LAST CLASS.

- The derivative of \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \) is the matrix of partial derivatives:
  \[
  Df = \begin{bmatrix}
  \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}
  \end{bmatrix}.
  \]

- The gradient of \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \) is the 3D vector
  \[\n  \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).\]

- The linear approximation of \( f \) at \((x_0,y_0,z_0)\), using the gradient:
  \[
  l(x,y,z) = f(x_0,y_0,z_0) + \nabla f(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0).
  \]
EXAMPLE. \( f(x,y) = \sin(y-x) \).

The linear approximation at \((0,0)\).
\[
\nabla f = (-\cos(y-x), \cos(y-x))
\]

\[
l(x,y) = f(0,0) + \nabla f(0,0) \cdot (x-0, y-0) \\
= 0 + (-1,1) \cdot (x,y) = -x+y.
\]

THE DERIVATIVE in HIGHER DIMENSIONS (a preview)

A function \( F : \mathbb{R}^n \to \mathbb{R}^m \) is given by

\[
F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).
\]

\( F \) is differentiable at a point if each \( f_1, \ldots, f_m \) is differentiable there. The derivative is the \( m \times n \) matrix

\[
DF = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}
\]

\( m \) rows, \( n \) columns.
A **scalar function** is \( f : \mathbb{R}^2 \to \mathbb{R} \) (or \( \mathbb{R}^2 \to \mathbb{R} \)).

A **path** is a function

\[
\mathbf{z} : \mathbb{R} \to \mathbb{R}^2 \quad \text{or} \quad \mathbb{R} \to \mathbb{R}^2.
\]

\[
\mathbf{z}(t) = (x(t), y(t)).
\]

A **curve** is the range or image of a path.

**Example**

\[
\mathbf{z}(t) = (2 \cos(t), 5 \sin(t)).
\]
The derivative (or velocity) of a path is the vector:

\[ \overrightarrow{C}'(t) = (x'(t), y'(t)) \].

**Remark.** If \( \overrightarrow{C}'(t) = (0, 0) \), then it is tangent to the curve at time \( t \).

**Example.**

We can parametrize the tangent line at \( t = \pi/4 \) by:

\[ \overrightarrow{C}'(\pi/4) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \].
\[ \ell(t) = \Xi(\pi/4) + t \Xi'(\pi/4) \]
\[ = (\sqrt{2}/2, \sqrt{2}/2) + t(-\sqrt{2}/2, \sqrt{2}/2) \]
\[ \ell(t) = (\sqrt{2} - t\sqrt{2}, \sqrt{2} + t\sqrt{2}). \]

The **speed** of a path at time \( t \) is the magnitude: \( \| \Xi'(t) \| \).

**Example.** \( \Xi(t) = (\cos(t), t, \sin(t)) \).
Exercise 1. Find the speed of \( \mathbf{c}(t) \) as a function of time.

2. Parametrize the tangent line to the helix at the point:

\[(x, y, z) = (0, \pi/2, 1).\]

ANS #1

\[\mathbf{c}'(t) = (-\sin(t), 1, \cos(t))\]

\[\|\mathbf{c}'(t)\| = \sqrt{\sin^2 t + 1^2 + \cos^2 t} = \sqrt{2}.\]

ANS #2

\[t = \pi/2.\]

\[\mathbf{c}'(\pi/2) = (-1, 1, 0).\]

Parametrized by:

\[\mathbf{l}(t) = \mathbf{c}(\pi/2) + t\mathbf{c}'(\pi/2) = (0, \pi/2, 1) + t(-1, 1, 0).\]

\[= (-t, \pi/2 + t, 1). \checkmark.\]
A. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, and $\lambda \in \mathbb{R}$. Then $\lambda f(x,y)$ is differentiable and:

$$D \lambda f(x,y) = \lambda D f(x,y)$$

$$= \left[ \lambda \frac{\partial f}{\partial x} \quad \lambda \frac{\partial f}{\partial y} \right].$$

(AND $\nabla \lambda f(x,y) = \lambda \nabla f(x,y)$)

B. If $f$ and $g$ are differentiable, then:

$$D (f(x,y) + g(x,y)) = D f(x,y) + D g(x,y).$$

$$= \left[ \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \quad \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} \right].$$
C. \[ D(f(x,y)g(x,y)) = f(x,y)Dg(x,y) + g(x,y)Df(x,y). \]

D. Quotient Rule. If \( g(x,y) \neq 0 \) then
\[
D\left(\frac{f(x,y)}{g(x,y)}\right) = \frac{gDf - fDg}{g^2}.
\]

**EXAMPLE.**

\[ f(x,y) = x^2 \quad g(x,y) = \sin(y). \]

\[ f \cdot g = x^2 \sin(y) \]

\[ Df \cdot g = \begin{bmatrix} 2x \sin(y) & x^2 \cos(y) \end{bmatrix}. \]

\[ Df = \begin{bmatrix} 2x \end{bmatrix} \quad Dg = \begin{bmatrix} 0 & \cos(y) \end{bmatrix} \]

\[
\sin(y)Df + x^2Dg = \begin{bmatrix} 2x \sin(y) & 0 \end{bmatrix} + \begin{bmatrix} 0 & x^2 \cos(y) \end{bmatrix}.
\]

\[
= \begin{bmatrix} 2x \sin(y) & x^2 \cos(y) \end{bmatrix}.
\]