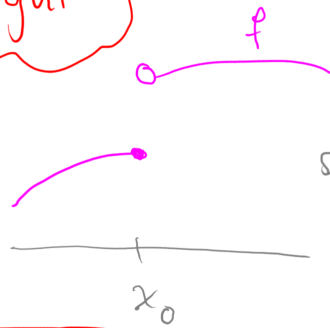
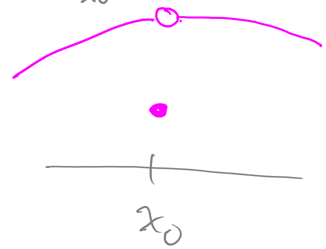
11/22: Limits, §20 part 1/3

Slogan: Not quite right $\to$ f is cts. at $x_0 \iff \lim_{x\to x_0} f(x) = f(x_0)$



f is not cts at x_0 since $\lim_{x\to x_0} f(x)$ DNE

$\lim_{x\to x_0} f(x) \neq f(x_0)$



Almost right definition: $\lim_{x\to x_0} f(x) = L$ if $\forall$ sequences (x_n) in domain(f) ??? converging to x_0, have $\lim_{n\to\infty} f(x_n) = L$.

Also want $\lim_{x\to x_0^+} f(x)$, $\lim_{x\to x_0^-} f(x)$

(Not quite right) Idea: restrict sequences to have values $\geq x_0$

Idea: restrict to $\leq x_0$.

Warm-up

$f: \mathbb{Z} \to \mathbb{R}$, $f(x) = (-1)^x$

T/F: f is continuous on $\mathbb{Z}$

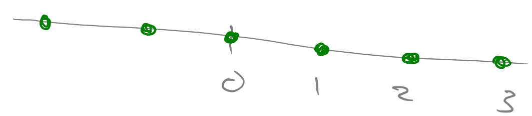
E.g. $x_0 = 0$

$\forall \varepsilon$ (assume $\varepsilon < 1$) show $\exists \delta$ (assume $\delta < 1$) s.t. if $|x| < \delta \leq 1$ & $x \in$ domain(f), then $|f(x) - f(0)| < \varepsilon$

Only x to consider is $x = 0$.

$\Rightarrow f$ is cts. at 0.

T/F: f is uniformly cts. on $\mathbb{Z}$.



Def: Given $S \subseteq$ domain(f).

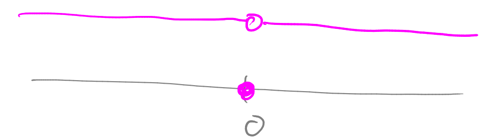
$\lim_{x\to x_0}^{S} f(x) = L$ if $\forall$ sequences (x_n) in S converging to x_0 then $\lim_{n\to\infty} f(x_n) = L$.

Note: f is continuous at $x_0 \iff \lim_{x\to x_0}^{S} f(x) = f(x_0)$ where $S = $ domain(f).

Why can't I define $\lim_{x \to x_0} f(x)$ as $\lim_{x \to x_0}^{S} f(x)$ wh. $S = \text{domain}(f)$?

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$

Want $\lim_{x \to 0} f(x) = 1$



Claim $\lim_{x \to 0}^{\mathbb{R}} f(x)$ DNE.

$x_n = 1/n \to 0$, $\lim_{n \to \infty} \underbrace{f(x_n)}_{1} = 1$

$y_n = \underbrace{0} \to 0$, $\lim_{n \to \infty} \underbrace{f(y_n)}_{0} = 0$

Takeaway: $\lim_{x \to x_0} f(x)$ should not depend on $f(x_0)$.

Possible fix: consider $\lim_{x \to x_0}^{S} f(x)$ where $S = \text{domain}(f) \setminus \{x_0\}$.

Other things that can go wrong:

$f: \mathbb{Z} \to \mathbb{R}$, $f(x) = (-1)^x$. What is $\lim_{x \to 0}^{S} f(x)$, where $S = \mathbb{Z} \setminus \{0\}$.

Look at sequences in $S = \mathbb{Z} \setminus \{0\}$ that converge to 0.
There aren't any :(

We give up on defining the limit in this case. (But recall $f(x)$ is cts. at 0.)

Ex: $S = \mathbb{R}$, $f(x) = x^2 + 1$. What is $\lim_{x \to 2}^{\mathbb{R}} f(x)$? 5

f is continuous everywhere, so $\lim_{x \to 2}^{\mathbb{R}} f(x) = f(2) = 5$.

Q: To what extent does $\lim_{x \to x_0} {}_S f(x)$ depend on S?

$f: \mathbb{R} \to \mathbb{R}$, $\quad f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

$\lim_{x \to 0} {}_S f(x)$ where $S = \mathbb{Q} \setminus \{0\}$? $\quad (x_n) \subseteq \mathbb{Q} \setminus \{0\}$, $\quad \lim_{n \to \infty} \overbrace{f(x_n)}^{1} = 1$

$\lim_{x \to 0} {}_S f(x)$ where $S = \mathbb{R} \setminus \mathbb{Q}$? $\quad (x_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$ $\quad \lim_{n \to \infty} \underbrace{f(x_n)}_{0} = 0$

T/F: If $S \subseteq T$, $0 \in S$, and $\lim_{x \to 0} {}_{T \setminus \{0\}} f(x)$ exists, Then $\lim_{x \to 0} {}_{S \setminus \{0\}} f(x)$ exists and equals $\lim_{T \setminus \{0\}}$.

$\lim_{S \setminus \{0\}}$ is for fewer sequences

T/F: The converse is true.

Maybe $\exists$ seq. in T but not S s.t. $\lim f(x_n)$ DNE.

Ex above: $\quad T = \mathbb{R}$, $\quad S = \mathbb{Q} \setminus \{0\}$.

no limit

had limit