This is the end.

**FINAL EXAM DETAILS**

1. Date/Time: Thursday 6/10
   11:30 - 2:30 pm
2. Open book, open notes.

3. There will be a practice final posted online this weekend.

4. I will have office hours next week (3 total hours; time TBA).

5. If you are happy with your QUIZ + HW grades, you don’t need to take the final.

6. For the final exam, you need to show up to Zoom meeting with camera ON.

7. There will be an email with asynchronous instructions.

¿QUESTIONS?
LAST CLASS.

Changing the order of integration.

$$\int_a^b \left( \int_{g_i(x)}^{g_2(x)} f(x,y) \, dy \right) \, dx = \int_{g_1(x)}^{g_2(x)} \left( \int_a^b f(x,y) \, dx \right) \, dy.$$

**Step 1** Draw the region of integration:

![Diagram showing a region of integration](image)

(A y-simple region)

**Step 2** Express this as an x-simple region.

**Step 3** Integrate over the y-simple region.

$$\cdots = \int_c^d \left( \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \right) \, dy.$$
Exercise: Change the order of integration and evaluate the following integrals.

(I) \[ \int_{-1}^{1} \left( \int_{0}^{x^2} dy \right) dx \]

(II) \[ \int_{0}^{6} \left( \int_{\sqrt[3]{x}}^{2} x \sqrt{1+y^3} \, dy \right) dx \]

Ans. (I):

- Region of Integration.
- As an x-simple region.
  \[ y - 1 \leq x \leq 1 - y \]
  \[ 0 \leq y \leq 1 \]
Integrate

\[ \int_0^1 \int_{0}^{1-y} x^2 \, dx \, dy = \int_0^1 \left( \frac{x^3}{3} \right)_{x=1-y}^{x=0} \, dy \]

\[ = \int_0^1 \frac{(1-y)^3}{3} - \frac{(y)^3}{3} \, dy = \int_0^1 \frac{2}{3} (1-y)^3 \, dy. \]

Let \( u = 1-y \), \( du = -dy \).

\[ \int \frac{2}{3} u^3 \, du = \int \frac{2}{3} u^3 \, du = -\frac{2}{12} u^4 = -\frac{1}{6} (1-y)^4. \]

\[ = \left( -\frac{1}{6} (1-y)^4 \right)_{y=0} - (-\frac{1}{6}) = \frac{1}{6} \]

\textbf{Ans. (I)}. 

\[ \int_0^1 \left( \int_{x/3}^{2} x \sqrt{1+y^3} \, dy \right) \, dx. \]

Region of Integration.

As an \( x \)-simple region.

\( 0 \leq x \leq 3 \)

\( 0 \leq y \leq 2. \)
New integral.

\[
\int_0^2 \left( \int_0^{3y} \frac{x^2}{2} \sqrt{1+y^3} \, dx \right) \, dy
\]

\[
= \int_0^2 \left[ \frac{x^2}{2} \sqrt{1+y^3} \right]_{x=0}^{x=3y} \, dy
\]

\[
= \int_0^2 \frac{9y^2}{2} \sqrt{1+y^3} \, dy.
\]

\[u = 1+y^3, \quad du = 3y^2 \, dy.\]

\[
\int \frac{9y^2}{2} \sqrt{1+y^3} \, dy = \int \frac{3}{2} \sqrt{u} \, du
\]

\[
= u^{3/2} = (1+y^3)^{3/2}.
\]

\[
= \left( (1+y^3)^{3/2} \right)_{y=0}^{2} = (1+2^3)^{3/2} - (1)^{3/2}
\]

\[
= 9^{3/2} - 2^{3/2} = 27 - \sqrt{8}.
\]
A 2D trick for solving 1D integrals.

**Differentiating under the integral.**

1. Consider \( \int_0^\infty x^n e^{-x} \, dx = \Gamma(n+1) \) (Gamma function)

We want to find \( \Gamma(n+1) \).

A. \( \Gamma(1) = \int_0^\infty x^0 e^{-x} \, dx = \left[-e^{-x}\right]_0^\infty = 1 \)

B. \( \Gamma(2) = \int_0^\infty x e^{-x} \, dx \)

**Option 1:** Integration by parts. BORING.

**Option 2:** Consider:

\[
\int_0^\infty e^{-tx} \, dx = \text{constant} + 20
\]

\[u = tx \]

\[du = t \, dx\]

\[= \int_0^\infty \frac{e^{-u}}{t} \, du = \frac{1}{t} \int_0^\infty e^{-u} \, du\]
\[
= \frac{1}{t} \Gamma(1) = \frac{1}{t}.
\]

\[
\int_0^\infty e^{-tx} \, dx = \frac{1}{t}.
\]

\[
\frac{d}{dt} \left[ \int_0^\infty e^{-tx} \, dx \right] = \int_0^\infty \frac{d}{dt} e^{-tx} \, dx
\]

\[
= \int_0^\infty e^{-tx} \, dx.
\]

\[
\frac{d}{dt} \left( \frac{1}{t} \right) = -\frac{1}{t^2}.
\]

So...

\[
\frac{1}{t^2} = \int_0^\infty e^{-tx} \, dx.
\]

When \( t = 1 \),

\[
1 = \int_0^\infty e^{-x} \, dx = \Gamma(2).
\]
C. Take mth derivatives of both sides

\[ \frac{1}{t} = \int_{0}^{\infty} e^{-tx} \, dx \]

and set \( t=1 \) to find \( \Gamma(n+1) \).

**Left hand side**

\[
\frac{d}{dt} \left( \frac{1}{t} \right) = -\frac{1}{t^2} = \frac{d}{dt} \left[ \int_{0}^{\infty} e^{-tx} \, dx \right] = \int_{0}^{\infty} -xe^{-tx} \, dx
\]

\[
\frac{d}{dt} \left( \frac{d}{dt} \left( \frac{1}{t} \right) \right) = \frac{2}{t^3} = \frac{d}{dt} \left[ \frac{d}{dt} \left( \int_{0}^{\infty} e^{-tx} \, dx \right) \right] = \int_{0}^{\infty} x^2e^{-tx} \, dx
\]

\[
\vdots
\]

\[
\frac{d}{dt} \left( \frac{d^{m-1}}{dt^{m-1}} \left( \frac{1}{t} \right) \right) = \frac{(-1)^m n!}{t^{n+1}} = \frac{d}{dt} \left[ \frac{d^{m-1}}{dt^{m-1}} \left( \int_{0}^{\infty} e^{-tx} \, dx \right) \right] = \int_{0}^{\infty} (-1)^m x^ne^{-tx} \, dx.
\]

**Right hand side**

**nth derivative**

Get \( \frac{(-1)^m n!}{t^{n+1}} = \int_{0}^{\infty} (-1)^m x^ne^{-tx} \, dx \)

Set \( t=1 \):

\[ n! = \int_{0}^{\infty} x^ne^{-x} \, dx = \Gamma(n+1) \]
That's all! Have a great summer!