LAST TIME.

- Optimizing functions on closed and bounded sets.

\[ A \subseteq \mathbb{R}^2, \quad f : \mathbb{R}^2 \rightarrow \mathbb{R} \]

\[ \text{Pt1. Find all the critical points on the interior of } A \ ( \nabla f = 0 ). \]

\[ \text{Pt2. Parametrize the boundary } \partial A \text{ (maybe in multiple parts) } \]

\[ \text{Use 1D methods to find the max/min on } \partial A. \]
Pt 3. Compare the value at all critical pts from Pt 1. with the extrema from pt 2.

Exercise. Find the minimum and maximum value of \( f(x, y) = xe^{xy} \) on the square:

\[ f(x, y) = xe^{xy} \]

Pt 1. 
\[ \nabla f = (1 + xy)e^{xy}, -xe^{xy} \]

\[ = e^{xy} (1 + xy, -x^2) \]

\[ \nabla f = (0, 0) \text{ means } x = 0. \]

but then \((1 + xy) \neq 0.\)
SO NO SOLUTIONS.

P+2. Parametrize $A=4$
parts:

\[0 \leq x \leq 1\, , \ y = 0 \ or \ 1\]
\[x = 0 \ or \ 1\, , \ 0 \leq y \leq 1.\]

(A) $0 \leq x \leq 1\, , \ y = 0$.  
\[f(x, 0) = xe^0 = x\]
\[\min f = 0.\]
\[\max f = 1.\]

(B) $0 \leq x \leq 1\, , \ y = 0$.  
\[f(0, y) = 0 \cdot e^0 = 0\, \text{constant.}\]

(C) $0 \leq x \leq 1\, , \ y = 1$.  
\[f(x, 1) = xe^x\]
\[\frac{ef(x, 1)}{dx} = (x+1)e^x.\]
\[\text{(No crit pts for } 0 \leq x \leq 1).\]
So check boundary value:
\[f(0, 1) = 0, \ f(1, 1) = e.\]

(D) \[x = 1\, , \ 0 \leq y \leq 1\, , \ f(1, y) = e^y.\]
\[ \begin{align*}
\min f(x, y) &= 1 
\max f(x, y) &= e.
\end{align*} \]

So...

- global min = 0.
- global max = e.

\section*{3.4 Constrained Extrema and Lagrange Multipliers.}

Last section: Needed to optimize functions on \( \partial A \).

LAGRANGE MULTIPLIERS help solve this problem when \( \partial A \) is a level set.
Suppose \( C \subset \mathbb{R}^2 \) is a curve \( f : \mathbb{R}^2 \to \mathbb{R} \).

The idea:

If \( \vec{v} \) is a small tangent vector to \( C \) at \( (x_0, y_0) \), then:

\[
\nabla f(x_0, y_0) \cdot \vec{v} = \left( \text{the directional derivative at } (x_0, y_0) \text{ in the direction of } \vec{v} \right)
\]
If $\nabla f \cdot \vec{v} > 0$ then $f$ increases in the $\vec{v}$ direction

AND $\nabla f \cdot (-\vec{v}) < 0$ so $f$ decreases in the $-\vec{v}$-direction.

So... $f$ does not have a local min/max at $(x_0, y_0)$.

The only way $(x_0, y_0)$ can be a local min/max is if $\nabla f(x_0, y_0)$ is normal to $C$ at $(x_0, y_0)$. 

$\nabla f(x_0, y_0) \approx \left( \text{the change of } f \text{ when you move from } (x_0, y_0) \rightarrow (x_0, y_0) + \vec{u} \right)$. 

The notation $\nabla f(x_0, y_0)$ represents the gradient of $f$ at the point $(x_0, y_0)$.
**Example**

Where is the function 

\[ f(x,y) = x + 2y \]

maximized/minimized on the unit circle?

\[ \nabla f = (1, 2) \]

Where is \((1, 2)\) normal to \(C\)?

\(C\) is given by \(x^2 + y^2 = 1\) and has normal vector \((2x, 2y)\).
So... $(1,2)$ is normal to $C$ at $(x,y)$ when $(1,2)$ is a scalar multiple of $(2x,2y)$.

$$(1,2) = \lambda (2x,2y).$$

$$1 = \lambda 2x$$
$$2 = \lambda 2y.$$ 

So

$$\frac{1}{\lambda} = \begin{array}{c}
2x = y.
\end{array}$$

AND...

$$x^2 + y^2 = x^2 + 2x^2 = 1$$

So

$$3x^2 = 1 \quad x = \pm \frac{1}{\sqrt{3}}$$

$$y = \pm \frac{2}{\sqrt{3}}.$$
**Theorem (LAGRANGE MULTIPLIERS)**

Let \( g : \mathbb{R}^n \to \mathbb{R} \) be a scalar function, let

\[
S = \text{level set of } g \text{ of level } c
\]

Suppose \((x_1, \ldots, x_n) \in S\) (so \( g(x_1, \ldots, x_n) = c \)). Assume there is another function:

\[
f : \mathbb{R}^n \to \mathbb{R}
\]

AND that \((x_1, \ldots, x_n)\) is a local maximum for \( f \) on \( S \). If

\[
\nabla g(x_1, \ldots, x_n) \neq (0, \ldots, 0)
\]

then

\[
\nabla f(x_1, \ldots, x_n) = \lambda \nabla g(x_1, \ldots, x_n)
\]

for some \( \lambda \in \mathbb{R} \).