Finding mins & maxs of $f(x,y)$ on $A$.

**Strategy:**

1. Find all critical points in $A$.

2. Parametrize the boundary points of $A$ & use 1D methods to find the max/mins on the boundary.

3. Compare the values from 1 & 2 to find the max/min.
Exercise: Find the max & min of 

\[ f(x, y) = xe^{xy} \]

on the set:

\[ \begin{align*}
\text{side 1:} & \quad (x, 0) \quad 0 \leq x \leq 1 \\
\text{side 2:} & \quad (1, y) \quad 0 \leq y \leq 1 \\
\text{side 3:} & \quad (x, 1) \quad 0 \leq x \leq 1 \\
\text{side 4:} & \quad (0, y) \quad 0 \leq y \leq 1 
\end{align*} \]

(Hint: parametrize the boundary points in 4 parts.)

Critical points:

\[ \nabla f = \left( e^{xy} + xy e^{xy}, x^2 e^{xy} \right) \]

\[ x^2 e^{xy} = 0 \quad \rightarrow \quad x = 0. \]

\[ (1+xy)e^{xy} = 0 \quad \rightarrow \quad \text{impossible.} \]

No critical points.

Boundary:
\[
\begin{align*}
\text{(Side 1)} & \quad f(x,0) = xe^0 = x. \\
& \quad \max = 1 \quad \min = 0.
\end{align*}
\]

\[
\begin{align*}
\text{(Side 2)} & \quad f(1,y) = e^y = e^y. \quad 0 \leq y \leq 1. \\
& \quad \min = e^0 \quad \max = e^1 = e.
\end{align*}
\]

\[
\begin{align*}
\text{(Side 3)} & \quad f(x,1) = xe^x. \\
& \quad \frac{dxe^x}{dx} = e^x + xe^x = (1+x)e^x. \\
& \quad \text{Critical point at } x = -1. \quad \text{(not in the set)} \\
& \quad \min = f(0,1) = 0 \quad f(1,1) = e = \max
\end{align*}
\]

\[
\begin{align*}
\text{(Side 4)} & \quad f(0,y) = 0. \quad \min = \max = 0.
\end{align*}
\]
53.4. Constrained Extrema & Lagrange Multipliers.

Last section we saw it was useful to find max/mins on the boundary sets.

Sometimes these sets are level sets (or several level sets).

**Question.**

**What is a critical point of a function on a level set?**

**Answer.**

LAGRANGE MULTIPLIERS

(this is the hardest topic in the class).
Want to find critical points on $g(x,y) = c$

If the directional derivative:
$$\nabla f(x_0, y_0) \cdot \vec{v} > 0$$
then $f$ increases in the direction of $\vec{v}$ (it decreases in the opposite direction).

So: the critical points are where $\nabla f(x_0, y_0)$ is perpendicular to the level set.

BUT: if $\nabla g(x_0, y_0) \neq 0$ then we know it is also perpendicular to the level set.
THEOREM (LAGRANGE MULTIPLIERS)

Let \( g(x,y)=0 \) be a level set. Let \( f(x,y) \) be a function. A local max or min of \( f(x,y) \) on \( S \) occurs at a point \((x_0,y_0) \in S\) either where:

1. \( \nabla g(x_0,y_0) = 0 \), or
2. \( \nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0) \). (\( \lambda \in \mathbb{R} \) is a scalar).
EXAMPLE. \( f(x,y) = x^2 - y^2 \). Want to maximize \( f \) on the set \( 4x^2 + y^2 = 1 \).

A. Find where \( \nabla g = (0,0) \) on \( S \).
\[
\nabla g = (8x, 2y) = (0,0)
\]
(only when \( x, y = (0,0) \))

BUT \( (0,0) \) is not on \( S \).

B. \( \nabla f = (2x, -2y) = \lambda (8x, 2y) \).

\[
2x = \lambda 8x \quad (2-\lambda 8)x = 0 \quad x = 0 \text{ or } \lambda = \frac{1}{8}
\]

\[-2y = \lambda 2y \quad (2+2\lambda)y = 0 \quad y = 0 \text{ or } \lambda = -1.
\]