Lecture 1: Graphs and their properties

Chapters 1.1, 1.3, 1.4

Topics for today:

- What is a graph?
- Terminology and basic properties
- Degrees and neighborhoods
- Special types of graphs

What is a graph?
- A pair of sets \((V, E)\)
  - \(V\) contains nodes or vertices (singular vertex)
  - \(E\) contains edges or unordered pairs of vertices

- Example:

\[
V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \quad E = \{(v_1, v_2), (v_2, v_6), (v_3, v_6), (v_4, v_5)\}
\]

Which relationships can be modeled by a graph?

- Group of people, people sharing a favorite sports team
- Mathematicians, co-authors (of papers)
- Intersections, roads (in a city)
- Book, words

Simple graphs
Drawing graphs:
- Vertices are points or nodes
- Edges are
  - Segments connecting vertices
  - Arcs connecting vertices
  - Curves connecting vertices

Example:

- Graphs can be drawn in more than one way

Adjacency and incidence:
- \((u,v)\) edge then \(u\) and \(v\) are adjacent
- \(e = (u,v)\) we say \(u\) and \(v\) are endpoints of \(e\)
- \( e = (u,v) \) we say \( u \) and \( v \) are **endpoints** of \( e \)

- \( e \) is a edge **incident** to \( v \) if \( v \) is an endpoint for \( e \)

Other types of graphs:

- **Multigraphs**, allow for **multiple edges** between same pair of vertices

- **Pseudographs**, allow for loops

- **Digraphs**, edges are **ordered**. Removing ordering yields the underlying graph.

- **Hypergraphs**, edges are sets of **more** than 2 vertices

Degrees and neighborhoods:

- \( v \) a vertex, the **neighborhood of** \( v \), \( N_G(v) \), is the set of vertices adjacent to \( v \)

\[
N_G(v) = \{w, w_2, x\} \quad \text{and} \quad N_G(y) = \{w, x, z\}
\]

\[
d_G(v) = |N_G(v)| = 3 \quad \text{and} \quad d_G(y) = |N_G(y)| = 4
\]

\[
\Delta(G) = 4 \quad \text{and} \quad \delta(G) = 1
\]

- **Degree of** \( v \), denoted \( d_G(v) \), is the size of the neighborhood of \( v \), \( d_G(v) = |N_G(v)| \)
- If \( d_G(v) = 0 \), we say \( v \) is isolated.

- We may drop \( G \) if it is clear from the context.

\[
\mathcal{N}(v) = \mathcal{N}_G(v), \quad d_G(v) = d(v)
\]

- **Minimum** degree of a graph, \( \delta(G) = \min\{d_G(v) : v \in V\} \)

- **Maximum** degree of a graph, \( \Delta(G) = \max\{d_G(v) : v \in V\} \)

Special types of graphs:

- **Complete graphs** or **cliques** \( K_n \)

  - Edge set is all possible edges.
  
  \[
  n = \# \text{ of vertices} = |V|
  \]

  \[
  (\binom{n}{2}) = \frac{n(n-1)}{2}
  \]

- **Bipartite** graphs

  - \( V = V_1 \cup V_2 \) is a partition of \( V \)
  
  \[
  |V_1| = m, \quad |V_2| = n
  \]

  - Edges occur only between \( V_1 \) and \( V_2 \)

- **Complete** bipartite graph \( K_{m,n} \)

  \[
  E \text{ consists of all possible edges between } V_1, \quad |V_1| = m
  \]
  
  \[
  \text{and } \quad V_2, \quad |V_2| = n
  \]

  Total number of edges = \( mn \)
- A **k-cycle** $C_k$ with $V = \{1, 2, 3, \ldots, k\}$ has $E = \{(1, 2), (2, 3), \ldots, (k-1, k), (k, 1)\}$

  $k = 5$

  ![Graph of a 5-cycle](image)

  **$C_5$ has length 5**

- A **k-path** $P_k$ with $V = \{1, 2, 3, \ldots, k+1\}$ has $E = \{(1, 2), (2, 3), \ldots, (k, k+1)\}$

  $k+1 = 6$

  ![Graph of a 6-path](image)

  **$P_6$ has length 5**