullet)$. The simplicial structure maps are just the simultaneous simplicial structure maps on the product:

$$(X \times Y)_n = X_n \times Y_n$$

21.1 Theorem.

$$|X_\bullet \times Y_\bullet| \to |X_\bullet| \times |Y_\bullet|$$

is a homeomorphism, provided that the category of spaces is replaced by the category of compactly generated spaces.

We won't talk about the proof.

21.2 Example. What is the geometric realization of $\Delta[1] \times \Delta[1]$? The set of k-simplices is the set of order-preserving maps $[k] \to [1] \times [1]$. So $[1]$ is the set $\{0, 1\}$. Visualize this as the points $(0, 0), (0, 1), (1, 0), (1, 1)$ in $\mathbb{R}^2$, say. A k-simplex is something that weakly increases. There are two nondegenerate 2-simplices: $(0, 0) \mapsto (0, 1) \mapsto (1, 1)$ and $(0, 0) \mapsto (1, 0) \mapsto (1, 1)$. All higher-degree simplices are degenerate. So the geometric realization has two 2-simplices, and perhaps some lower ones. So you get

$$
\begin{array}{ccc}
(0,1) & \rule[0.5ex]{1.5em}{0.4pt} & (1,1) \\
\Big| & \diagup & \Big| \\
(0,0) & \rule[0.5ex]{1.5em}{0.4pt} & (1,0)
\end{array}
$$

where the faces have opposite orientations.

21.3 Example. Now try $\Delta[1] \times \Delta[2]$. You have to check that every nondegenerate simplex can be extended to one of dimension three. This is kind of obvious: a simplex is just a weakly increasing path. So a smaller one either doesn't start at $(0, 0)$ or doesn't end at $(2, 1)$. There are three non-degenerate 3-simplices, which are the three

north-east paths.

$$(0,1) \text{——} (1,1) \text{——} (2,1)$$
$$| \qquad\qquad | \qquad\qquad |$$
$$(0,0) \text{——} (1,0) \text{——} (2,0)$$
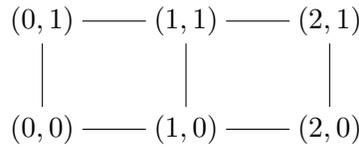
Draw triangles $(0,0), (1,0), (2,0)$, and another one for the top row of this diagram. Connect them to make a triangular prism. Drawing the northeast paths (nondegenerate simplices) in this way cuts the prism into three pieces.

Suppose we have $\Delta[k] \times \Delta[\lambda]$. The maximal non-degenerate simplices are of dimension $k + \lambda$, and correspond to all strictly-increasing paths of length $k + \lambda$, starting at $(0,0)$ and ending at (k, λ). (Basically, these are all the north-east walks, etc.) What is a good way to write this? These are in 1-1 correspondence with what are called (k, λ)-shuffle permutations. Imagine you had a deck with $k + \lambda$, cards, which you cut into a pile of k cards and a pile of λ cards, and then you shuffle them. We're interested in permutations of $\{1 \cdots k, k+1 \cdots k+\lambda\}$ that preserve the order of the first k and last λ.
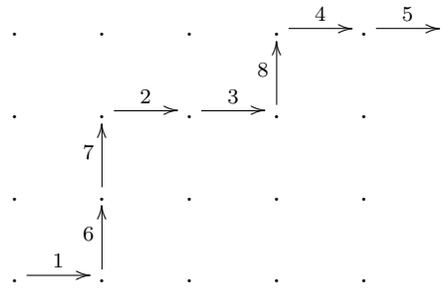
The horizontal elements are the first k items, and the vertical elements are the last λ. So this permutation is $[1, 6, 7, 2, 3, 8, 4, 5]$

21.4 Definition. A *simplicial homotopy* is a map $X_\bullet \times \Delta[1] \to Y_\bullet$.

Remark. After geometric realization, a simplicial homotopy becomes a homotopy.

This just isn't that much data (there aren't that many non-degenerate simplices), and it is very easy to work with. But there is a problem: simplicial homotopy is not an equivalence relation.

If we're looking at maps $X_\bullet \times \Delta[1] \to Y_\bullet$, we might as well talk about maps $\Delta[1] \to Y_\bullet^{X_\bullet}$.

21.5 Definition (Simplicial function space)**.** This consists of simplicial sets $(Y_\bullet^{X_\bullet})$, where $(Y_\bullet^{X_\bullet})_n$ are maps $sSet(X_\bullet \times \Delta[n]_\bullet, Y_\bullet)$ where simplicial set maps have the property that $sSet(K_\bullet, Y^X)$ is the same as the simplicial maps $sSet(K \times X, Y)$.

If we just have the simplicial set $0 \to 1$, $0 \sim 1$ but $1 \not\sim 0$. We also have the simplicial

set:

$$
\begin{array}{ccc}
 & & 2 \\
 & & \uparrow \\
0 & \longrightarrow & 1
\end{array}
$$
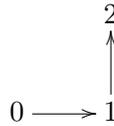
which is a part of $\Delta[2]$. There is a homotopy from 0 to 1 and from 1 to 2, but not from 0 to 2. So you need an extra condition to make homotopy work:

21.6 Definition (Kan extension condition)**.** Define another simplicial set $\partial\Delta[n]_\bullet$ as the union of all codimension 1 faces. But we need a definition *as a simplicial set*. Define $(\partial\Delta[n])_k$ as all the maps $[k] \to [n]$ which are not surjective. For example, the boundary of the standard n-simplex has simplices $0 - 1$, $0 - 2$, $1 - 2$, as well as some degenerate simplices, but not $0 - 1 - 2$.

For example, consider the k-horn $\Lambda_k[n] \subset \delta\Delta[n]$ which is the union of all faces containing the vertex k. For the 2-simplex, there is a 0-horn, which is the union of the simplices $0 - 1$ and $0 - 2$.

21.7 Definition. A simplicial set X is a Kan-complex if it satisfies the Kan extension condition: for every map $\Lambda_k[n] \to X$ there is an extension $\Delta[n] \to X_\bullet$:

$$
\begin{array}{ccc}
\Lambda_k[n] & \overset{\forall}{\longrightarrow} & X_\bullet \\
\Big\uparrow & \nearrow & \\
\Delta[n] & &
\end{array}
$$
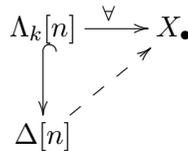
You can say this as "every horn has a filler." (That is, you want to think about filling in the 0 horn $0 - 1 \cup 0 - 2$.)

21.8 Proposition. *(1) If X is a Kan complex, so is X^K for any K.*

(2) Homotopy is an equivalence relation on $sSet(K, X)$ when X is a Kan complex.

[Look at Curtis' paper "Simplicial homotopy theory" in *Advances in Math*. But, almost all the proofs have an incorrect detail. Another good reference is Hovey's book *Model Categories*.]

21.9 Example. If X is a space, $Sing(X)$ is a Kan complex. Why? We want:

$$
\begin{array}{ccc}
\Lambda_k[n] & \overset{\forall}{\longrightarrow} & X_\bullet \\
\Big\uparrow & {}^{?}\nearrow & \\
\Delta[n] & &
\end{array}
$$
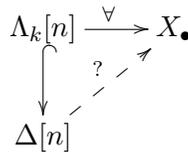
This is the same as asking for this diagram.

$$
\begin{array}{ccc}
|\Lambda_k[n]| & \overset{\forall}{\longrightarrow} & |X_\bullet| \\
\Big\uparrow & \nearrow & \\
|\Delta[n]| & &
\end{array}
$$
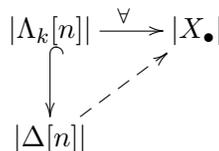
After geometric realization the horn is a retract of the existing complex.

Let's prove the second part of the proposition, assuming the first part:

Proof. Homotopies of maps $K \to X$ is the same as homotopies of maps from $\Delta[0] \to X^K$. If $K \to X$ is Kan, then so is the other map. Map the zero-horn of the two-simplex to the 1-simplex by killing $0 - 2$. The map $1 - 2$ is a homotopy in the other direction. $\mathscr{Q.E.D.}$

Lecture 22
3/23/2011

So, we're going to keep talking about simplicial sets for another couple of lectures. There is a lot of material we could present, and I am trying to strike a balance. I want to introduce this purely algebraic model of rational homotopy theory, and the easiest way is to go through simplicial sets. That's one thing. From that point of view, it's just the model structure on simplicial sets and the Quillen equivalence that is important. Then again, if I just make a geodesic to that result, we don't get a feel for the material. All right, so I need to strike a balance. I've set up enough that I could discuss the model structure, but I want to pause on that today and to talk about life is like when you live in the homotopy category of simplicial sets. It's a pretty good world.

Last time, we introduced the notion of a *Kan complex.* This is a simplicial set where every horn has a filler. That is, every map $\vee_k[n] \to K$ extends $\Delta[n] \to K$. These are going to turn out to be the fibrant objects in this model structure. They're good for receiving maps. All the things we want to do in homotopy theory should work out well for Kan complexes.

Let's try to define the *homotopy groups* of K for K a Kan complex with a chosen basepoint (0-simplex $* \in K_0$, which gives a basepoint $*$ in K_n for each n). That should be the set of base-point preserving homotopy classes of maps

$$\Delta[m]/\partial\Delta[m] \to K,$$

or, alternatively,

$$\pi_m(K)$$

is the set of maps $\Delta[m] \to K$ sending the boundary to $*$ modulo some equivalence relation that we have to construct. So this is equivalently the set of n-simplices in K_n all of whose faces are the basepoint. The equivalence relation is also kind of cool. We could do this modulo homotopy. That is, a map

$$\Delta[n] \to \Delta[1] \to K$$

that restricts on the right places to the two maps $\Delta[n] \rightrightarrows X$, and which sends $\partial \Delta[n] \times \Delta[1] \to *$.

We can work out the condition in a simple case. For instance, let's work out what homotopy of 2-simplices means. We can draw $\Delta[2] \times \Delta[1]$ and its 3 nondegenerate 3-simplices h_0, h_1, h_2. We can say that $x, y \in K_2$ are equivalent if the zero-face of h_0 is y, we have equations

$$\partial_0 h_0 = y, \ \partial_1 h_0 = \partial_1 h_1, \quad \partial_2 h_1 = \partial_2 h_2, \partial_3 h_2 = x.$$

There are further conditions taht the boundaries are basepoints. In a combinatorial way, you can define a simplicial homotopy in this way. This is purely combinatorial for a homotopy group, as long as you have lots of Kan complexes.

The thing is, $\Delta[1]$ was supposed to be some kind of a cylinder object, at least in the category of pointed simplicial sets but we could have picked a smaller one. We could take $\Delta[2]$ modulo the zero-skeleton, which receives a map from two copies of $\Delta[1]$. This turns out to be an acyclic fibration of pointed simplicial sets. So we can get a definition of π_n consisting of all K_n whose faces are at the basepoint modulo the relation $x \sim y$ if there is $h \in K_{n+1}$ such that $\partial_0 h = x, \partial_1 h = y$ and all the other faces of h are at the basepoint. This is even easier to write down.

There is a strictly combinatorial way of relating these different approaches.

We will show: **SSet** is a model category where:

1. Cofibrations are monomorphisms.

2. Fibrations are *Kan fibrations*

3. Weak equivalences will be worked out later.

So everything is cofibrant, and the Kan complexes are the fibrant things.

Suppose that K is a simplicial abelian group (i.e. each K_n is an abelian group and all the boundary and face maps are group-homomorphisms). There is an easy exercise that K is a Kan complex.

What are the homotopy groups of K? This is the quotient of the set of all $x \in K_n$ all of whose faces are zero (zero being the basepoint), modulo the equivalence relation that $x \sim y$ if and only if there exists a h as above. In abelian groups, we just need to know when $x \sim 0$. So we can say that $x \sim 0$ if and only if there is $h \in K_{n+1}$ such that $\partial_0 h = x$ and all other faces of h are zero.

Now from K, we get a chain complex NK such that $NK_n = \{x \in K_n : \text{all but the zeroth face is zero}\}$.

The differential is given by ∂_0. You have to check that $\partial_0^2 = 0$, but this is clear: $\partial_0\partial_0 = \partial_0\partial_1$. We learn that $\pi_n(K)$ is the nth homology $H_n(NK)$.

As long as we're here, let's say a few words about simplicial abelian groups. The functor from simplicial abelian groups to chain complexes $K \mapsto NK$ is an equivalence of categories. In fact, it has a nice right-adjoint: if you want to make a simplicial abelian group from a chain complex C, we define $\mathrm{Sing}C$ in dimension n to have chain complex maps $N(\mathbb{Z}\Delta[n]) \to C$. We do know some interesting things about simplicial abelian groups. Let's do an example for instance. Let's let K be the free abelian group on $\Delta[n]$ modulo $\mathbb{Z}[\partial\Delta[n]]$. This is zero up to $n-1$, and there is a single nondegenerate n-simplex. In dimension $n+k$, we have the set of surjective maps $[n+k] \to n$.

Let's look at this in a low dimension. When $n = 2$, the complex looks like

$$0 \quad 0 \quad \mathbb{Z}[\{0,1,2\}] \quad \mathbb{Z}[\{0,0,1,2\}, \{0,1,1,2\}, \{0,1,2,2\}] \quad \ldots.$$

Now let's understand the kernel of d_1; that corresponds to omitting the element in position 1. One can check that the normalized chain complex on $\mathbb{Z}[\Delta[n]]/\partial\Delta[n]$ are just $\mathbb{Z}$ in degree n and zero everywhere else. This thing is thus an Eilenberg-MacLane space $K(\mathbb{Z}, n)$. By far, this is the easiest way of getting your hands on an Eilenberg-MacLane space.

Let's talk about the group structure on π_n. Where does it come from? If we have two maps $\Delta[1] \to K$, we can piece them together into a map $\vee_1[2] \to K$, and we can restrict that to the other 1-face to get a third map $\Delta[1] \to K$. This composite is the composition in the fundamental group.

Here's a cool theorem. Let F be a simplicial group (not necessarily commutative). The same argument applies. We learn that $\pi_n(F)$ is the intersection of the kernel of the ∂_i modulo the same equivalence relation. This leads to a description of NF as before, $\pi_n(F)$ becomes the "homology."

Suppose X is a simplicial set. Then we can form FX such that in dimension n, it is the free group on X_n. There is a theorem, due to Milnor, which states that the geometric realization of FX has the homotopy type of the loops on the suspension of $|X|$. Thus you can get a combinatorial formula for the homotopy groups of spheres, but it's not easy to use. There's a fantastic thing where you can filter a group by the lower central series and take successive quotients. The associated graded is a bunch of simplicial abelian groups.

Lecture 23
March 25, 2011

We were still talking about simpicial sets. Let's talk about the skeleton of a simplicial set. Suppose X is a simplicial set.

Suppose $x \in X_n$ is an n-simplex, which we could also think of as a map $\Delta[n] \to X_\bullet$. For example suppose x is $\Delta[1]/\partial\Delta[1]$; this is just a loop, and in this situation $d_0 x = d_1 x$. Break up the simplices into the set of degenerate simplices and nondegenerate simplices. Recall that x is degenerate if there is a degeneracy map $\varphi : [n] \to [p]$ where $p < n$ and $y \in X_p$ such that $\varphi^* y = x$. Every map can be factored as an injective map followed by a surjective map. The surjective maps are degeneracies. So we can assume that φ is surjective. We can always factor any surjective φ as a series of surjective maps

$$[n] \twoheadrightarrow [n-1] \twoheadrightarrow \cdots \twoheadrightarrow [p]$$

(Basically, identify things one at a time.) The degenerate simplices can be expressed as $\cup_{i=0}^{n-1} Im(S_i : X_{n-1} \to X_n)$, where the dual S^i is the map that repeats the ith index.

23.1 Lemma. *Let X be a simplicial set. Suppose $x \in X_n$ is a non-degenerate simplex. There can still be relations among the faces, but the degeneracies of this simplex are all distinct. More precisely, if $s, s' : [m] \twoheadrightarrow [n]$ are surjective, and $s(x) = s'(x)$ then $s = s'$.*

Proof. Not hard, but annoying to write down. For simplicity assume that $x \in X_2$ is non-degenerate. Suppose $s, s' : [3] \to [2]$, where $s = [0, 0, 1, 2]$ and $s' = s^1 = [0, 1, 1, 2]$. For the sake of contradiction assume $s^0 x = s^1 x$. Find some index (here index zero) where removing this entry from s' produces a degenerate simplex (here $[1, 1, 2]$), but removing it from s produces a non-degenerate simplex (here $[0, 1, 2]$). So $d_2 s_0 x = d_2 s_1 x$ but $d_2 s_0 x = s_0 d_1 x$. This contradicts x being non-degenerate. $\mathcal{Q.E.D.}$

Let $X_\bullet$ be a simplicial complex and $Y_\bullet \subset X_\bullet$ be a simplicial subset: i.e., $Y_n \subset X_n$ for each n. Suppose $x \in X_n$ is a nondegenerate simplex such that $d_i x \in Y_{n-1}$ for all $i \leq n$. Construct the pushout:

23.2 Proposition. *D is an inclusion.*

Proof. In dimension m we have

$$
\begin{array}{ccc}
(\delta\Delta[n])_m & \hookrightarrow & (\Delta[n])_m \\
\downarrow & & \downarrow \\
Y_m & \longrightarrow & Y'_m
\end{array}
$$

For $m < n$, $Y_m = Y'_m$, and we're done. If $m = n$ then $Y'_n = Y_n \sqcup \{x\}$, and this is OK. For $m > n$, let i_n be the simplex $[0, \cdots, n]$, which is the unique non-degenerate simplex of $\Delta[n] \to X$. So $\Delta[n]_m$ is the degeneracies of i_n, disjoint union the degeneracies of the faces of i_n. So

$$
Y'_m = Y_m \sqcup \text{the degeneracies of } i_n
$$

Y_m is just a subset, and we just proved that there are no relations in the map sending the degeneracies into X_m. $\mathcal{Q.E.D.}$

§1 Skeleton Filtrations

Define the zero-skeleton $Sk^0 X$ as just the zero-simplices $X_0 \subset X$ regarded as a constant simplicial set. Equivalently, this is the zero-simplices and all of their degeneracies. Define this as a pushout:

$$
\begin{array}{ccc}
\sqcup \partial\Delta[n]_\bullet & \longrightarrow & \sqcup \Delta[n]_\bullet \\
\downarrow & & \downarrow \\
Sk^{n-1}X & \hookrightarrow & Sk^n X
\end{array}
$$

where the disjoint union is over all nondegenerate $x \in X_n$. In the left corner we have all the degenerate n-simplices, and in the right corner we have all the non-degenerate n-simplices. So note that $(Sk^n X)_j = X_j$ for $j \leq n$.

The geometric realization commuted with disjoint unions, so we get a pushout diagram:

$$
\begin{array}{ccc}
\sqcup \partial\Delta[n] & \longrightarrow & \sqcup \Delta[n] \\
\downarrow & & \downarrow \\
|Sk^{n-1}X| & \longrightarrow & |Sk^n X_\bullet|
\end{array}
$$

where $|X_\bullet| = \varinjlim_n |Sk^n X_\bullet|$. (Basically we are just building CW complexes.)

23.3 Corollary. *$|X|$ is a CW complex with one n-cell for every non-degenerate n-simplex. (So basically you only have to draw the non-degenerate simplices.)*

$$
C_n^{cell}|X| = \mathbb{Z}\{\text{non-degenerate } n\text{-simplices}\}
$$

We would like to find a cell map to $C_{n-1}^{cell}|X|$.

Now talk about simplicial abelian groups and chain complexes. Let $C_\bullet$ be a simplicial abelian group. We had introduced $(NC)_n = \cap_{i=1}^n ker(d_i : C_n \to C_{n-1})$. This made a chain complex with differential d_0. Last class, we said that there is an equivalence of categories between simplicial abelian groups and chain complexes. Equivalently, $(N'C)_n = C_n/\cup_{i=1}^{n-1} Im(S_i)$. In other words, this is C_n, modulo the subgroup generated by the degenerate simplices. Define a differential on this

$$(N'C)_n \overset{d}{\to} (N'C)_{n-1} \text{ where } d = \sum(-1)^i d_i$$

23.4 Proposition. *$NC \to N'C$ is an isomorphism of chain complexes.*

$(NC)_1 = ker(C_1 \overset{d_1}{\to} C_0)$ and $(N'C)_1 = coker(C_0 \overset{s_0}{\to} C_1)$. Because $d_1 s_0 = Id$ the sequence splits and we can write $C_1 = ker(d_1) \oplus Im(s_0)$.

These are both called the normalized chain complex. There is a third complex C_n where $d = \sum(-1)^i d_i$, where

$$
\begin{array}{c}
C_n \\
\uparrow \quad \searrow^{f} \\
NC \overset{\cong}{\longrightarrow} N'C
\end{array}
$$

We claim that f is a homology isomorphism.

23.5 Proposition. $C_*^{cell}|X| = N'\mathbb{Z}X$

23.6 Corollary. *There is a canonical map $|Sing(X)| \to X$. This is an isomorphism in homology. In fact, it is a weak homotopy equivalence. In fact, it is a functorial equivalence.*

The homology of X is what you get by taking the free abelian group on $Sing(X)$, and taking alternating sums of face maps. We've almost already proved this; you just have to check it is the right map.

23.7 Corollary (of the construction)**.** *For every chain $Z \in C_n^{Sing}(X)$ there is a CW complex A and an n-chain $Z' \in C_n^{Sing}(A)$ and a map $A \to X$ such that $Z' \to Z$.*

23.8 Corocorollary (of the previous corollary)**.** *A weak homotopy equivalence of arbitrary spaces induces an isomorphism in homology.*

This tells us how to define the homology of a simplicial set. Let $X_\bullet$ be a simplicial set.

23.9 Definition. $H_* X = \pi_* \mathbb{Z}\{X_\bullet\}$

If I had given a different definition, this would be the Dold-Kan Theorem. The LHS only makes sense in spaces, but the RHS is categorical: all the information is contained in $\mathbb{Z}\{X_\bullet\}$. Quillen used this to define homology on an object to be the free abelian group on that object. Every model category thus has an intrinsic notion of homology.

Lecture 24
March 28, 2011

So we've been talking about simplicial sets for a few lectures. We're probably not going to do a lot with them. We want to finish this model of rational homotopy theory.

Let us state without proof a theorem about simplicial sets, and then return to model categories.

We talked about this notion of a **Kan complex,** which is a simplicial set X with the property athat every horn $\vee_k[n] \to X$ has a filler $\Delta[n] \to X$. There is a related notion of a **Kan fibration.**

24.1 Definition. A map $X \to Y$ is a **Kan fibration** if for every diagram

$$
\begin{array}{ccc}
\vee_k[n] & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[n] & \longrightarrow & Y
\end{array}
$$

has lifting $\Delta[n] \to X$ as in the dotted arrow. That is, p has the RLP with respect to all inclusions $\vee_k[n] \to \Delta[n]$.

24.2 Theorem. SSet *forms a model category when equipped with the following structure:*

1. *The cofibrations are the monomorphisms.*

2. *The fibrations are the Kan fibrations.*

3. *The weak equivalences are forced.*

This is called the **Kan model structure.** It is not the only one. It is compactly generated, and the set $\mathcal{A} = \{\vee_k[n] \to \Delta[n]\}$ while $\mathcal{B} = \{\partial\Delta[n] \to \Delta[n]\}$. It turns out that the fibrations have the RLP with respect to $\mathcal{A}$ and the acyclic ones have the RLP with respect to $\mathcal{B}$; this is in fact, by the skeleton thing we talked about last time, equivalent to having the RLP with respect to all monomorphisms. (Indeed, any monomorphism is a transfinite composition of push-outs of things in $\mathcal{B}$; if we have an inclusion $A_\bullet \subset X_\bullet$, we can form a transfinite sequence $A_\bullet = X(0)_\bullet \subset X(1)_\bullet \subset \ldots$ whose colimit is X and such that each inclusion $X(i)_\bullet \hookrightarrow X(i+1)_\bullet$ is a push-out of a coproduct of maps $\partial\Delta[i-1] \to \Delta[i]$.)

The hard part is to say what the weak equivalences are. There are three different ways of doing so, each of which leads to a different proof.

1. A map $A \to B$ is a weak equivalence iff for all Kan complexes X, the set of simplicial homotopy classes $[B, X] \to [A, X]$ is a bijection. (We restrict to Kan complexes because then homotopy is an equivalence relation.) This is a very robust way of introducing the model structure. (If you introduce a different filling structure, you can get another model structure. If you weaken the Kan condition so that only *inner horns* $\vee_i[n] \to \Delta[n]$ (so $0 < i < n$) have fillers, then you get a notion of *quasicategory*; Joyal showed that if Kan complexes are replaced by quasicategories, then a new model structure emerges. One lesson that people learned after defining model categories was that they are really useful in a lot of different places.)

2. We want to say that a map $A \to B$ is a weak equivalence iff the map of geometric realizations is a weak equivalence. This is only natural if we want the Quillen equivalence.

3. We already showed that the *homology* of an arbitrary simplicial set could be calculated efficiently. There is another approach. A map of topological spaces that are simply connected is a weak equivalence iff it is an isomorphism in homology. Motivated by this, we can define π_1 of a simplicial set, and define a map to be a weak equivalence iff it is an isomorphism on π_1 (and π_0) and an isomorphism on homology with all local coefficient systems.

24.3 Definition. The **fundamental groupoid** of A has as objects the zero-simplices of A and as morphisms generated by the 1-simplices modulo the relation that going around a 2-simplex gives the identity. A **local system** is a functor from the fundamental groupoid of A to the category of abelian groups.

24.4 Theorem. *The pair of adjoint functors*

$$\mathbf{SSet} \to \mathbf{Top}$$

given by the geometric realization and the singular simplicial set functors form a Quillen equivalence.

This is an important theorem, and was really one of the motivating examples for a model category. Everyone knew that you could do homotopy theory in simplicial sets. We won't prove these theorems. They are rather difficult to prove.

There is a deep theorem of Quillen (the SA theorem):

Let $\mathcal{C}$ be a category of algebras of some kind. It is supposed to be cocomplete and complete. There is a set of small generators, or even with a single small[2] (or compact) generator x. (This means that if $y \in \mathcal{C}$, there is an effective epimorphism from a coproduct of copies of x to y.) For instance, $\mathcal{C}$ might be groups (x a free group), $\mathcal{C}$ might be commutative rings (x a polynomial ring $\mathbb{Z}[t]$).

---

[2]Homming out of x commutes with filtered colimits.

Given a generator x, you get a functor $\mathcal{C} \to \mathbf{Sets}$ sending $y \mapsto \mathrm{Hom}(x, y)$. This is to be thought of as the "underlying set" or *forgetful* functor. In the above examples, this is the forgetful functor. It is faithful because x is a generator. This is the abstract way of having an "underlying set" functor. There is a

24.5 Theorem. *Simplicial objects in $\mathcal{C}$ have a model structure in which "fibration" and "weak equivalences" are created by the forgetful functor to simplicial sets.*

Thus the forgetful functor "creates" the model structure. When the forgetful functor takes values in the category of groups, then everything in simplicial $\mathcal{C}$ is fibrant.

Lecture 25
March 30, 2011

We got on this model category story because I wanted to tell you about a model for rational homotopy theory.

§1 $\mathbb{Q}$-homotopy theory of spaces

25.1 Theorem. *The category of spaces is a model category when equipped with the following structure:*

- *Weak equivalences: rational homology isomorphisms, as well as the weak equivalences from the Serre model structure;*

- *Cofibrations: retracts of cellular maps (what we had before, as well);*

- *Fibrations: forced.*

This is an example of making a new category out of an old one. let C be a model category, and S be a collection of maps in C. You would like to form a new model structure category $S^{-1}C$ on the same category, where the new cofibrations are the old fibrations, and the elements in S become weak equivalences. That is, you want it to satisfy a universal property, where if F is a Quillen functor and Ff is a weak equivalence in D for $f \in S$, we want a universal model category $S^{-1}C$ such that there is a map τ:

$$
\begin{array}{ccc}
C & \xrightarrow{\ F\ } & D \\
\big\downarrow & \nearrow\!\!\!\nearrow^{\tau} & \\
S^{-1}C & &
\end{array}
$$

There is a general theory, in which you can guarantee this exists given some logic criteria. If you're specifying a set S, you might as well assume that you have cofibrations between cofibrant objects, because you're just getting the old ones.

Notation: call the new weak equivalences $\mathbb{Q}$-weak equivalences, and denote them by $\overset{\Sigma_{\mathbb{Q}}}{\to}$; the $\mathbb{Q}$-fibrations will be denoted by $\overset{\mathbb{Q}}{\twoheadrightarrow}$. The cofibrations are the old ones, and hence so are the acyclic fibrations. However, there are more weak equivalences, and more acyclic cofibrations; equivalently there are fewer fibrations. (It is harder for things to be fibrant.) For example, take a space $K(\pi, n)$ for $n \geq 2$. Everything is fibrant in the Serre model structure; but in this structure what happens? The degree-p map $S^n \overset{p}{\to} S^n$ is a $\mathbb{Q}$-equivalence. Also the map $\pi_n K(\pi, n) \to \pi_n K(\pi, n)$ has to be multiplication by p. So $\pi = p_n K(\pi, n)$ is a $\mathbb{Q}$-vector space. A map that is a weak homology equivalence is a homology isomorphism, i.e. a map into an Eilenberg-Maclane space. You can show that $K(\pi, n)$ is fibrant iff π is a $\mathbb{Q}$-vector space. More generally, $K(\pi, n) \to K(\pi \otimes \mathbb{Q}, n)$ is a fibrant replacement for $n \geq 2$. It is an isomorphism of homotopy groups mod Serre class of p-torsion. By the mod C Serre theorem, it is an isomorphism in homology...

You want to find some generators for the model structure. We will introduce two classes of maps: generating acyclic cofibrations are

$$A = \{V \hookrightarrow D : \ \mathbb{Q}\text{-equivalence cofibration }\}$$

V is cellular, it has at most countably many cells, $V \hookrightarrow D$ is cellular (it is made by attaching cells, maybe not in order), and D has at most countably many cells. The cofibrations are the same, so a generating set is

$$B = \{S^{n-1} \hookrightarrow D^n : n \geq 0\}$$

25.2 Proposition (9). *The sets A and B generate. That is, a map $f : X \to Y$ is a $\mathbb{Q}$-fibration iff f has the RLP for all maps in A, and it is a $\mathbb{Q}$-acyclic fibration iff it has the RLP for B.*

$\mathbb{Q}$-acyclic fibrations are the same as cofibrations; so we have already shown the second statement. So what is left is to show that the RLP for A is the same as having a $\mathbb{Q}$-fibration. Any $\mathbb{Q}$-fibration by definition has the RLP for all maps, not just A. We need to show that any map that has the RLP for A has the RLP for all $\mathbb{Q}$-acyclic cofibrations. We need a way to reach every $\mathbb{Q}$-acyclic cofibration from the ones in A. We need:

25.3 Proposition ("$9\frac{3}{4}$"). *Suppose B is cellular (in any haphazard order), and $A \subset B$ is a subcomplex. Suppose that $H_*(B, A; \mathbb{Q}) = 0$. (This is almost a random acyclic cofibration, except that A is a cell complex.) Given any $B_0 \subset B$ with at most countably many cells, there exists some K with countably many cells, where $B \supset K \supset B_0$ and $H_*(KK \cap A; \mathbb{Q}) = 0$. (Every cellular pair with vanishing rational homology can be written as a filtered colimit of pairs in A.)*

Proof. $H_*(_0, B_0 \cap A; \mathbb{Q})$ is a countable rational vector space. Choose a basis $\{e_1, e_2, \cdots\}$. Now, $0 = H_*(B, A; \mathbb{Q}) = \varinjlim_{L \subset B \text{ countable}} H_*(L, L \cap A; \mathbb{Q})$. At some point, e_1 has to go to zero. Choose a countable L_1 such that $e_1 \mapsto 0 \in H_*(L_1, L_1 \cap A)$. Continue, choosing a countable $L_2 \supset L_1$ such that $e_2 \mapsto 0 \in H_*(L_2, L_2 \cap A; \mathbb{Q})$. So we end up with a filtration $L_1 \subset L_@ \subset L_3 \subset \cdots$ where $e_n \mapsto 0 \in H_*(L_n, L_n \cap A)$. Let

$$K_1 = \cup_i^\infty L_i$$

This is countable, because everything in it was. So $B_0 \subset K_1$ where $H_*(B_0, B_0 \cap A) \xrightarrow{0} B_*(K_1, K_1 \cap A)$ where all $e_i \mapsto 0$.

Repeat this process, so you get $B_0 \subset K_1 \subset \cdots$. K_n is countable, so $H_*(K_{n-1}, K_{n-1}) \to H_*(K_n, K_n \cap A)$ is the zero map. Finally, set $K = \cup K_n$. Then

$$H_*(K, K \cap A) = \varinjlim H_*(K_n, K_n \cap A) = 0$$

You could do this for homology theories other than over $\mathbb{Q}$, where countability is replaced by the cardinality of the coefficient ring, or something. What is important is that you're writing arbitrary things as a colimit of things of bounded size. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$.

25.4 Proposition ("$9\frac{1}{2}$"). *If $X \to Y$ has RLP for A then it has RLP for all maps $A \hookrightarrow B$, where B is cellular, $A \subset B$ is a subcomplex, and the map is an isomorphism in rational homology.*

Proof. This is a Zorn's Lemma problem. Consider the set S of:

where there is a lift. Suppose we're in the situation

where there are lifts. S is a partially ordered set, where every chain in S has a maximal element.

By Santa Claus [Zorn], there is a maximal element

We claim that $B' = B$. Suppose not. Then we can find a cell $e \subset B - B'$ with $\delta e \subset B'$ (say, of minimal dimension). Then $B_0 \subset B$ is a countable subcomplex containing e. (The notation here is not going to match Proposition "9". B_0 here, is A there.) By proposition $9\frac{3}{4}$ there is a countable $K \subset B$ containing b_0 with $H_*(K, K \cap B') = 0$. Call the pushout B'':

$$
\begin{array}{ccccc}
K \cap B' & \hookrightarrow & B' & \longrightarrow & X \\
\uparrow & & \uparrow & & \downarrow \\
K & \hookrightarrow & B'' & \longrightarrow & Y
\end{array}
$$

The map $K \cap B' \to K$ is in A. Since it is a pushout, you have a lift $B'' \to X$. This contradicts the maximality of B'. Therefore, $B' = B$. We're almost done.

$$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$$

Lecture 27
April 4, 2011

The model structure on spaces turns out to extend quite easily to simplicial sets.

27.1 Definition. $sSets^{\mathbb{Q}}$ is a model category, where the objects are simplicial sets and

- the cofibrations are monomorphisms;

- the weak equivalences are isomorphisms in rational homology;

- the fibrations are maps with the RLP w.r.t. acyclic cofibrations.

Take the generators to be

$$A = \{K_\bullet \subset L_\bullet \ : \ L \text{ is countable and } H_*(-;\mathbb{Q}) \text{ is an isomorphism }\}$$

$$B = \{\delta\Delta[n] \subset \Delta[n] \ : \ n \geq 0\}$$

27.2 Theorem. *$sSets^{\mathbb{Q}}$ is a model category with A and B as generators.*

27.3 Theorem.
$$| \ | : sSets^{\mathbb{Q}} \leftrightarrows \tau^{\mathbb{Q}} : Sing$$

form a Quillen equivalence, where $\tau^{\mathbb{Q}}$ is spaces, with the model structure above.

§1   Commutative Differential Graded Algebras over $\mathbb{Q}$

Call this category **DGA**. We will start out working purely algebraically. But this will be related to the previous things by a *contravariant* functor. In other words, we will get a model for the cochains of a space.

An object in this category looks like

$$A_\bullet = \bigoplus_{n \geq 0} A_n$$

with multiplication given by $x \cdot y = (-1)^{|x||y|} y \cdot x$, where $|x|$ is the dimension of x (i.e. $x \in A_n$). If you forget the differential and the ring, this is a cochain complex. There is also a differential that satisfies Leibniz' rule

$$d(xy) = dx \cdot y + (-1)^{|x|} x \cdot dy$$

First we will give it a model category structure. There is a forgetful functor

$$\mathbf{DGA} \xrightarrow{F} \text{Cochain complexes over } \mathbb{Q}$$

Going backwards, you can always give a cochain complex the symmetric algebra, so there is a map backwards. These form a pair of adjoint functors.

27.4 Example. Suppose you have the cochain complex $D(n)^*$ where you have $d : \mathbb{Q} \to \mathbb{Q}$ in dimension $n - 1 \to n$. Let $y = dx$. Because we want this to be graded commutative, take the tensor algebra over $\mathbb{Q}$ on the two generators x, y, modulo some equivalence relation: we want

$$x \cdot y = (-1)^{n(n-1)} yx = yx$$
$$x \cdot x = (-1)^{|x|} x \cdot x$$
$$y \cdot y = (-1)^{|y|} y \cdot y$$

which implies $x^2 = 0$ when its dimension is odd, and $y^2 = 0$ when its dimension is odd. Write $\mathbb{Q}[x, y]$ for the free graded commutative algebra on x and y. More generally, if $\{x_\alpha\}$ is a homogeneous basis for a graded vector space, write $\mathbb{Q}[x_\alpha]$ for the free graded commutative algebra on these generators. In other words, we have $\mathbb{Q}[X_\alpha] = P[x_\alpha : |x| = even]$, and $\mathbb{Q}[X_\alpha] = E[x_\beta : |x| = odd]$. So $Sym(D(n))^* = \mathbb{Q}[x, dx]$ where $|x| = n - 1$.

Here is a model category structure on cochains: the fibrations are surjective, the weak equivalences are cohomology isomorphism, and the cofibrations have the RLP w.r.t. acyclic fibrations. The generators are:

$$A = \{0 \to D(n) : n \geq 1\}$$

$$
\begin{array}{ccc}
0 & \longrightarrow & X_{\bullet} \\
\downarrow & \nearrow & \downarrow \\
\mathbb{Q} \overset{n-1}{\to} \mathbb{Q} & \longrightarrow & Y_{\bullet}
\end{array}
$$

To say I have lifting of degree $n-1$ just means that $X \to Y$ is surjective. Also declare

$$B = \{S(n)^* \hookrightarrow D(n)^*\} \cup \{\mathbb{Q} \to 0\} \cup A$$

where

$$
S(n-1)^* = \begin{cases} \mathbb{Q} & * = n-1 \\ 0 & \text{else} \end{cases}
$$

We declare $S(-1)^* = 0$ and $D(0)^*$ is $\mathbb{Q}$ at zero and zero elsewhere.

27.5 Theorem. *The functor F creates a model category structure on DGA, where $X \to Y$ is a fibration on weak equivalences iff $FX \to FY$ is one too. The generators are*

$$A = \{sym(\text{generating acyclic cofibrations})\}$$

$$B = \{sym(\text{generating cofibrations})\}$$

A map of underlying cochain complexes is either surjective or an isomorphism in cohomology. Check that a cobase change along one of these is a weak equivalence.

Suppose n is odd. Take $\mathbb{Q}[x_n]$, where x_n is in dimension n. Everything is fibrant in this category; the claim is that this is cofibrant. This is because it is $sym(S(n)^*)$, and $S(n)^*$ was cofibrant. Form the pushout

$$
\begin{array}{ccc}
S(n-1)^* & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
D(n) & \longrightarrow & S(n-1)
\end{array}
$$

This is a cobase change of a generating cofibration. $S(n-1)$ is cofibrant. Any free graded commutative algebra on non-negative-degree generators, with $d=0$ is cofibrant. We'd like to associate a space X to the graded algebra $H^*(X; \mathbb{Q})$, or some DGA with this cohomology. Take an even sphere S^{2n}. Then $H^*(X; \mathbb{Q}) = \mathbb{Q}[x_{2n}]/x_{2n}^2$. This is the correct answer, but it is not cofibrant. So we need a cofibrant resolution. We know that $\mathbb{Q}[x_{2n}]$ works and it maps onto $\mathbb{Q}[x_{2n}]/(x_{2n})^2$. Extend this to $\mathbb{Q}[x_{2n}, y_{4n-1}] : dy = x^2$. Claim that when you take homology you get $1, x, x-2, x^2, \cdots$ and $\cdots y, xy, x^2$ that cancel. Is this cofibrant?

$\mathbb{Q}[x_{2n}, y_{4n-1}, dy]$ is cofibrant. It is $\mathbb{Q}[x_{2n}] \otimes \sum(D(4n)^*)$. Now add the relation. Map

$$
\begin{array}{ccc}
\mathbb{Q}[x_{4n}] & \overset{0}{\longrightarrow} & \mathbb{Q} \\
{\scriptstyle dy-x^2}\downarrow & & \downarrow \\
\mathbb{Q}[x, y, dy] & \longrightarrow & \mathbb{Q}[x, y \, : \, dy = x^2]
\end{array}
$$

The map on the left is cofibrant so the one on the right is too. When we did homotopy groups of spheres we had

$$
\begin{array}{ccc}
 & & X \\
 & \nearrow & \downarrow \\
S^{2n} \longrightarrow & K(\mathbb{Q}, 2n) \xrightarrow{\ i^2\ } & K(\mathbb{Q}, m)
\end{array}
$$

Look at Serre spectral sequence with $K(\mathbb{Q}, 2n)$, $K(\mathbb{Q}, 4n-1)$, where $y_{4n=1} \to x_{2n}^2$. But the spectral sequence is a differential graded algebra that is the same as the one we had before.

The point of the Postnikov tower is you take a sphere, map to an Eilenberg MacLane space, and then make a Postnikov towers where you try to kill cohomology groups until you get back to the cohomology of the sphere.

<h1 align="center">Lecture 28
April 6, 2011</h1>

There are a few more things to do with this rational homotopy stuff. We are studying this situation where we have spaces and its rational model structure, simplicial sets with their model structure (and these two are Quillen equivalent, by the singular set functor and the geometric realization). We are going to construct a Quillen equivalence:

$$\mathbf{Set}^{\mathbb{Q}} \rightleftarrows \mathbf{DGA}^{op}$$

We have to discuss this, but before that, we'll linger in the homotopy theory of **DGA**. We'd like to look at formulas for the topological and homotopy-theoretic invariants in **DGA**. Later, we'll discuss the Quillen equivalence in detail.

If X is a simplicial set (or a space), we let $A(X)$ be the corresponding differential graded algebra (or the DGA of the associated singular simplicial set). We want to work out what $A(X)$ looks like.

The key thing about $A(X)$ is that the cohomology of $A(X)$ is isomorphic (functorially) to $H^*(X; \mathbb{Q})$ as a ring, which we'll eventually prove, once we've defined A. We want to use this fact to figure out $A(X)$ for a bunch of spaces.

28.1 Example. $X = S^{2n-1}$ is an odd-dimensional sphere. Whatever $A(X)$ is, we know that the cohomology of it is an exterior algebra on one generator. [Recall that in odd dimensions, $x^2 = 0$, so this relation is present, just implicit.] So it is $\mathbb{Q}[x_{2n-1}]$ (with our previous conventions). This class x_{2n-1} must be represented in $A(X)$ by a cocycle, which we'll call x_{2n-1}, by abuse of notation. This gives a map

$$\mathbb{Q}[t_{2n-1}] \to A(X)$$

sending $t_{2n-1} \mapsto x_{2n-1}$. By construction, this is an isomorphism in cohomology, i.e. a weak equivalence. That's great. We have gotten our hands on A of an odd sphere. We know that it must be weakly equivalent to $\mathbb{Q}[t_{2n-1}]$.

28.2 Example. Let's take X to be an even sphere. In this case, the cohomology of $A(X)$ is $\mathbb{Q}[x_{2n}]/x_{2n}^2$. Again, let's choose a cocycle x_{2n} of $A(X)$ representing x_{2n}. We get a map

$$\mathbb{Q}[x_{2n}] \to A(S^{2n}).$$

Now x_{2n}^2 is not necessarily zero in $A(S^{2n})$, but it is zero in cohomology. So there is some y_{4n-1} with $dy_{4n-1} = x_{2n}^2$. So we get a map

$$\mathbb{Q}[x_{2n}, y_{4n-1}]/(dy = x^2) \to A(S^{2n}),$$

which we easily check to be a weak equivalence. [Unlike with the odd sphere, the "commutativity" does not already force $x_{2n}^2 = 0$, so we have to explicitly do something to make this relation happen.]

We can certainly consider a map $\mathbb{Q}[x_{2n}, y_{4n-1}]/(dy = x^2) \to \mathbb{Q}[x_{2n}]/x_{2n}^2$, sending $y \mapsto 0$. This is a weak equivalence. So the $A[S^{2n}]$ and the cohomology ring are weakly equivalent by a chain of weak equivalences. This works when the cohomology ring is a complete intersection. That means it is a free graded commutative ring modulo a regular sequence. There is a whole industry that relates the homotopy theory of spaces to the commutative algebra of the cohomology rings.

28.3 Definition. A space (or simplicial set) X is called **formal** if $A(X)$ is weakly equivalent to a DGA with $d = 0$ (which is necessarily the cohomology ring $H^*(X; \mathbb{Q})$, as it has to have that for the cohomology).

As stated, a space is formal iff its cohomology ring is a complete intersection. This means that the cohomology ring determines the rational homotopy type.

28.4 Theorem (Deligne-Griffiths-Morgan-Sullivan). *A Kähler manifold (e.g. smooth complex projective variety) is formal.*

This theorem has a fairly easy proof.

A space whose $A(X)$ is $\mathbb{Q}[x_3, x_5, y_7]/(dy_z = x_3x_5)$ is not formal. If there are three classes a, b, c and $dx = ab$ is cohomologous to zero, and $dy = bc$ is cohomologous to zero, then we can get a cohomology class $< a, b, c >$ which isn't formal...

Now let's do another example.

28.5 Example. Let $X = K(\mathbb{Q}, n)$; then $H^*(X)$ is a free graded ring on one generator $\mathbb{Q}[\iota_n]$, as we calculated with the spectral sequence. Just by choosing a representing cocycle, we can map $\mathbb{Q}[\iota_n] \sim A(K(\mathbb{Q}, n))$.

In spaces, we are interested in $[X, K(\mathbb{Q}, n)]$, which we want to be related in the homotopy category of
$$\text{Ho}_{\mathbf{DGA}}(AK(\mathbb{Q}, n), A(X)).$$

The other thing that is interesting to us are the rational homotopy groups, so the homotopy classes of maps $S^n \to X$ in the rational homotopy category. This should correspond to homotopy classes of maps in pointed DGAs $A(X) \to A(S^n)$. It would be nice to understand what these things are.

To do this, we are going to have to work out cofibrant and fibrant replacements, and path and cylinder objects. Which means that we have to determine

When is a DGA cofibrant?

Cofibrant means that we can build it out of cells. Since our generators are $S(n)^* = \mathbb{Q}[x_n]$ and $D(n)^* = \mathbb{Q}[x_n, y_{n-1}]/(dy = x)$ and the maps $S(n)^* \to D(n)^*$ generate the cofibrations, then anything that fits into a diagram

$$
\begin{array}{ccc}
\bigotimes S(n) & \longrightarrow & \bigotimes D(n) \\
\downarrow & & \downarrow \\
A_k & \longrightarrow & A_{k+1}
\end{array}
$$

gives a cofibration $A_k \to A_{k+1}$. To get A_{k+1} from A_k, we just have to take

$$A_k[\{y_n\}]/\{dy_n = x_n \in A_k\}.$$

So we add new polynomial generators but make the derivatives in the old ring.

We find:

28.6 Proposition. *A DGA A is cofibrant if there is a filtration $A_0 \subset A_1 \subset \ldots$ by DGAs such that A_{k+1} is A_k adjoined some list of variables with derivatives in A_k.*

Let us now try to make a path object in DGAs. We are supposed to take an algebra A, and consider the diagonal map

$$A \to A \times A,$$

which we need to factor as an acyclic cofibration by a fibration (i.e. surjective weak equivalence). To do this, we adjoin x_0 to A in degree zero, and dx_0 is also adjointed. The map $A[x_0, dx_0] \to A \times A$ sends $x_0 \mapsto (0, 1)$ and $dx_0 \mapsto 0$. The first map

$$A \to A[x_0, dx_0]$$

is the same as

$$A \otimes \mathbb{Q} \to A \otimes D(1).$$

So it is a trivial cofibration. It is easy to see that $A[x_0, dx_0] \to A \times A$ is surjective.

If we have two maps $B \rightrightarrows A$, then a homotopy between the two would be a lift

$$B \to A[x_0, dx_0] \to A \times A.$$

That's only a good notion if B is cofibrant.

28.7 Example. Let us compute homotopy classes of maps $\mathbb{Q}[\iota_n] \to A$. The first is cofibrant, and A is clearly fibrant, so we are good. A map $\mathbb{Q}[\iota_n] \to A$ can go to any cycle. So these maps are in bijection with n-cocycles of A. Now we have two n-cocycles $z_0, z_1 \in A_n$. Let us suppose they are cohomologous by some h. Then we can get a homotopy

$$\mathbb{Q}[\iota_n] \stackrel{\iota_n \mapsto z_0(1-x_0)+z_1 x_0 - h x_0}{\to} A[x_0, dx_0]$$

so two cohomologous cocycles are homotopic; conversely, homotopic cocycles are cohomologous because cohomology is a homotopy invariant.

So maps in the homotopy category $\mathbb{Q}[\iota_n] \to A$ are just $H^n(A)$.

Now let's turn to the question about homotopy groups. For there, look at $\pi_n(X, *)$, a space with a basepoint. An n-sphere has a preferred basepoint, and everything preserves the basepoint. What is a basepoint in algebraland? Since all the arrows are reversed, we see that a basepoint is a ring-homomorphism

$$A \to \mathbb{Q}.$$

(We will find that $A(*) = \mathbb{Q}$.) So a pointed space corresponds to an augmented algebra. There is then a kernel $I \subset A$, called the *augmentation ideal.*

So now we are looking at copointed maps from A to any fibrant model of the n-sphere, for instance $\mathbb{Q}[x_n]/x_n^2$. Copointed means that the obvious diagram commutes. A map $A \to \mathbb{Q}[x_n]/x_n^2$ like that will be of the form $f(a) = \varepsilon(a) + D(a)x_n$ for some D. For this to be a ring-homomorphism, then $D(ab) = D(a)\varepsilon(b) + \varepsilon(a)D(b)$. That's called a derivation of $A \to \mathbb{Q}$. So we are looking at derivations of $A \to \mathbb{Q}$.

In the case $n > 0$, then $\varepsilon(a) = 0$, so that $D(ab) = 0$ is the simple condition. So D is just a linear map from $I/I^2 \to \mathbb{Q}$ where I is the augmentation ideal.

So it works out that π_n can be calculated as follows.

1. Find a cofibrant approximation to A, call it A_c.

2. There is an augmentation $A_c \stackrel{\varepsilon}{\to} \mathbb{Q}$, which has an augmentation ideal I.

3. Form $\mathrm{Hom}_{\mathbb{Q}}(I/I^2, \mathbb{Q})$, which is a chain complex (as one can check).

4. The homology groups of this complex are the rational homotopy groups.

Now let's go back to do a calculation that we've done a whole bunch of times. We figured out that $A[S^{2n}]$ is $\mathbb{Q}[x_{2n}, y_{4n-1}]/(dy = x^2)$. Thus I/I^2 has a basis x_{2n}, y_{4n-1} and the differentials of these are zero, so we get the rational homotopy groups of spheres.

28.8 Definition. A DGA is **minimal** if it is a free graded commutative algebra, $A_0 = \mathbb{Q}$, $A_1 = 0$, and $d = 0$ on I/I^2. A minimal DGA is always cofibrant, and the homotopy groups are just the duals of I/I^2.

Lecture 29
April 8, 2011

We had been talking about the homotopy theory of differential graded algebras. Today I will explain the Quillen equivalence with simplicial sets. But first I will go over some examples.

PROBLEM: Compute $\pi_*(S^3 \vee S^5) \otimes \mathbb{Q}$. We have a model $A(S^3 \vee S^5)$ that has cohomology $\mathbb{Q}[x_3, x_5]/x_3x_5$, with a basis $1, x_3, x_5$. x_3 is represented by some cocycle. Start with $\mathbb{Q}[x_3, x_5] \to A$. In cohomology $x_3x_5 \to 0$. So there is something y_7 of degree 7 that hits x_5.
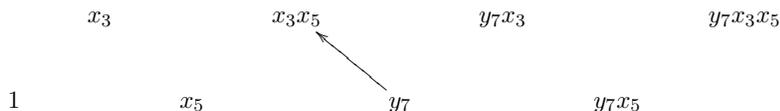
We need z_9 and z_{11} so that $z_9 \to y_7x_3$ and $z_{11} \mapsto x_7x_5$. Since z_9x_5 and $z_{11}x_3$ both map to the same class, their difference go to zero. In theory we could keep doing this. We would have to keep introducing new variables to kill combinations of the other ones. The main thing is that *we will get an answer*. People refer to this as *working out the minimal model*. You're creating products of things, and the new variables you're introducing always hit those products. So I/I^2 has basis $x_3, x_5, y_7, z_9, z_{11}, \cdots$ The next one will probably be in dimension 13. But this is good through dimension 11. The homotopy group $\pi_*(S^3 \vee S^5) \otimes \mathbb{Q}$ has basis dual to the above basis.

There's a theorem in rational homotopy. Suppose X is a pointed space, simply connected. Look at ΩX. Now $H_*(\Omega X; \mathbb{Q})$ is a Hopf algebra. This is an algebra coming from the loop product $\Omega X \times \Omega X \to \Omega X$. It is also a coalgebra coming from the diagonal $\Omega \overset{\Delta}{\to} \Omega X \times \Omega X$.

29.1 Theorem. $\pi_{*+1}X \otimes \mathbb{Q}$ *is the primitives in* $H_*(\Omega X; \mathbb{Q})$*: that is, the elements* $\Delta_*(X) = x \otimes 1 + 1 \otimes x$.

The primitives form a Lie algebra. If I take the coproduct of primitives x and y I get:

$$\Delta_*(xy) = (x \otimes 1 + 1 \otimes x) \vee y(\otimes 1 + 1 \otimes y)$$
$$= xy \otimes 1 + 1 \otimes xy + x \otimes y + y \otimes x$$
$$\Delta_*(yx) = yx \otimes 1 + 1 \otimes yx + y \otimes x + x \otimes y$$
$$\Delta_*(xy - yx) = (xy - yx) \otimes 1 + 1 \otimes (xy - yx)$$

This corresponds to the Whitehead product. $H_*(\Omega S^3 \vee S^5, \mathbb{Q}) = \text{Tensor-alg}(a_2, a_4)$. This is a free non-commutative algebra on two generators, both primitive. In Lie algebras you learn that $Tensor(a_2, a_4)$ is the free Lie algebra on $\{a_2, a_4\}$. So basically what this does is: shift the generators down one degree, take the free Lie algebra, shift it up again. If we have a_2, a_4 we have

$$a_2, a_4, [a_2, a_4], [a_2, [a_2, a_4]] \cdots$$

These correspond to dimensions $3, 5, 7, 9, \cdots$ back in I/I^2. Index these by commutators in the free Lie algebra; then you will "see the theorem appear in front of you."

This is an example of a formal space, where we can start with a space and just compute homotopy from the cohomology. If you can get to the situation where you've created all the cohomology, but you've created it in too many dimensions, then killing it is an entirely algorithmic process that will work like this.

Here's an example of a space that is not formal. Suppose $H^*(X, \mathbb{Q}) = 1, x_3, x_5, x_3 x_5 = 0, a_{10}$. Let's make the minimal model. We have $A(X)$ and can certainly map the free graded commutative $\mathbb{Q}[x_3, x_5, y_7] \to A(X)$. Where does $x_3 y_7$ go? It might go to to something cohomologous to zero, in which case we would have to hit it with something. But it might go to a_{10}. This is an example of a space where just knowing the cohomology is not enough to calculate the rational homotopy groups.

§1   The Quillen Equivalence $sSets^{\mathbb{Q}} \rightleftarrows DGA^{op}$

I had functors $A : sSets^{\mathbb{Q}} \rightleftarrows DGA^{op} : Sing$. Since A is a left adjoint, by the Yoneda lemma it is determined by $A(\Delta[n])$ and all the simplicial structure maps. This ends up being elegantly simple. I have to say what $A(\Delta[n])$ is. We want this to be the polynomial de Rham complex. Take the ring of polynomial functions on the standard n-simplex $(\mathbb{Q}[t_0 \cdots t_n]/\sum t_i = 1$, where the degree of each t_i is zero). Make this into a differential graded algebra, by adjoining $dt_0 \cdots dt_n$, where dt_i has degree 1. Now mod out by the relation: assert $\sum dt_i = 0$. Leibniz' rule comes from it being a differential graded algebra.

When I have a simplicial map of these (say, the inclusion of the ith face), the map of de Rham complexes will go in the other way. So $[n] \to A(\Delta[n])$ forms a simplicial DGA. All the maps are homomorphisms of DGA's. Call this $\Omega_\bullet^*$, so $\Omega_n^* = A(\Delta[n])$, where the

$*$ stands for the degree. When I put them all together, I get a single differential graded algebra. When you unravel all of this, you see that

$$A(X) = sSet(X_\bullet, \Omega_\bullet^*)$$

Think of this as the piecewise (linear) polynomial de Rham complex of X. If X was a product of simplices, such as

we would put a form in each triangle, we only require that they agree on the edge. They are not polynomial forms that extend. If X is a smooth manifold, I can compare this with the actual de Rham complex.

So that was one of the functors. The right adjoint is determined by this. $Sing(B)_n = DGA^{op}(A(\Delta[n]), B) = DGA(B, A(\Delta[n])) = DGA(B, \Omega_n^*)$ (these are DGA-maps). I don't expect this to have the right cohomology unless A was fibrant in the opposite category, or B was cofibrant (like a polynomial algebra). We need to show that A is a left Quillen functor. That is, we need to show that A takes cofibrations to cofibrations, and acyclic cofibrations to acyclic cofibrations. To show it's a Quillen equivalence, there's another thing to check. We won't go into this.

Let's see what we need to do to show that A takes cofibrations to cofibrations. Suppose $X_\bullet \hookrightarrow Y_\bullet$ is a cofibration of simplicial sets. Then we have a map $A(X_\bullet) \to A(Y_\bullet)$ that we want to be a cofibration in $(DGA)^{op}$. In actual DGA, we want $A(Y_\bullet) \to A(X_\bullet)$ to be a fibration. But fibrations were just surjections. But a typical element $a \in A(X)$ is a map $X \to \Omega_\bullet^*$
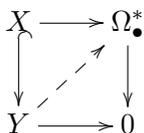
Asking it to be a surjection is the same thing as asking for a lift $Y \to \Omega_\bullet^*$, i.e. $\Omega_\bullet^*$ is an acyclic fibration. Fibration is easy, because $\Omega_\bullet^*$ is a simplicial abelian group (in fact, a $\mathbb{Q}$-vector space), and therefore fibrant. We need to show that $\pi_k \Omega_\bullet^* = 0$ for all k. Alternatively, if we make a chain complex

$$\Omega_0^* \overset{d_0 - d_1}{\leftarrow} \Omega_1^* \overset{d_0 - d_1 + d_2}{\leftarrow} \Omega_2^* \leftarrow \cdots$$

This is the standard complex for calculating $Tor^{Q[x,dx]}(\mathbb{Q}, \mathbb{Q})$ ($|x| = 0$), where the first $x \mapsto 1$ and the second $x \mapsto 0$. Let $h \in \Omega_1^0$ where $d_0 h = 0$ and $d_1 h = 1$ where $h = t_0$ (or t_1?). Suppose $\omega \in \Omega_n^*$ and $d_i \omega = 0$ for $i = 0 \cdots n$. We need to show that ω is a boundary (we need to write down a contracting homomorphism).

$\omega = d\omega_n$ where $\omega_n = \sum_{\text{shuffles a,b}} \pm s^a h \cdot s^b \omega$. This comes from the Alexander-Whitney

map. Suppose $\omega \in \Omega_2^*$. Then sum over all the higher-dimensional (degenerate) simplices

$$\omega_n = s^{0111}h \cdot s^{0012}\omega - s^{0011}h \cdot s^{0112}\omega + s^{0001}h \cdot s^{0122}\omega$$

(There's just an explicit contracting homotopy that you can write down.) This proves it takes cofibrations to cofibrations. The deeper fact is that it is a Quillen equivalence. But I'll leave these points for now.

Lecture 30
April 4, 2011

So, today we want to talk about the formality of Kähler manifolds. Suppose M is a smooth manifold. Then we claim that the differential graded algebra $A(M)$ is such that $A(M) \otimes_{\mathbb{Q}} \mathbb{R}$ is weakly equivalent to the de Rham complex Ω_M. This is supposed to be at least plausible. We can come back and justify this later, but it is supposed to be intuitive.

Recall that $A(X)$ consists of polynomial forms on each simplex that glue together appropriately. So ordinary differential forms seem kind of close to this. We'll clarify this on Friday.

Now we want to talk about how geometric structures on M will interact with the de Rham complex Ω_M.

First, let us suppose that M has a (Riemannian) metric $\langle , \rangle$. Suppose M is oriented as well, so there is a unique globally defined volume form. So we have the de Rham complex $\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots$. Each of these spaces Ω^i will then acquire a metric as well, so they are all inner product spaces. If these were finite-dimensional, then there would be an adjoint $d^* : \Omega^2 \to \Omega^1$. If that's the case, then we can write Ω^i as a decomposition $\ker(d) \oplus \ker(d)^\perp = \mathbb{H}^i(M; \mathbb{R}) \oplus \mathrm{im}(d) \oplus \ker d^\perp$, where $\mathbb{H}^i(M; \mathbb{R})$ is defined as an orthogonal complement. To get this in general, we have to use some theorems on differential equations. But let's assume that these decompositions all exist and work out.

Note that $\ker d / \mathrm{im} d = H^*(M; \mathbb{R})$ by the de Rham theorem. So $\mathbb{H}^i(M; \mathbb{R}) = H^i(M; \mathbb{R})$.

Now it is elementary linear algebra that $(\ker d)^\perp = \mathrm{im} d^*$. So we get a decomposition

$$\Omega^i = \mathbb{H}^i(M; \mathbb{R}) \oplus \mathrm{im}(d) \oplus \mathrm{im}(d^*).$$

The forms in $\mathbb{H}^i(M;\mathbb{R})$ (i.e., those orthogonal to $\operatorname{im}(d) \oplus \operatorname{im}(d^*)$) are called **harmonic forms** and are the ones that are killed by the **Laplacian** $\pm(dd^* + d^*d)$. (On functions, that turns out to be the usual Laplacian.)

30.1 Lemma. *A form is harmonic iff it is killed by* d, d^*.

Proof. One direction is clear. If $\nabla\omega = 0$, then take inner products with ω to get

$$(dd^*\omega + d^*d\omega, \omega) = (d\omega, d\omega) + (d^*\omega, d^*\omega) = 0,$$

implying $d\omega = d^*\omega = 0$ by positive-definiteness of the bilinear form. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$.

From this, we want to think about what the de Rham complex looks like. If we have the Ω^{k-1}, then this is isomorphic to $d\Omega^{k-2} \oplus d^*\Omega^k \oplus \mathbb{H}^{k-1}$. If we look at Ω^k, then this is isomorphic to $d\Omega^{k-1} \oplus \mathbb{H}^k \oplus d^*\Omega^{k+1}$. So the de Rham differential induces an isomorphism of $d^*\Omega^k$ with $d\Omega^{k-1}$.

If M is smooth with a Riemannian metric, then we get a map

$$H^*(M) \to \mathbb{H}^*(M) \subset \Omega^*$$

i.e. every cohomology class is canonically represented by a harmonic form. However, this is not enough. This map is *not* a ring-homomorphism. The wedge product does not preserve harmonic forms.

However, we could take the *closed* forms. This is a subalgebra $\Omega_{cl} \subset \Omega$, and we could map $\Omega_{cl} \to \mathbb{H}^*$. We could give $\mathbb{H}^*$ a structure of a DGA by considering it as the quotient of Ω_{cl} by the exact forms. However the inclusion $\Omega_{cl} \to \Omega$ is not a quasi-isomorphism.

Suppose now the manifold M has an almost complex structure J. This is a map $J : TM \to TM$ such that $J^2 = -1$. Every tangent space thus has a complex structure.

Then we can get another differential d_c as $J^{-1}dJ$. Multiplication by J gives an isomorphism of cochain complexes

$$(\Omega, d) \xrightarrow{J} (\Omega, d_c).$$

Since they are isomorphic, we can study the Hodge decomposition for d_c instead. We will just end up getting a new Hodge decomposition that is just J applied to the old ones. We also get a new Laplacian ∇_{d_c}.

We now describe the Kähler condition. We want to see how J interacts with d. Since J has eigenvalues $\pm i$, we should complexify. Let $\Omega_{\mathbb{C}}^* = \Omega_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, so $\Omega_{\mathbb{C}}^1$ is the space of complex-valued 1-forms. Now J acts on $\Omega_{\mathbb{C}}^1$ by $\mathbb{C}$-linear maps, so we can decompose it into eigenspaces of J. We let $\Omega^{1,0}$ for the i eigenspace and $\Omega^{0,1}$ for the $-i$ eigenspace. The de Rham differential can be then written as $d = \partial + \bar{\partial}$ in two components. If we

look locally, and imagine that $M = \mathbb{C}^n$ with the usual J, with coordinates $z_1, \ldots, z_n$ with $z_i = x_i + iy_i$, then

$$df = \sum \frac{\partial f}{\partial x_i} dx_i + \sum \frac{\partial f}{\partial y_i} dy_i,$$

$$\partial f = \sum \frac{\partial f}{\partial z_i} dz_i$$

and

$$\overline{\partial} f = \sum \frac{\partial f}{\partial \overline{z_i}} d\overline{z}_i.$$

So f is holomorphic iff $\overline{\partial} f = 0$, i.e. if df can be expressed only involving z's. Similarly we can decompose $\Omega^2 = \wedge^2(\Omega^{1,0} \oplus \Omega^{0,1}) = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$. The $\overline{\partial}$ component is that raises the thing by 1.

Having just a J on every tangent space doesn't mean you get a complex structure.

30.2 Theorem (Newlander-Nirenberg). *J is integrable (i.e. M has a complex structure s.t. J is multiplication by i) iff*

$$\overline{\partial}^2 = 0.$$

The integrability condition gives a relation between d_c and these other operators. **Fact:** If J is integrable, then $\overline{\partial}^2 = 0$, then $d_c = i(\partial + \overline{\partial})$.

Now suppose M has a metric and an almost complex structure J which is integrable. Suppose that this metric $\langle \rangle$ and J are compatible, i.e. J is an isometry. Then, there is a unique hermitian metric $(.,.)$ on the complexified tangent space, and therefore on all the complexified $\Omega^*_{\mathbb{C}}$, whose real part is the Riemannian metric. It is given by:

$$(u, v) = \langle u, v \rangle + i \langle u, Jv \rangle.$$

Let us focus on this imaginary part $\langle u, Jv \rangle = \omega(u, v)$. We don't have this until we have the metric and the thing J. Then ω is a 2-form, and one can check that $\omega \in \Omega^{1,1}$.

30.3 Definition. M is **Kähler** if ω is closed.

This has the beautiful consequence that if M is Kähler and $Z \subset M$ is a complex submanifold, then Z is Kähler too. An example is $\mathbb{CP}^n$ with the **Fubini-Study metric,** so that any smooth complex projective variety is Kähler.

This is equivalent to:

30.4 Proposition. *On a Kähler manifold, d and d_c satisfy the relation*

$$dd_c^* + d^*cd = 0$$

and the Laplacians ∇_d, ∇_{d_c} coincide.

That's something that only involves the real de Rham complex, and not the complex-ified one. This is what we're going to use to prove the formality of Kähler manifolds.

30.5 Proposition. *Kähler manifolds are formal.*

Proof. We use the Hodge decomposition using d_c. We have a decomposition $d_c\Omega^{k-2} \oplus \mathbb{H}^{k-1} \oplus d_c^*\Omega_k = \Omega^{k-1}$. Here the fact that the two Laplacians are the same means that the harmonic forms coincide. Since d, d_c^* anticommute, it respects this decomposition.

Now consider how d acts on this decomposition of Ω^*. All of the homology comes from the harmonic forms. If we get rid of that, the complex becomes acyclic. In fact, the top row of $d_c\Omega^*$ and the bottom row $d_c^*\Omega^*$ are acyclic with respect to d.

We have an inclusion

$$\ker d_c^* \to \Omega^*$$

and $\ker d_c^* \to \mathbb{H}^*$ is a map of DGAs, where the second is a quotient. However, the kernel of d_c^* is the harmonic things and the bottom row, so the quotient is acyclic. It follows that these maps of DGAs are quasi-isomorphisms. So we get a zigzag of weak equivalences relating Ω^* and $\mathbb{H}^*$.

To recapitulate the story, there was nothing we could do on an ordinary manifold. The decomposition that we got there didn't interact well with d. However, with the Kähler condition we got a decomposition that did interact well.   *Q.E.D.*

Lecture 31
April 15, 2011

§1 Principal G-bundles

31.1 Definition. A map $p : E \to B$ is a fiber bundle if there is an open cover $\sqcup U$ such that when you pull it back:

$$
\begin{array}{ccc}
E & \longleftarrow & \sqcup U \times F \\
\downarrow & & \downarrow \\
B & \longleftarrow & \sqcup U
\end{array}
$$

you get a direct product.

Let G be a topological group. For each $g \in G$, you have a map $g' \mapsto g'g$ that is a homeomorphism and a morphism of left G-sets. $G \subset Homeo(G)$; U determines $C^1(U, homeo(F))$. If F was a group we have a distinguished subset.

31.2 Definition. A principal G-bundle is a fiber bundle with fiber G represented by $C^1(U, G) \subset C^1(U, homeo(G))$.

Remark. We have a map

$$
\begin{array}{ccc}
G \times E & \longrightarrow & E \\
\downarrow & \swarrow & \\
B &
\end{array}
$$

which is an action of G on E.

More intrinsically:

31.3 Definition. A principal G-bundle is a fiber bundle $E \to B$ with fiber G and an action $G \times E \to E$ over B such that the fibers are identified with G: that is,

$$
\begin{array}{ccc}
G \times E & \overset{f}{\longrightarrow} & E \times_B E \\
& \searrow & \downarrow \\
& & B
\end{array}
$$

where $f : (g, e) \mapsto ge \times e$, is an isomorphism. Equivalently,

$$
\begin{array}{ccc}
G \times E & \overset{\mu}{\longrightarrow} & E \\
\downarrow & & \downarrow \\
E & \longrightarrow & B
\end{array}
$$

is a pullback.

Remark. If G acts freely on X, then $p : X \to X/G$ is a principal G-bundle if p admits local sections. That is, for all $x \in X/G$ we have $U \subset X/G$ such that f has a section:

$$
\begin{array}{ccc}
X \times_{X/G} X & \longrightarrow & X \\
f \downarrow \quad \uparrow s & & \downarrow \\
x \in U & \hookrightarrow & X/G
\end{array}
$$

31.4 Definition. Let $Princ_G(X)$ be the isomorphism classes of principal G-bundle on X.

Even though this asks for a characterization up to homeomorphism, it turns out to be a problem in homotopy theory.

Note: Given a principal G-bundle $Z \to Y$ and a map $X \overset{f}{\to} Y$ we have a pullback $Princ_G(Y) \overset{f^*}{\to} Princ_G(X)$. The claim is that this depends on f only up to homotopy.

31.5 Theorem. *Given homotopic maps $f_0, f_1 : X \to Y$ ($f_0^* = f_1^*$) i.e. $f_0^* Z \cong f_1^* Z$.*

Proof. There are two maps H_0 and $H_1 = f_1$ from $X \times I \to Y$. It is sufficient to show that $H^*Z \cong (f_0 \times Id)^*Z$. Then you would get an isomorphism between f_0^* and f_1^*. Each of these gives you a principal G-bundle, E and E'. We want to build an isomorphism, but we start with an isomorphism on $X \times \{0\}$ because both of these are H^*Z. We construct an isomorphism of principal G-bundles $\widetilde{H} : E \to E'$. In the definition we have identifications $E_{X \times t}$ that is the fiber over $X \times t$. Given a map of bundles, $\widetilde{H}_{X \times t}$ will denote the map $E_{X \times t} \to E'_{X \times t}$. $\widetilde{H}_{X \times \{0\}}$ is the identity, so $E_{X \times \{0\}} = f_0^* Z \overset{Id}{\to} f_0^* Z_X = E' X_{X \times \{0\}}$. If you have a fiber bundle, you have canonical identifications of the fiber.

Choose some open set over which E and E' are trivial: $E = U \times G$ and $E' = U \times G$ on U. (You can forget that G is a group for now.) You can create $\widetilde{H}_{X \times t}$ will be the composite

$$E_{X \times t} \overset{triv.}{\to} E_{X_0 \times t_0} \overset{\widetilde{H}_{x_0 \times t_0}}{\to} E'_{X_0 \times t_0} \overset{triv.}{\to} E'_{X \times t}$$

This can be done given Y over which E and E' are trivial, and a choice $X_0 \times t_0 \in U$ where $\widetilde{H}$ is defined.

Question (in case you're bored): Is $\widetilde{H}$ unique, given H?

Let's continue the proof; we have to show that all of this is compatible. Assume X is compact and normal. We will be invoking Urysohn's lemma. Choose a cover U_α of X, and a cover I_α of I by consecutive intervals, such that E and E' are trivial over $U_\alpha \times I_\alpha$. So we have some grid, and we've taken one square in which $E = U \times G$ and $E' = U \times G$. Take the interval, and subdivide it by writing down t_i. Assume inductively that we have defined $\widetilde{H}$ on all of $X \times [0, t_i]$. For each $x \in X$ choose $x \in W \subset W'$ such that $\overline{W} \subset W'$. Take finitely many W_i and W_i' such that the W_i cover X. By Urysohn's Lemma, there are functions $\overline{w}_i : X \to [t_i . t_{i+1}]$ such that $\overline{w}_i|_W = t_{i+1}$ and $\overline{w}_i|_{X-W} = t_i$. Let $\tau_k : X \to [t_i, t_{i+1}]$ such that $\tau_k(x) = max\{\overline{w}_1(X), \cdots, \overline{w}_k(X)\}$. Let $X_k = \{(X \times t) : t \leq \tau_k(X)\}$. These give bundles E_k, with associated E_k'. Assume inductively that $\widetilde{H}$ is defined on E_{k-1}. Then $\widetilde{H}(X, t) = \widetilde{H}_{X, \tau_{k-1}(t)}$. This gives $\widetilde{H}$ on E_k. $\qquad \mathcal{Q.E.D.}$

Remark. This is true for X with the homotopy type of a CW complex.

So the functor does not depend on homotopy, and is furthermore representable. Now construct the universal principal G-bundle.

31.6 Definition. A principal G-bundle $E \to B$ represents $Prin_G(-)$ if $\forall X$, $[X, B] \to Princ_G(X)$ is a bijection.

31.7 Theorem. *Suppose $E \to B$ is a principal G-bundle such that $\pi_n(E) = 0$ for all $n \geq 0$. THen $E \to B$ represents $Princ_G()$.*

For point-set reasons we will prove this for spaces that look like CW complexes.

31.8 Definition. X is a G-CW complex if $X = \varinjlim X^{(k)}$ is given by

$$X^{(k+1)} = X^{(k)} \cup_\alpha (D_\alpha^{k+1} \times G/H_\alpha)$$

via attaching maps $S_\alpha^k \to X^{(k=1)}$.

31.9 Fact. *Actions of compact Lie groups on manifolds are of this form, as are algebraic actions on projective varieties.*

Proof. Suppose $E \to B$ is such that E is a G-CW complex, $B = E/G$, $P \to X$ is a principal G-bundle, and P is a principal G-bundle. We'll find $f : X \to B$ such that $f^*E = P$.

It is sufficient to find a G-equivariant map $H : P \to E$ because quotienting by G gives a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

which must be a pullback. The reason is that you get a map to the pullback; since we have an injective surjective homeomorphism that is locally trivial, then inverse is also a homeomorphism. Build H inductively over $X^{(k)}$. Assume $H|_{X^{(k)}}$ is defined. We ned to define this on $E_\alpha^{k+1} \times G$. We need a non-equivariant extension on D_α so $H|_{\partial D; a} \in \pi_k(\varepsilon) = 0$. Choosing a nullhomotopy of $H|_{\partial D^{k+1}}$ gives an extension.

If we can get a contractible space with a free G-action, then we're done. Here is a construction.

Let Δ be the category whose objects are $0, 1, \cdots$. We have a map $EG \to BG$, where $EG_\cdot : \Delta \to Top$; $EG_n = EG_\cdot(m) = maps\{\Delta, G\} = G^{n+1}$. Define $|EG_/| = \sqcup_n \Delta^n \times EG_n$ modulo $f^*\delta_m \times t \sim \delta_m \times f_* t$. (We were taking $f : n \to m$ where $\delta_m \in EG_m$; $t \in \Delta^n$). G acts on $|EG|$. To check EG is contractible. $EG_\cdot$ is the nerve of the category with objects in G, and there is a unique morphism between any two objects. Any object is final, which implies that EG is contractible. $\mathcal{Q.E.D.}$

Lecture 33
April 20, 2011

We were talking about $BGL_n\mathbb{R}$. This is the limit $\lim_{N \to \infty} Gr_n(\mathbb{R}^N)$. Also, $[X, BGL_n\mathbb{R}]$ is the space of n-dimensional vector bundles over X.

Last class we calculated $H^*(BGL_n\mathbb{R}; \mathbb{Z}/2) = \mathbb{Z}/2[w_1 \cdots w_n]$, where w_i is in degree i. These are called Stiefel-Whitney classes.

How do you compute these w_i? Let V be a vector bundle over X. We will state some properties without proving them.

33.1 Proposition (Cartan formula). *(1) $w_m(V \oplus W) = \sum_{i+j=m} w_i(V) \cdot w$, where V and W are vector bundles. This is the Cartan formula. Here is another way of writing this: if the total Stiefel-Whitney class is $w_t(V) = 1 + w_1(v) + \cdots$. Then $w_t(V \oplus W) = w_t(V) \cdot w_t(W)$. This is equivalent to saying $w_m(V \oplus W) = \sum_{i+j=m} w_i(V) \cdot w_j(W)$*

Denote fibers by $(V \oplus W)_x = V_x \oplus W_x$. This is called the Whitney sum.

(2) $w_m(V) = 0$

(3) $w_1(\ell) \neq 0 \in H^i(\mathbb{R}P^\infty)$, where ℓ is the tautological line bundle.

33.2 Theorem. *These three properties determine the w_i.*

The original definition of the Stiefel Whitney classes was different. This definition is due to Grothendieck.

33.3 Lemma. *Suppose $V' \hookrightarrow V \twoheadrightarrow V''$ is an exact sequence of vector spaces over X. Then there is a splitting $V'' \to V$ and $V = V' \oplus V''$.*

This isn't true for algebraic or holomorphic vector bundles. But if the functions are continuous then every short exact sequence of vector bundles splits.

Proof. Idea: do it locally, and then patch using a partition of unity. Alternatively, you could come up with a Riemannian metric for every vector bundle, and use orthogonality to get the splitting.   *Q.E.D.*

Suppose I have a vector bundle $p : V \to X$, and associate to this the projective bundle $P(V) \xrightarrow{\pi} X$. Over $P(V)$ there is a tautological line bundle on every fiber, which is classified by the maps ℓ into $\mathbb{R}P^\infty$. We showed that if $\dim V = n$, then $H^*(P(V); \mathbb{Z}/2) = H^(X'\mathbb{Z}/2)\{1 \cdots x^{n-1}\})$ (that is, it is a module over $H^*(X; \mathbb{Z}/2)$ with this basis).

Look at the vector bundle V. Every fiber is the projective space of V; in the projective space we have a line bundle. So we have a short exact sequence of vector bundles over $P(V)$

$$\ell \hookrightarrow \pi^* V \twoheadrightarrow H$$

The dimension of H is $n-1$, so $w_n(H) = 0$. Let's use our other axioms. We know that $(1+x)(1+w_1(H)+\cdots+w_{n-1}(H)) = (1+w_1(V)+\cdots+w_n(V))$. So using the Cartan formula:

$$1 + w_1(H) + \cdots + w_{n-1}(H) = (1 + x + x^2 + \cdots)(1 + w_1(V) + \cdots + w_n(V))$$

Look at the terms of degree n:

$$0 = w_n(V) + xw_{n-1}(V) + \cdots + x^n$$

This is the relation we used to define the w_i's. Starting with this definition, you could derive the properties above. (Recall we are working over $\mathbb{Z}/2$; if we were working on complex vector bundles, this would not happen, and this would have some alternating terms.)

33.4 Example. Let's calculate the Stiefel-Whitney classes of the tangent bundle to $\mathbb{R}P^n$. This argument will work for the Grassmannian as well.

On $\mathbb{R}P^n$ we have a short exact sequence

$$\ell \hookrightarrow \mathbb{R}^{n+1} \twoheadrightarrow H$$

ℓ is a point in $\mathbb{R}P^n$. Imagine that I have a line ℓ, and the orthogonal complement H; suppose I make an infinitesimal change. Interpret this as the graph of a homomorphism $\ell \to H$. In other words, $T\mathbb{R}P^n = Hom(\ell, H)$. We get a sequence

$$Hom(\ell, \ell) \hookrightarrow Hom(\ell, \mathbb{R}^{n+1}) \twoheadrightarrow T\mathbb{R}P^n$$

$Hom(\ell, \ell)$ is canonically the real numbers (maps are just scalars). $Hom(\ell, \mathbb{R}^{n+1})$ is the same as $Hom(\ell, \mathbb{R})^{n+1}$. So this can be rewritten

$$\mathbb{R} \hookrightarrow Hom(\ell, \mathbb{R})^{n+1} \twoheadrightarrow T\mathbb{R}P^n$$

We claim that

$$Hom(\ell, \mathbb{R}) = \ell$$

Pick a metric on ℓ. ℓ is a line through the origin in $\mathbb{R}^{n+1}$, so take the metric that came with $\mathbb{R}n + 1$. This gives us a map from $\ell \otimes \ell \overset{metric}{\hookrightarrow} \mathbb{R}$. In fact this map is an isomorphism. $\ell \overset{\cong}{\to} Hom(\ell, \mathbb{R})$. Now just do this fiberwise. We get

$$\mathbb{R} \hookrightarrow \ell^{n+1} \twoheadrightarrow T\mathbb{R}P^n$$

We find that $w_t(T\mathbb{R}P^n) \cdot w_t(\mathbb{R}) = (1+x)^{n+1}$, and $w_t(\ell) = 1 + x$.

What is $w_1(\mathbb{R})$? It is a pullback of the trivial bundle on a point. If you have the map $X \overset{f}{\to} pt$ there is no more cohomology here, so $w_t(\mathbb{R}^n) = 1$.

So

$$w_t(T\mathbb{R}P^n) = (1+x)^{n+1}$$

Here is an application: how many linearly independent vector fields can there be on $\mathbb{R}P^n$? The answer is known, but we can get a lower bound.

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Partial answer: suppose there are k. Then $T\mathbb{R}P^n = V \oplus \mathbb{R}^k$ where $\dim V = n - k$. Using the Cartan formula, this implies that $w_i(T\mathbb{R}P^n) = 0$ for $i > n - k$. V doesn't have any Stiefel-Whitney classes greater than $n - k$.

Look at $\mathbb{R}P^2$, given by $1 + x + x^2$. There is not even one vector field, or else the coefficient of x^2 would be zero.

In $\mathbb{R}P^3$, $(1 + x)^4 = 1$, so there could be three.

In $\mathbb{R}P^5$, we have $(1 + x)^6 = 1 + x^2 + x^4$. There is at most one vector field.

Suppose a manifold M^n is immersed in $\mathbb{R}^{n+k}$. At every point we have the tangent bundle, and it has an orthogonal complement called the normal bundle. The tangent bundle has dimension n, and he normal bundle has dimension k. So $T^n \oplus \nu^k - \mathbb{R}^{n+k}$. [The exponents just denote dimension.] If M immerses in $\mathbb{R}^{n+k}$ there is a k-dimensional vector bundle with $T \oplus \nu = \mathbb{R}^{n+k}$. Actually, the converse is true.

Can $\mathbb{R}P^5$ be immersed in $\mathbb{R}^{5+k}$? We know it can be embedded in $\mathbb{R}^9$. If so, then $T \oplus \nu = \mathbb{R}^{k+5}$. Then $\dim \nu = k$ and $w_t(T) \cdot w_t(\nu) = 1$ in the ring Rewrite this: $(1 + x)^6 \cdot w_t(\nu) = 1$, or in other words $w_t(\nu) = (1 + x)^{-6}$. $(1 + x)^8 = 1 + x^8 = 1$. So $(1 + x)^{-6} = (1 + x)^{8-6} = (1 + x)^2 = 1 + x^2$. So $\dim \nu \geq 2$ since $w_2(\nu) \neq 0$. If $\mathbb{R}P^5$ immerses into $\mathbb{R}^{5+k}$ then $k \geq 2$. So it doesn't immerse in something less than 7. So the optimal number is 7,8, or 9. But we don't know which.

[Read Milnor-Stasheff, *Characteristic Classes.*]

Let's try to prove the Cartan formula. I'm going to imagine that I have two vector bundles V and W. Write the projective space $P(V \oplus W)$ in two ways. Look at all the lines in this space that are not in W: $P(V \oplus W) - P(W) := U_V$. This is a covering for $P(V \oplus W)$. Lines that aren't in W can be projected down to V. This is a homotopy equivalence. I can rewrite this up to homotopy:

$$P(W) \hookrightarrow P(V \oplus W) \text{ and } P(V) \hookrightarrow P(V \oplus W)$$

Let $X = U_1 \cup U_2$. Suppose $\alpha \in H^*(X) \to H^*(U_1)$ that goes to zero under this map. Let β be the analogous thing for $H^*(X) \to H^*(U_2)$. Then $\alpha \cdot \beta = 0$. In $H^*(X, Y_1) \to H^*(X)$ $\alpha' \mapsto \alpha$, and there is an analogous β'. Sp $\alpha'\beta' \in H^*(X, U_1 \cup U_2) = 0$ goes to $\alpha\beta \in H^*(X)$. The Cartan formula is equivalent to

$$(x^n + w_1(V)x^{n-1} + \cdots + w_n(V))(x^m + w_1(W)x^{n-1} + \cdots + w_m(W)) = 0$$

But this is really $\alpha\beta$ and each of these goes to zero in either V or W.

Lecture 35
April 25, 2011

§1 Étale homotopy theory

The initial motivation for algebraic geometry is to understand the zeroes of polynomials, in one or several variables, over fields or even rings. This is the algebraic side. The idea is to consider these zeroes as points in some sort of space. Here is the prototype of an algebraic-geometrical object: start with a field K and consider the ring of polynomials $K[T_1 \cdots T_n]$. Take some polynomials $f_1 \cdots f_m$. An algebraic variety is the set of common zeroes in K^n of all the f_i's. If the field is small, this set can be empty! (For example, $T_1^2 + T_2^2 = 1$ over $\mathbb{Q}$.) Consider the set of zeroes of f_i's in the algebraic closure $\overline{K}$. We consider $\overline{K} = \mathbb{A}_{\overline{K}}^m$ as the affine n-space over $\overline{K}$. Hilbert's Nullstellensatz says that there is a 1-1 correspondence between points in $\overline{K}^n$ and maximal ideals in $\overline{K}[T_1 \cdots T_n]$. These look like $(T_i - a_i)$ for $a_i \in \overline{K}$. What you're really interested in are the prime ideals, or quotients of them by ideals.

We're also interested in projective varieties: consider homogeneous polynomials in $K[T_0 \cdots T_n]$ (where every monomial term has the same degree), and let V be the set of zeroes of the f_i. This is sitting inside of $\mathbb{P}_{\mathbb{C}}^n$, the set of lines in $\mathbb{C}^{n+1}$ through the origin.

You can always embed the zeroes in Euclidean space. Is it a manifold? If $K = \mathbb{C}$, then consider the set V of zeroes as a subset of $\mathbb{C}^n$ and equip it with the subspace topology. Call this topological space V_{an}, and call this the analytic topology. Now we can ask about its homotopy groups, etc. First we observe two things. If V is nonsingular (smooth) (i.e. when you consider the matrix of partials $(\frac{\partial f_i}{\partial T_j})_{ij}$, it has maximal rank), then V is a complex manifold. If V is a rojective non-singular variety, then V_{an} is a complex-analytic complex manifold. So these are really nice over $\mathbb{C}$. What if the field had characteristic zero? Then you can embed the field into $\mathbb{C}$ and ask about the resulting analytic manifold. For a projective non-singular algebraic variety V we can study $V_{an,K \hookrightarrow \mathbb{C}}$. We consider the coefficients of our embedding as in $\mathbb{C}$, and the zeroes as in $\mathbb{C}$.

This object really depends on the choice of embedding $K \hookrightarrow \mathbb{C}$. Serre showed that there are different kinds of number fields K and projective non-singular varieties such and embeddings $\varphi : K \hookrightarrow \mathbb{C}$ and $\psi : K \hookrightarrow \mathbb{C}$ such that the analytic variety associated to φ, and that associated to ψ, are not homeomorphic, or even homotopy-equivalent. That is, $\pi(V_{an}, \varphi) \neq \pi_1(V_{an}, \psi)$. Here is the idea for the construction. (Assume that the class group is nontrivial.) Start with $k = \mathbb{Q}(\sqrt{-p})$, where p is prime, $p \equiv 1$ modulo 4. Anyway, let K be the class field of k. We have $[K : k] = \#\mathcal{O}_k$. Take one embedding that corresponds to the trivial element, and one that does not. Study elliptic curves, and embed them into $\mathbb{C}$ by these embeddings. Take products of these; they cut out projective curves. Let groups act on them. . .

Anyway, $\pi_1(\hat{V_{an}}, \varphi) = \pi_1(\hat{V_{an}}, \psi)$. (Take all normal subgroups with finite index. Take all the quotients, and take the limit of these normal subgroups. This is the profinite completion.) The point is that both $V_{an,\varphi}$ and $V_{an,\psi}$ come from a common étale homotopy type of V. Roughly speaking, the étale homotopy type of V is defined over K itself, and it can be defined over any field, even in characteristic $p > 0$.

We need to define a new topology on V. So far, I haven't told you a topology, except whenever I embed it somewhere, it gets the subset topology. There is an intrinsic topology for every algebraic variety that is called the Zariski topology. But this is too coarse; there are not enough open subsets for what we want to do. The analytic topology depends on embeddings, and is not available in positive characteristic. But there *is* something that does the job, and it is called the étale topology. Here is the idea for the construction. Assume we are on a manifold. Instead of considering open subsets U of our object X, consider them as open immersions $U \hookrightarrow X$. Instead of open immersions, consider local diffeomorphisms from some manifold $U \to X$. This can be used for an algebraic variety with the right choice of local diffeomorphisms, to define something which is almost like a topology. What do I mean? There is something that plays the role of local diffeomorphisms in algebraic geometry, and these are called étale morphisms.

An étale map $f : U \to X$ induces isomorphisms of the tangent space $T_{U,u} \to T_{X,f(x)}$. Equivalently, $f : U \to X$ is etale for the following situation: you have a ring A, a quotient of a polynomial ring $A[T_1 \cdots T_n]/(P_1 \cdots P_n) = B$. B is étale over A if the Jacobian matrix $\det(\frac{\partial P_i}{\delta T_j})_{ij}$ is a unit (invertible) in B. This is just what you expect from the earlier definition. So we've defined a new kind of local diffeomorphism. Now we have to define a new kind of topology; this is not given by defining open subsets.

Define a "topology" by defining an *étale covering* of X as a family of étale morphisms $U_i \overset{f_i}{\to} X$ such that $X = \cup_i f_i(U_i)$. An étale neighborhood of a point $x \in X$ is an étale morphism $U \top f X$ such that $x \in f(U)$. So we allow the open thing to be a map such that our point is in the image of our thing. In a usual topology, this would be an open embedding, and this is exactly what an open neighborhood is. In general, call this the Grothendieck topology. This satisfies certain properties:

- Isomorphisms should be coverings

- If $\{U_i \to X\}$ is a covering, and $\{V_{ij} \to U_i\}_j$ is a covering, then $\{V_{ij} \to X\}$ is a covering of X.

- If $\{U_i \to X\}$ is a covering and $Y \to X$ is any morphism, then the base change $\{U_i \times_X Y \to Y\}$ should be a covering of Y.

Grothendieck's original motivation was not to study homotopy theory; he was interested in defining sheaf theory, which gets you étale cohomology for algebraic varieties. But

next time we will use this to define the étale homotopy type. The thing that ties these two is Čech cohomology and Čech coverings.