Wednesday, November 25
press record.

Last time:

• Introduced the arc length of a curve.

\[ \ell_c = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

Length of \( C \) = \( \sum_{i=0}^{n-1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, \Delta t \)

= \[ \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

(Note: this formula works for curves in any dimension.)
COMMENT: Sometimes the arclength can differ from the actual length of $C$ …

**Example 1:** The arclength of the path:

$$
\mathbf{C} : [0, 4\pi] \to \mathbb{R}^2
$$

\[ t \mapsto (\cos(t), \sin(t)) \]

can be computed by:

\[
\text{Arclength} = \int_0^{4\pi} \| \mathbf{C}'(t) \| \, dt
\]

\[
= \int_0^{4\pi} \| (-\sin t, \cos t) \| \, dt
\]

\[
= \int_0^{4\pi} \sqrt{\sin^2 t + \cos^2 t} \, dt
\]

\[
= \int_0^{4\pi} 1 \, dt = 4\pi.
\]
In this example... we are parametrizing the unit circle, but we go around twice.

BUT. If a path

\[ C : [a, b] \rightarrow \mathbb{R}^n \]

parametrizes a curve \( C \) and is one-to-one (or: one-to-one with finitely many exceptions), then

\[ \text{Arc length} = \text{length of } C. \]
Exercise: The helix is parametrized by the path

\[ \vec{r}(t) = (\cos(t), \sin(t), t). \]

1. X-y-coordinates are on the unit circle.
2. z-coordinate goes up linearly.

Find the arclength of the helix for

\[ 1 \leq t \leq 2\pi. \]
\[
\text{Arc length} = \int_0^{2\pi} \sqrt{\left(2e^t\right)^2 + 4} \, dt \\
= \int_0^{2\pi} \sqrt{4 + 4r^2} \, dt = \sqrt{2} \sqrt{r^2 \pi}.
\]
**Example.** Consider the path:

\[ \mathbf{c}(t) = (2t, t^2, \log t) \]

for \( t > 0 \).

We want the arclength between \( t = 1 \) and \( t = 2 \).

\[ \text{Arclength} = \int_1^2 \| \mathbf{c}'(t) \| \, dt \]
\[
\int_1^2 \sqrt{4 + (2t)^2 + \left(\frac{1}{t}\right)^2} \, dt.
\]

\[
= \int_1^2 \sqrt{4t^2 + 4 + \left(\frac{1}{t}\right)^2} \, dt
\]

\[
= \int_1^2 \sqrt{(2t + \frac{1}{t})^2} \, dt
\]

\[
= \int_1^2 2t + \frac{1}{t} \, dt
\]

\[
= \left. \left( t^2 + \log t \right) \right|_1^2
\]

\[
= 4 + \log 2 - 1 - 0
\]

\[
= 3 + \log(2).
\]
HAPPY THANKSGIVING!