MONDAY, NOVEMBER 2ND.

Please record.

Tomorrow is election day. Go vote!

LAST TIME:

General version of the chain rule.

1. For functions
   \[ F: \mathbb{R}^2 \to \mathbb{R}^3 \]
   and \[ G: \mathbb{R}^2 \to \mathbb{R}^3 \],

   The chain rule says

   \[ D(G \circ F)(x,y) = DG(F(x,y)) \cdot DF(x,y) \]

   \[ 3 \times 3 \text{ matrix} \times \ 3 \times 2 \text{ matrix} \]

2. Discussed matrix multiplication:

   If \[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]
   and \[ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \]

   then \[ C = A \cdot B \] is \( 3 \times 2 = (3 \times 3) \cdot (3 \times 2) \)
DON'T FORGET TO VOTE!

The entries of $C$ are the dot products of (row vectors) of $A$ and (column vectors) of $B$.

3. Explicitly: the entry in the $i$th row and the $j$th column of $C$ is the dot product of the $i$th row vector of $A$ and the $j$th column vector of $B$.

(You can write this:

$$c_{ij} = \sum_{k=1}^{3} a_{ik} \cdot b_{kj}$$

)
DON'T FORGET TO VOTE!

13.6.0 Gradients and Directional Derivatives

(recall)...
The gradient of \( f(x, y, z) \) is the vector:

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).
\]

Example. \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \).

Then \( \nabla f = \left( \frac{2x}{\sqrt{x^2 + y^2 + z^2}}, \frac{2y}{\sqrt{x^2 + y^2 + z^2}}, \frac{2z}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{1}{\sqrt{x^2+y^2+z^2}} \nabla f(x, y, z).

Geometrically, at each point \( (x, y, z) \in \mathbb{R}^3 \),

\( \nabla f \) points in the same direction as \( (x, y, z) \).

\( 2 \) has length \( = 1 \).
Don't forget to vote!

Let $\mathbf{v} \in \mathbb{R}^3$ be a vector.
We can make the dot product:

$$\nabla f(x_0, y_0, z_0) \cdot \mathbf{v}.$$  

(A). This is a real number, called THE DIRECTIONAL DERIVATIVE.

At $(x_0, y_0, z_0)$ IN THE DIRECTION OF $\mathbf{v}$.

(B). Geometrically, the directional derivative represents how fast $f$ is changing at $(x_0, y_0, z_0)$ if you move in the direction of $\mathbf{v}$. 

Example. Consider the composition

\[ \mathbf{c} : \mathbb{R} \rightarrow \mathbb{R}^3 \]
\[ \mathbf{c}(t) = (1, 2, 3) + t(-1, 0, 1) \]

(a line through (1,2,3) with velocity (-1,0,1))

and \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \).
\[ f(x, y, z) = \sin(xyz). \]

Then \[ \frac{df \circ \mathbf{c}}{dt} (0) = \nabla f(\mathbf{c}(0)) \cdot \mathbf{c}'(0) \]

(the Chain Rule).

\[ = \nabla f(\mathbf{c}(0)) \cdot (-1, 0, 1) \]

(the directional derivative at (1,2,3) in the (-1,0,1) direction)
\[ \nabla f = (yz \cos(xy^2), xz \cos(xyz), xy \cos(yxz)) \]
\[ \nabla f(1,2,3) = (6 \cos(6), 3 \cos(6), 2 \cos(6)) \]
\[ \nabla f(1,2,3) \cdot (0,0,1) = -6 \cos(6) + 2 \cos(6) = -4 \cos(6). \]

**Theorem.** The gradient \( \nabla f \) at \((x_0, y_0, z_0)\) points in the direction that \( f \) is increasing the fastest.

---

**QUIZ #3**

---

**DON'T FORGET TO VOTE!**