**LAST CLASS.**

- $f: \mathbb{R}^3 \to \mathbb{R}$. $\nabla f$ is normal to the level set of $f$.

\[ f(x, y, z) = c. \]

**Equation for tangent plane to level set...**

If $f(x_0, y_0, z_0) = c$ and $\nabla f(x_0, y_0, z_0) \neq 0$ then the level $c$ level set of $f$ has tangent plane:

\[ \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0. \]

§3.1 2 ND DERIVATIVES

If $F: \mathbb{R}^2 \to \mathbb{R}$

Four 2nd derivatives:

\[
\begin{bmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{bmatrix}
\]

Hessian of $f$
§3.3. Extrema of functions

Want to find mins/maxs of multivariable functions.

\[ Z = f(x, y) \]

How do we find the top?

Observation: The tangent plane at the top is flat. The directional derivative in every direction is constant.

\[ \nabla f(x_0, y_0) = (0, 0). \]

A critical point of \( f(x, y) \) is a point where

\[ \nabla f(x_0, y_0) = (0, 0) \]

(or where \( f \) is not differentiable).
**THEOREM.** Local mins/maxs occur at critical points.

3 IMPORTANT EXAMPLES.

\[ f(x,y) = -x^2 - y^2. \]
\[ \nabla f(0,0) = (0,0). \]

Local max!

downwards paraboloid

\[ f(x,y) = x^2 + y^2. \]
\[ \nabla f(0,0) = (0,0) \]

Local min!

\[ f(x,y) = x^2 - y^2. \]
\[ \nabla f(0,0) = (0,0) \]

Saddle point!
THEOREM (2ND DERIVATIVE TEST)

Let \( f(x,y) \) be a function with critical point at \((x_0, y_0)\). Let

\[
H = \text{Hess}(f)(x_0, y_0) = \begin{bmatrix}
\frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\
\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0)
\end{bmatrix}.
\]

A. \text{If } \frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0 \text{ AND } \det H > 0 \text{ then } (x_0, y_0) \text{ is a local min.}

B. \text{If } \frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0 \text{ AND } \det H > 0 \text{ then } (x_0, y_0) \text{ is a local max.}

C. \text{If } \det H < 0 \text{ then } (x_0, y_0) \text{ is a saddle point (neither a local min or a local max).}
3 IMPORTANT EXAMPLES.

1. \( f(x, y) = -x^2 - y^2 \)
   \( \nabla f(0,0) = (0,0) \)
   \( H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \)
   \( \frac{\partial^2 f}{\partial x^2} < 0 \)
   \( \det H = 4 > 0 \)
   local max!
   downwards paraboloid

2. \( f(x, y) = x^2 + y^2 \)
   \( \nabla f(0,0) = (0,0) \)
   \( H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \)
   \( \frac{\partial^2 f}{\partial x^2} > 0 \)
   \( \det H = 4 > 0 \)
   local min!

3. \( f(x, y) = x^2 - y^2 \)
   \( \nabla f(0,0) = (0,0) \)
   \( H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \)
   \( \det H = -4 < 0 \)
   saddle point!
QUIZ TIME!