1. A linear function
\[ f: \mathbb{R}^2 \to \mathbb{R} \]
has the form: \( f(x,y) = ax + by + cy \).

2. A differentiable function:
\[ f(x,y) : \mathbb{R}^2 \to \mathbb{R} \]
can be "well-approximated" at every point \((x_0, y_0)\) by a linear function \( l(x,y) \):
\[
\lim_{(x,y) \to (x_0,y_0)} \frac{f(x,y) - l(x_0,y_0)}{||(x,y) - (x_0,y_0)||} = 0.
\]

\( l(x,y) \) is the linear approximation of \( f(x,y) \) at \((x_0,y_0)\).
3. If \( f(x, y) \) has linear approximation
\[ l(x, y) = a + bx + cy \]

at \((x_0, y_0)\), then \( l(x, y_0) = f(x_0, y_0) \)

AND

\[
\frac{\partial f}{\partial x}(x_0, y_0) = b \quad \frac{\partial f}{\partial y}(x_0, y_0) = c.
\]

4. The partial derivative \( \frac{\partial f}{\partial x}(x, y) \) can be computed by treating \( y \) as a constant and differentiating with respect to \( x \).

5. The derivative of a scalar function
\[ f(x, y) : \mathbb{R}^2 \to \mathbb{R} \quad (\text{or} \quad g(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}) \]
is the \( (2 \times 1) \) (or \( 1 \times n \) matrix) \( x \)

\[
\begin{bmatrix}
\frac{\partial f}{\partial x}(x, y) \\
\frac{\partial f}{\partial y}(x, y)
\end{bmatrix}
\]

(or \( \begin{bmatrix}
\frac{\partial g}{\partial x_1} \\
\vdots \\
\frac{\partial g}{\partial x_n}
\end{bmatrix} \))

The gradient of a scalar function is the vector:

\[
\nabla f(x, y) = \left( \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right)
\]

(or \( \nabla g(x_1, \ldots, x_n) = \left( \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n), \ldots, \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n) \right) \)).
**Example.** \( f(x, y) = x^2 \sin(y) \)

\[ \nabla f(1, 1) = \begin{pmatrix} 2 \cdot 1 \cdot \sin(1) \\ 1^2 \cos(1) \end{pmatrix} = \begin{pmatrix} \sin(1) \\ \cos(1) \end{pmatrix} \]

**Example.** The linear approximation

\[ g(w, x, y, z) = w^2 + xy + z^2 \]

At \((2, 1, 0, 3) = (w, x, y, z)\),

\[ \frac{\partial g}{\partial w} = 2w, \quad \frac{\partial g}{\partial x} = y, \quad \frac{\partial g}{\partial y} = x, \quad \frac{\partial g}{\partial z} = 2z \]

\[ \nabla g(2, 1, 0, 3) = (4, 0, 1, 6) \]

\[ l(w, x, y, z) = g(2, 1, 0, 3) + 4(w - 2) + 0(\overline{x} - 1) + 1(\overline{y} - 0) + 6(\overline{z} - 3) \]

\[ = 4 + 0 + 9 + 4w - 8 + 0 + y + 6z - 18 \]

\[ = 4w + y + 6z - 13 \]
Gew"{u}l Formula

The linear approximation of \( f(x, y) \) at \((x_0, y_0)\) is:

\[
ell(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot ((x, y) - (x_0, y_0))
\]

QUESTIONS?

ASIDE: derivatives in higher dimensions ...

(not important yet ...)

A function \( F: \mathbb{R}^n \to \mathbb{R}^m \)

1. takes \( n \) inputs \( x \)s. 2. outputs \( m \) \( y \)s.

\( F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \)

\( F \) is differentiable if each \( f_1, \ldots, f_m \) is differentiable.

The derivative is an \( m \times n \) matrix.
§2.4 - INTRODUCTION TO PATHS & CURVES

A path is a function \( \ell(t) : [a, b] \rightarrow \mathbb{R}^2 \) (or \( \mathbb{R}^3 \)).

A curve is the range of a path.

"The path parameterizes the curve"
**Example:**

\[ f(t) = (t+1, 3-t) \]

\[ t \in [0, 1). \]

**Example:**

\[ f(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]. \]
The derivative of a path $\overline{C}(t) = (x(t), y(t))$ is the vector:

$$\overline{C}'(t) = (x'(t), y'(t)).$$

- If $\overline{C}'(t_0) = (x'(t_0), y'(t_0)) \neq (0,0)$ then it is tangent to the curve at $\overline{C}(t_0)$.
- As a consequence, we can parameterize the tangent line with the equation:

$$f(t) = \overline{C}(t_0) + t \overline{C}'(t_0).$$

**Example**

$$\overline{C}'(\frac{3\pi}{2}) = (-\sin(\frac{3\pi}{2}), \cos(\frac{3\pi}{2})),$$

$$= (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}).$$

$$\overline{C}(\frac{3\pi}{2}) = (\cos(\frac{3\pi}{2}), \sin(\frac{3\pi}{2}))$$

$$= (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}).$$

$l(t) = (-\frac{\sqrt{2}}{2}t, \frac{\sqrt{2}}{2}t) - t(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}).$