11/29: More on $\lim_{x\to a} f(x)$, §20 part 3/3

$\lim_{x\to a^S} f(x) = L \iff \forall$ sequence (x_n) in S converging to a, $\lim_{n\to\infty} f(x_n) = L$.

- $\lim_{x\to a} f(x)$ — $S =$ (interval con't a) $\setminus a$
- $\lim_{x\to a^-} f(x)$ $S =$ (t, a)
- $\lim_{x\to a^+} f(x)$ $S =$ (a, t)

Thm 20.10: Spse f that's defined on $J\setminus\{a\}$ where J is an open interval containing a. Then $\lim_{x\to a} f(x)$ exists $\iff$ $\lim_{x\to a^+} f(x)$ & $\lim_{x\to a^-} f(x)$ exist and are $=$.
In this case, all three are equal.

Limits commute with addition, multiplication, & composition.

f is continuous at a $\iff$ $f(a) = \lim_{x\to a} f(x)$ ← assuming domain is reasonable

Warm-up

(a) Suppose f is continuous on $[0,2]$.

(T)/F: $\lim_{x\to 2^-} f(x)$ exists

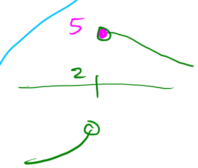
T/(F): $\lim_{x\to 2^+} f(x)$ exists. (not in general)

(b) Spse $f: \mathbb{R}\setminus\{2\} \to \mathbb{R}$ is cts and $\lim_{x\to 2} f(x) = 5$.

Is there a continuous extension of f to $\tilde{f}: \mathbb{R}\to\mathbb{R}$? Yes

Define $\tilde{f}(2) = 5$

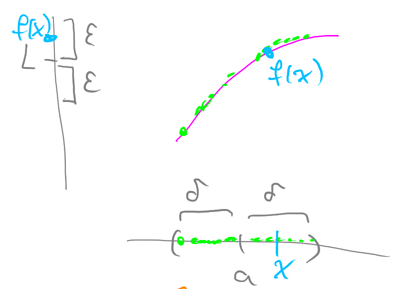
(c) What if we only know $\lim_{x\to 2^+} f(x) = 5$. What can you say?



$\exists$ continuous function $\tilde{f}: [2,\infty) \to \mathbb{R}$ that agrees w/ f where they're both defined.

Alternative definition of $\lim_{x\to a^S} f(x)$: Assume $S \subseteq \text{domain}(f)$

$\lim_{x\to a^S} f(x) = L$ (finite) $\iff$ $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in S$ s.t. $|x-a| < \delta$, $|f(x) - L| < \varepsilon$



↑ look at $S \cap (a-\delta, a+\delta)$

Given an ε-interval around L, all points $x \in S$ sufficiently close (w/in δ) to a have $f(x)$ ε-close to L.

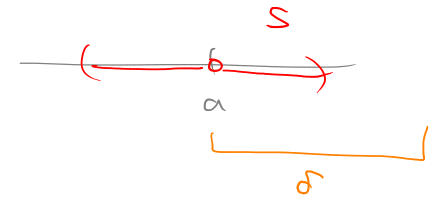
$\lim_{x\to a^S} f(x) = +\infty \iff \forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $\forall x \in S$ w/ $|x-a| < \delta$, $f(x) > M$.

Recall $\lim_{x\to a}$, $\lim_{x\to a^+}$, $\lim_{x\to a^-}$ are special cases of this.

Suppose f is def'd on an open interval containing a. Then

$\lim_{x\to a} f(x) = L \iff \forall \varepsilon>0, \exists \delta>0$ s.t. $\forall x$ s.t. $0<|x-a|<\delta$, $|f(x)-L|<\varepsilon$

finite

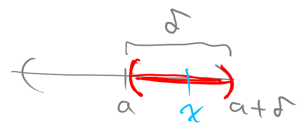


Might need to shrink δ to make $(a-\delta, a+\delta) \subseteq$ domain.

finite

$\lim_{x\to +\infty} f(x) = L \iff \forall \varepsilon>0, \exists \alpha$ s.t. $\forall x>\alpha$, $|f(x)-L|<\varepsilon$

$\lim_{x\to a^+} f(x) = +\infty \iff \forall M\in\mathbb{R}, \exists \delta>0$ s.t. $\forall x$ s.t. $a<x<a+\delta$, $f(x)>M$



Ex: Show $\lim_{x\to 0^+} \frac{1}{x} = +\infty$.

Let $M\in\mathbb{R}$. Let $\delta = \frac{1}{M}$. Let x satisfy $0<x<\delta$.

If $M\le 0$, $f(x)\ge 0$ since $x>0$.

$f(x) = \frac{1}{x} > \frac{1}{\delta} \ge M.$

works if $\delta \le \frac{1}{M}$ and $M>0$

<u>Sequence version of this proof</u>: Given $(x_n)\longrightarrow 0$ and $x_n>0$.

$\lim_{x\to 0^+}$

Need: $\lim_{n\to\infty} f(x_n) = +\infty$

Let $M>0$. Need: $\exists N$ s.t. $1/x_n > M$ $\forall n>N$.

Use $\lim_{n\to\infty}(x_n) = 0$ and $x_n>0$ to get N s.t. $\forall n>N$, $0<x_n<1/M$

Then $\forall n > N$, $\frac{1}{x_n} > M$.

Ex: Show $\lim_{x \to +\infty} \frac{1}{x} = 0$. $f(x) = 1/x$

Need: $\forall \varepsilon > 0$, $\exists \alpha$ s.t. $\forall x > \alpha$, $|f(x) - 0| < \varepsilon$.

Let $\varepsilon > 0$. Let $\alpha = \frac{1}{\varepsilon}$. Let $x > \alpha$.

$$|f(x)| = \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{\alpha} \leq \varepsilon$$

assume $\alpha > 0$ so $x > 0$

works if $\frac{1}{\varepsilon} \leq \alpha$