LAST TIME:

- Defined open the open disc, open sets, and open neighborhoods of points.

- The limit as \((x,y) \rightarrow (a,b)\) of \(f(x,y)\) is \(L\) if "as \((x,y)\) approaches \((a,b)\) \(f(x,y)\) approaches \(L\)."

- A function \(f(x,y)\) is continuous if for every point \((a,b) \in \mathbb{R}^2\)

\[
\lim_{(x,y) \to (a,b)} f(x,y) = L.
\]

Theorem: The composition of continuous functions is continuous.
An example of a function with no limit as \((x,y) \to (0,0)\)

\[ f(x,y) = \frac{x^2}{x^2+y^2}. \]

Along the line \(y=\alpha x\) the function we have:

\[ f(x,y) = f(x,\alpha x) = \frac{x^2}{x^2+\alpha^2 x^2} = \frac{x^2}{(1+\alpha^2)x^2} = \frac{1}{1+\alpha^2}. \]

1. \(f(x,y)\) is constant along \(L\)

2. But for different slopes, there are different constants.

**Ex.** Along \(y=x\)

\[ f(x,x) = \frac{1}{2}. \]

Along \(y=2x\).

\[ f(x,2x) = \frac{1}{5}. \]

So \( \lim_{(x,y) \to (0,0)} \frac{x^2}{x^2+y^2} \) does not exist.
2.3 • DIFFERENTIATION

SLOGAN: “Differential functions are the ones that can be approximated by linear functions.”

This begs the question...

Q: What is a linear function???

A linear function:
\[ f: \mathbb{R}^2 \rightarrow \mathbb{R}. \]
\[ f(x,y) = ax + by + c \]
\[ a, b, c \text{ are scalars.} \]

Other:
\[ g: \mathbb{R}^3 \rightarrow \mathbb{R}. \]
\[ g(x, y, z) = ax + by + cz + d \]
A function \( f(x,y) \) is \textbf{DIFFERENTIABLE} at \( (x_0, y_0) \in \mathbb{R}^2 \) if there is a linear function \( l(x,y) \) so that:

\[
\lim_{{(x,y) \to (x_0, y_0)}} \frac{f(x,y) - l(x,y)}{||l(x,y) - (x_0, y_0)||} = 0.
\]

If \( f(x,y) \) is differentiable we say \( l(x,y) \) is the \textbf{LINEAR APPROXIMATION}.

If \( l(x,y) = a + bx + cy \) is the linear approximation, then

- the partial derivative \( \frac{\partial f}{\partial x} (x_0, y_0) = b \).
- the partial derivative \( \frac{\partial f}{\partial y} (x_0, y_0) = c \).
How do you compute partial derivatives?

If \( f(x,y) \) is differentiable at \( (x_0, y_0) \) then

\[
\frac{df}{dx}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}.
\]

(Note: \( f(x_0+h, y_0) \) is a 1-variable function of \( h \).)

**Note:** Treat \( y \) as constant and differentiate with respect to \( x \).

**Example:** \( f(x,y) = e^{xy} \).

\[
\frac{df}{dx}(x,y) = ye^{xy}.
\]
Example. \( g(x,y,z) = \sin(xz) + y^2z \).

\[
\frac{\partial g}{\partial z}(1,2,3) = 1 \cdot \cos(1 \cdot 3) + 2^2 \\
= 4 + \cos(4)
\]

Exercise. The function
\[ g(x,y,z) = 1 + x^2 + xy + yz \]
is differentiable on all of \( \mathbb{R}^3 \). Compute the linear approximation
\[ l(x,y,z) = a + bx + cy + dz \]
of \( g(x,y,z) \) at \((1,1,1) \in \mathbb{R}^3\).
\begin{align*}
\text{Ans.} & \quad \frac{\partial g}{\partial x} = 2x + y. & b & = \frac{\partial g}{\partial x}(1,1,1) = 2 + 1 = 3 \\
& \frac{\partial g}{\partial y} = x + z. & c & = \frac{\partial g}{\partial y}(1,1,1) = 1 + 1 = 2, \\
& \frac{\partial g}{\partial z} = y. & d & = \frac{\partial g}{\partial z}(1,1,1) = 1. \\
\end{align*}

\[ e(x,y,z) = a + 3x + 2y + z. \]

What is \( a \)? We know:

\[ 0 = \lim_{(x,y,z) \to (1,1,1)} \frac{g(x,y,z) - (a + 3x + 2y + z)}{||(x,y,z) - (1,1,1)||}. \]

This can only happen if:

\[ 4 = g(1,1,1) = a + 3 + 2 + 1 = a + 6. \]

So \( a = -2 \).
A shortcut:

The linear approximation of the function $g(x,y,z)$ at $(x_0, y_0, z_0)$ is

$$
e(x,y,z) = g(x_0,y_0,z_0) + \frac{\partial g}{\partial x}(x_0,y_0,z_0)(x-x_0)$$
$$+ \frac{\partial g}{\partial y}(x_0,y_0,z_0)(y-y_0) + \frac{\partial g}{\partial z}(x_0,y_0,z_0)(z-z_0).$$

Let $f(x,y)$ (or $g(x_1,\ldots,x_n)$) be a differentiable function. The derivative of $f(x,y)$ at $(x_0,y_0)$ is the 1x2 matrix:

$$Df(x_0,y_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \end{bmatrix}.$$

The gradient of $f(x,y)$ at $(x_0,y_0)$ is

$$\nabla f(x_0,y_0) = \left( \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \right).$$
Example \[ f(x,y) = x^2 \sin(y) \]

\[ \begin{align*}
\n0 \quad Df(x_0, y_0) &= \begin{bmatrix}
2x_0 \sin(y_0) & x_0^2 \cos(y_0)
\end{bmatrix}, \\
0 \quad \nabla f(1, 1) &= (2 \sin(1), \cos(1)).
\end{align*} \]

Example \[ g(\omega, x, y, z) = \omega^2 + xy + z^2 \]

\[ \nabla g(\omega, x, y, z) = (2\omega, y, x, 2z). \]