Gradients & Directional Derivatives.

\[ f : \mathbb{R}^2 \to \mathbb{R} \text{ (or } \mathbb{R}^3 \to \mathbb{R} \text{).} \]

The directional derivative of \( f \) at a point \((x_0, y_0) \in \mathbb{R}^2\) in the direction of a vector \( \mathbf{v} \) is the number

\[ \nabla f(x_0, y_0) \cdot \mathbf{v} \]

**Motto:** \( \nabla f(x_0, y_0) \cdot \mathbf{v} \) is the rate of change of \( f \) at the point \((x_0, y_0)\) if you move in the \( \mathbf{v} \)-direction.

**Theorem:** The gradient \( \nabla f \) points in the direction that \( f \) is increasing the fastest.

**Graphically:** If you graph \( z = f(x, y) \)

The gradient \( \nabla f \) points in the steepest ascending direction.
Exercise. Let \( f(x,y) = x + e^{x-y} \).

1. In what direction does \( f \) increase most quickly at \((x,y) = (0,0)\)?

2. Compute the directional derivative at \((0,0)\) in the direction \((1,2)\).

\[ \nabla f(0,0) = \left( \frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0) \right) \]
\[ \frac{\partial f}{\partial x} = 1 + e^{x-y} \quad \frac{\partial f}{\partial y} = -e^{x-y} \]
\[ \nabla f(0,0) = (2, -1). \]

**ANS #1** The direction that \( f \) is increasing most quickly is given by

**ANS #2** The directional derivative is:

\[ \nabla f(0,0) \cdot (1,2) = (2,-1) \cdot (1,2) \]
\[ = 2 - 2 = 0. \]
THEOREM. Let \( f: \mathbb{R}^2 \to \mathbb{R} \) be a function. Assume \( f(x_0, y_0, z_0) = c \) if \((x_0, y_0, z_0)\) lies on the level set of level \( c \), if
\[
\nabla f(x_0, y_0, z_0) \neq (0, 0, 0)
\]
then it is a normal vector to the level set of level \( c \).

EXAMPLE.
\[
f(x, y) = x^2 + y^2.
\]
\[
\nabla f = (2x, 2y)
\]
TANGENT LINES/PLANES TO LEVEL SETS...

If \( f(x_0, y_0, z_0) = c \) and \( \nabla f(x_0, y_0, z_0) \neq (0, 0, 0) \)
then it gives a normal vector to the level set...

**Equation for tangent plane:**

\[
\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.
\]

**Example:** \( x^2 + 2y^2 + z^2 = 10 \)
is the level 10 level set of \( f(x, y, z) = x^2 + 2y^2 + z^2 \).
The point \((2, 1, 2)\) is on this level set:

![Tangent plane at (2,1,2)](image)

\[
\nabla f(2, 1, 2) \cdot (x - 2, y - 1, z - 2) = 0.
\]

\[
\nabla f = (2x, 4y, 2z)
\]

\[
\nabla f(2, 1, 2) = (4, 4, 4)
\]

**Eqn for tangent plane:** \((4, 4, 4) \cdot (x - 2, y - 1, z - 2) = 0.\)

\[
4(x - 2) + 4(y - 1) + 4(z - 2) = 0.
\]
The gradient assigns each point in $\mathbb{R}^2$ a vector.

**Example**

\[
f(x, y) = y - \cos(x).
\]

\[
\nabla f = (\sin(x), 1).
\]
§3.1 Iterating Partial Derivatives

If $f(x,y)$ is twice differentiable (or a $C^2$-function) then there are 4 2nd derivatives we can take:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right).$$

We put these in a $2 \times 2$ matrix, called the Hessian of $f$:

$$\text{Hess}(f) = \begin{bmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{bmatrix}.$$
**Theorem.** If \( f \) is \( C^2 \), then:

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.
\]

So... order of differentiation doesn't matter for nice functions.

**Exercise.** Let \( f(x,y) = x^3 + xy \).

**Compute** \( \text{Hess}(f) \) at the point \((1,1)\).

**Ans.**

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= 6x, & \frac{\partial^2 f}{\partial x \partial y} &= 1, \\
\frac{\partial^2 f}{\partial y^2} &= 0.
\end{align*}
\]

\[
\text{Hess}(f) = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}
\]

\[
\text{Hess}(f)(1,1) = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}.
\]