Lecture 2: Digraphs, subgraphs, the Handshake Lemma

Covers chapters 1.5, 1.6, 1.7

Topics for today:

- Digraphs; what do neighborhoods and degrees look like?

- Subgraphs, induced subgraphs, removing edges and vertices

- A simple but powerful ("Handshake") Lemma

Digraphs (directed graphs, networks)

- $G = (V, E)$

- $V$ vertices, $E$ arcs (oriented pairs of vertices, loops, many arcs between two vertices)

- Every vertex $v$ has an in-neighborhood and an out-neighborhood:

$$N^{\text{out}}(v) = N^+(v) = \{ u : (v, u) \in E \}$$

$$N^{\text{in}}(v) = N^-(v) = \{ u : (u, v) \in E \}$$

- Consequently, the in-degrees and out-degrees of digraphs are given as

$$d^{\text{out}}(v) = d^+(v) = |N^+(v)|$$

$$d^{\text{in}}(v) = d^-(v) = |N^-(v)|$$

Subgraphs

- A graph $H = (V_1, E_1)$ is a subgraph of the graph $G = (V, E)$ if
- A subgraph $H$ of $G$ is **induced** by a subset of vertices $X$ if $E(H)$
  consists of all edges in $G$ between vertices in $X$

$$G$$

$$X = \{2, 4, 5, 6\}$$

$$E(H) = \{(2, 5), (5, 6), (4, 6), (2, 6)\}$$

- A subgraph $H$ of $G$ is **induced** by a subset of edges $E(H) \subseteq E$ if $V(H)$
  consists of all the vertices in $V$ which are **incident** to edges in $E(H)$.

$$E(H) = \{(1, 2, 3, 5), (5, 3), (6, 4)\}$$

$$X = \{1, 2, 3, 4, 5, 6, 7\}$$

- When we want to "remove" a set of vertices $X$, we obtain the subgraph
  induced by $V \setminus X$. We denote the resulting graph by $G - X$.

- When we want to "remove" a set of edges $E_1$, we take the subgraph that
  results when each of those edges is removed. We denote the resulting
  subgraph by $G - E_1$.
The Handshake Lemma (also known as the "handshaking" Lemma)

![Diagram of a graph with labeled vertices and edges]

\[
d(1) = 2 \\
d(2) = 3 \\
d(3) = 2 \\
d(4) = 1 \\
\]

\[d(1) + d(2) + d(3) + d(4) = 8 = 2|E|\]

**Lemma.** For any simple graph \( G = (V, E) \),

\[
\sum_{v \in V} d_G(v) = 2|E|
\]

**Proof.** A very useful technique in this course, based on counting something in two different ways.

We will count the number of (distinct) pairs \((e, v)\)

such that \(e \in E\), \(v \in V\), and \(v\) is incident to \(e\).

**count #1.** Based on edges. Let \(e \in E\).

How many pairs involve \(e\)?

How many vertices can form a pair with \(e\)?

2 vertices: \(e = (u, v)\)

2 pairs will be counted: \((e, u), (e, v)\).

Every edge appears in 2 pairs.

**TOTAL # of pairs:** \(2|E|\)

**count #2.** Based on vertices. Let \(v \in V\).

How many pairs involve \(v\)?

How many edges are incident to \(v\)?

As many as: \(|N_G(v)| = d_G(v)|\)
As many as \( |N_G(v)| = d_G(v) \)

Every vertex appears in \( d_G(v) \) pairs.

**TOTAL # of pairs:** \[ \sum_{v \in V} d_G(v) \]

Therefore \[ \sum_{v \in V} d_G(v) = 2|E| \].

**Corollary.** The number of vertices of odd degree in a graph is always even.

**Proof.** By contradiction. Suppose the number were odd.

\[
2|E| = \sum_{v \in V} d_G(v) = \sum_{d_G(v) \text{ odd}} d_G(v) + \sum_{d_G(v) \text{ even}} d_G(v)
\]

Summing an odd number of odd degrees

= odd number

But \( 2|E| \) is even! Contradiction!

Therefore the corollary is true.

**Example.** We will call a graph in which all degrees are equal to \( r \) a \( r \)-regular graph.

How many edges does a 3-regular graph on 5 vertices have?

**Answer.**

This is a **TRICK QUESTION**...

**No such graph exists!**

**No regular graph with odd degree**
No regular graph with odd degree and odd number of vertices exists!