1. If $\varphi \in \text{Aut}(\Gamma)$ and $v \in V$, then $\varphi(v)$ has the same degree as v (i.e. they are incident to the same number of edges). In particular, φ preserves the central vertex of Γ (which has degree 4, while the others have degree 3). However, φ can send the topmost vertex v to any of the four outer vertices (for example φ could be a rotation). The bottom vertex must be sent to the one opposite (i.e. not adjacent to) $\varphi(v)$. There are two places where φ can send the leftmost vertex. Both are possible: one is given by the rotation taking v to $\varphi(v)$, and the other is given by first flipping the graph horizontally. This gives a total of $4 \times 2 = 8$ choices for φ. In fact $\text{Aut}(\Gamma) \cong D_8$, because D_8 acts faithfully on Γ.

2. In Problem 1 we saw that the central vertex c is fixed by every automorphism of Γ, so $\text{Orb}(c) = \{c\}$. By rotating we see that the orbit of the topmost vertex consists of the four outer vertices. Therefore the action of $\text{Aut}(\Gamma)$ on V has two orbits.

3. Automorphisms of Γ preserve the central vertex c, so they also preserve the set of edges incident to c. By rotating we see that the orbit of such an edge consists of all such edges, and similarly the orbit of an "outer edge" (i.e. one not incident to c) consists of all the outer edges. Therefore the action of $\text{Aut}(\Gamma)$ on E has two orbits.

4. Let $S := \text{Fun}(V, \{1, \ldots, k\})$ be the set of colourings of V with k not necessarily distinct colours. The number of such colourings up to symmetry is the number of orbits of the natural action of $\text{Aut}(\Gamma)$ on S, which is given by Burnside's formula:

$$\frac{1}{|\text{Aut}(\Gamma)|} \sum_{\varphi \in \text{Aut}\,\Gamma} |S^\varphi|.$$

Note that $S^e = S$ has size k^5, because each of the five vertices can have any of k colours. For notational simplicity we will identify $\text{Aut}(\Gamma)$ with D_8 (so r is no longer a function $\mathbb{R} \to \mathbb{R}$, it is just the corresponding automorphism of Γ). If a colouring is fixed by r, then the outer vertices must have the same colour, because the colour of the leftmost vertex is the same as the topmost, which is the same as the rightmost colour and so on. This leaves only two choices of colour, so $|S^r| = k^2$. Similarly $|S^{r^3}| = k^2$. However $|S^{r^2}| = k^3$, because r^2 just swaps the two pairs of opposite vertices, so there are colourings fixed by r^2 in which the topmost and leftmost vertices (for instance) have different colours. Since s and sr^2 just swap two vertices, $|S^s| = |S^{sr^2}| = k^4$. On the other hand sr and sr^3 (which are reflections in the diagonal axes) behave more like r^2, so $|S^{sr}| = |S^{sr^3}| = k^3$. Adding these up gives the following number of colourings up to symmetry:

$$\frac{k^5 + 2k^4 + 3k^3 + 2k^2}{8} = \frac{k^2(k^3 + 2k^2 + 3k + 2)}{8} = \frac{k^2(k+1)(k^2 + k + 2)}{8}.$$

5. If $\varphi \in \text{Aut}(\Gamma)$ is not the identity, then it moves at least one outer vertex $v \in V$, so it also moves the edge connecting v to the central vertex c. Therefore φ does not preserve any

colouring of E with k distinct colours. In other words, the stabiliser of such a covering is $\{e\}$. By the orbit-stabiliser theorem, the corresponding orbit has size $|\mathrm{Aut}(\Gamma)| = 8$. The total number of colourings is $\frac{k!}{(k-8)!}$ (i.e. the number of injective functions $E \to \{1, \ldots, k\}$, which we interpret as 0 when $k < 8$), so the number of orbits is

$$\frac{k!}{8(k-8)!}.$$

6. The key is to notice that there is an automorphism $\psi \in \mathrm{Aut}(\Gamma)$ which sends each vertex to the opposite one (i.e. the one it is not adjacent to). In fact two vertices $u, v \in V$ are adjacent (or equal) if and only if $\psi(u) \neq v$, so a permutation $\varphi \in A(V)$ corresponds to an automorphism iff it preserves the relation $\psi(u) = v$, i.e. iff $\psi(u) = v \Leftrightarrow \psi(\varphi(u)) = \varphi(v)$. The latter statement just means that $\psi\varphi = \varphi\psi$, so $\mathrm{Aut}(\Gamma)$ is precisely the centraliser of ψ in $A(V)$ (and in particular $\psi \in \mathrm{Aut}(\Gamma)$). The orbit of the topmost vertex therefore contains the opposite vertex. By rotating, we see that this orbit also contains the other vertices. Therefore the action of $\mathrm{Aut}(\Gamma)$ on V has only one orbit.

7. As in Problem 6, the orbit of any vertex $v \in V$ has size 6. Any $\varphi \in \mathrm{Stab}(v)$ also fixes $\psi(v)$, because $\varphi(\psi(v))$ is not adjacent (or equal) to $\varphi(v)$. In other words φ is a permutation of $U := V - \{v, \psi(v)\}$, which commutes with the restriction $\psi|_U$ of ψ to U. Conversely, any such permutation f defines an element of $\mathrm{Aut}(\Gamma)$ that fixes v, essentially because $\psi f \psi^{-1}$ fixes v and $\psi(v)$. Therefore $|\mathrm{Stab}(v)| = |C(\psi|_U)| = |A(U)|/|\mathrm{Cl}(\psi|_U)| = 4!/3 = 8$ (since $\psi|_U$ swaps two pairs of vertices, it has the form $(a\ b)(c\ d)$, and hence three conjugates). By the orbit-stabiliser theorem, it follows that $|\mathrm{Aut}(\Gamma)| = 6 \times 8 = 48$.

   In fact, we could have shown directly that $|\mathrm{Aut}(\Gamma)| = |C(\psi)| = |A(V)|/|\mathrm{Cl}(\psi)| = 6!/15$, using the fact that ψ has the form $(a\ b)(c\ d)(e\ f)$, and hence 15 conjugates (5 choices for the image of one vertex, and 3 for the next one).

8. Like in Problem 6, the trick is to describe adjacency in a nice way. In this case, every pair of distinct vertices is adjacent, so every permutation of V corresponds to an automorphism of Γ. Therefore $|\mathrm{Aut}(\Gamma)| = |A(V)| = 4! = 24$.

9. You can use Burnside's formula here (as in Problem 4). The only issue is that $\mathrm{Aut}(\Gamma)$ has a lot of elements. But since permutations with the same cycle type act in much the same way, you only need to look at a representative from each of the five conjugacy classes.

   There is an even better way. The rough idea is that, because $\mathrm{Aut}(\Gamma)$ is all of $A(V)$, we can forget which vertex is which and just keep track of which colours appear, and how many times they do. To a colouring of Γ we can associate a string with 4 stars and $k - 1$ bars. The bars divide the string into k pieces, corresponding to the different colours. The number of

stars between each bar determines the number of times the corresponding colour appears. For example, if $k = 5$, two vertices have colour 1, one has colour 2 and one has colour 4, the corresponding string is $* * \,|\, * \,||\, * \,|$.

Applying a permutation to a colouring does not change the number of times a given colour appears, so this association gives a well-defined function f from the set of orbits of our action to the set of strings described above. Every such string arises from some colouring: just order the vertices in some way and put the colour of the ith star on the ith vertex. In other words, f is onto. If two colourings determine the same string, there is an automorphism (i.e. a permutation) sending one to the other: for each colour, the two sets of vertices with that colour (from the two colourings) have the same size, so we can pick a bijection between them, and combine these into a bijection $V \to V$. In other words, f is 1-1. So the number of orbits of our action is the number of strings with 4 stars and $k - 1$ bars, which is the number of ways to choose 4 elements of $\{1, \ldots, 4 + k - 1\}$ (corresponding to the positions of the stars in the string). This number is $\binom{4+k-1}{4} = \binom{k+3}{4}$.