# ***Lecture Notes*, Math 170A, Spring 2020**

## *Chapter 4.2: the SVD and the spectral norm and condition number*

**The spectral norm and the SVD.** Recall the singular value decomposition, $A = U\Sigma V^T$.

**Theorem 1.** *For any matrix $A \in \mathbb{R}^{n\times m}$, define the induced 2-norm $||A||_2 = \max_{x\neq 0} \frac{||Ax||_2}{||x||_2}$. (We have previously defined this solely for $m = n$, but there is nothing to prevent us from extending this definition to any $m, n$.)*

*Then $||A||_2 = \sigma_1$, where $\sigma_1$ is the largest singular value of $A$.*

*Proof.* Recall that $\{v_1, \ldots, v_m\}$ is an orthonormal basis for $\mathbb{R}^m$, and $\{u_1, \ldots, u_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Let $x \in \mathbb{R}^m$, $x \neq 0$, then there exist constants $c_i$ with $1 \leq i \leq m$ such that $x = \sum_{i=1}^{m} c_i v_i$.

Note that since $A = U\Sigma V^T$, $AV = U\Sigma$, and $Av_i = \sigma_i u_i$ for all $1 \leq i \leq m$.

Calculating, we obtain that

$$||x||_2^2 = \langle \sum_{i=1}^{n} c_i v_i, \sum_{i=1}^{m} c_i v_i \rangle = \sum_{i,j=1}^{m} c_i c_j v_i^T v_j \ ,$$

and due to the orthonormality of the set $\{v_1, \ldots, v_m\}$,

$$||x||_2^2 = \sum_{i=1}^{m} c_i^2 \ , \qquad \text{or } ||x||_2 = \sqrt{\sum_{i=1}^{m} c_i^2} \ .$$

Similarly, we show that

$$Ax = A\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{m} c_i A v_i = \sum_{i=1}^{m} c_i \sigma_i u_i \ ,$$

and $||Ax||_2 = \sqrt{\sum_{i=1}^{m} c_i^2 \sigma_i^2}$.

But then

$$\frac{||Ax||_2}{||x||_2} = \frac{\sqrt{\sum_{i=1}^{m} c_i^2 \sigma_i^2}}{\sqrt{\sum_{i=1}^{m} c_i^2}} \leq \frac{\sqrt{\sum_{i=1}^{m} c_i^2 \sigma_1^2}}{\sqrt{\sum_{i=1}^{m} c_i^2}} = \sigma_1 \ ,$$

where the inequality in the above is based on the fact that $\sigma_1$ is the largest singular value. Thus, by taking the maximum, we obtain also that

$$||A||_2 = \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \leq \sigma_1 \ .$$

On the other hand, by making $x = v_1$, and hence $Ax = Av_1 = \sigma_1 u_1$, we obtain that

$$\sigma_1 = \frac{||Av_1||_2}{||v_1||_2} \leq \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} = ||A||_2 \ .$$

These two inequalities taken together show the desired conclusion. $\square$

We thus have an expression of the induced 2-norm (also known as the *spectral* norm) for any matrix, as the largest singular value of the matrix.

**The SVD of the inverse and condition numbers.** For an $n \times n$ square, invertible matrix ($r = n$), is $A = U\Sigma V^T$ (then $U, V, \Sigma$ are all $n \times n$), $A^{-1} = V\Sigma^{-1}U^T$, and note that this means the singular values of $A^{-1}$ are $\frac{1}{\sigma_n} \geq \frac{1}{\sigma_{n-1}} \geq \ldots \geq \frac{1}{\sigma_1}$, and thus

$$||A^{-1}||_2 = \frac{1}{\sigma_n} \ .$$

Thus,

**Theorem 2.** *For an $n \times n$ invertible matrix $A$, in the 2-norm,*

$$\kappa(A) = ||A||_2 \cdot ||A^{-1}||_2 = \frac{\sigma_1}{\sigma_n} \ .$$

**Low-rank approximation.** Let us write $A$ as the sum of rank-one matrices (as we have seen in the previous lecture),

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T \ .$$

We will denote by $A_k$ the truncated sum $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$.

**Theorem 3.** *For any $k = 1, \ldots, r$, let $\mathcal{V}_k$ be the set of all $n \times m$ matrices of rank at most $k$. Then, with the notation above,*

$$\sigma_{k+1} = ||A - A_k||_2 = \min_{B \in \mathcal{V}_k} ||A - B||_2 \ .$$

*That is, among all $n \times m$ matrices of rank $k$ or less, $A_k$ is "closest" to $A$, and the difference in norm is $\sigma_{k+1}$.*

**Remark 1.** *Note that when $m = n$, $\{0\} \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_{n-1} \subset \mathcal{V}_n = \mathbb{R}^{n \times n}$, and that $\mathcal{V}_{n-1}$ is the set of all singular matrices (if a matrix is singular, then its rank is at most $n - 1$.)*

The proof for this theorem is in the textbook (4.2.15), and I invite you all to take a look. Although it's a bit long, it is quite elegant.

As a consequence, we have the following two results.

**Corollary 1.** *If $A$ is square and full-rank ($m = r = n$) and $B$ is a matrix for which $||B - A||_2 < \sigma_n$, then $B$ is full-rank.*

**Corollary 2.** *If $A$ is square and non-singular ($m = n = r$ and $\sigma_n > 0$), let $B$ be the matrix that is singular and closest to $A$ in 2-norm (so that $||A - B||_2$ is minimal among all singular matrices). Then, with the notations above, $B = A_{n-1}$ and*

$$\frac{||A - A_{n-1}||_2}{||A||_2} = \frac{1}{\kappa(A)} \ .$$

Note that the second corollary shows that the result we got in 2.3 about the distance to singularity and the condition number is, in fact, tight.