NAÏVE NONCOMMUTATIVE BLOWUPS AT ZERO-DIMENSIONAL SCHEMES: AN APPENDIX.

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Various peripheral results in the paper [RS2] were stated without proof and the aim of this appendix (which will not appear in print) is to provide the details. We refer the reader to [RS2] for the relevance of the results proved here.

The following assumptions will remain in force throughout the appendix, while undefined terms can be found in [RS2].

Assumptions 1.1. Let X be a integral projective scheme of dimension $d \geq 2$. Fix $\sigma \in \mathrm{Aut}(X)$ and a σ-ample invertible sheaf $\mathcal{L}$. Finally assume that $Z \subseteq X$ is a saturating 0-dimensional subscheme of X. We will always write the *bimodule algebra* as

$$\mathcal{R} = \mathcal{R}(X, Z, \mathcal{L}, \sigma) = \mathcal{O}_X \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots ,$$

where $\mathcal{R}_n = \mathcal{L}_n \otimes_{\mathcal{O}_X} \mathcal{I}_n$, for $\mathcal{L}_n = \mathcal{L} \otimes \sigma^*\mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L}$ and $\mathcal{I}_n = \mathcal{I} \cdot \sigma^*\mathcal{I} \cdots (\sigma^{n-1})^*\mathcal{I}$. This bimodule algebra has a natural multiplication and the *naïve blowup algebra of X at Z* is then the algebra of sections

$$R \;=\; R(X, Z, \mathcal{L}, \sigma) \;=\; \mathrm{H}^0(X, \mathcal{R}) \;=\; k \oplus \mathrm{H}^0(X, \mathcal{R}_1) \oplus \mathrm{H}^0(X, \mathcal{R}_2) \oplus \cdots$$

By [RS2, Theorem 3.1], R is noetherian with qgr-$R \simeq$ qgr-$\mathcal{R}$.

Proposition 1.2. [RS2, Proposition 3.20]. *Keep the above assumptions and assume that $\mathcal{L}$ is also ample and generated by its global sections. Then there exists $M \in \mathbb{N}$ such that, for $m \geq M$:*

(1) *$\mathcal{I}_n \otimes \mathcal{L}_n^{\otimes m}$ is generated by its global sections for all $n \geq 1$.*

(2) *$R(X, Z, \mathcal{L}^{\otimes m}, \sigma)$ is generated in degree* 1.

Proof. (1) The proof of [KRS, Proposition 4.12(1)] can be used without change, except that the reference to [KRS, Proposition 4.6] is replaced by one to [RS2, Theorem 3.1(1)].

(2) This requires slightly more argument than that of [KRS, Proposition 4.12(2)]. For the rest of the proof we set $\mathcal{N} = \mathcal{L}^\sigma$, and for any coherent sheaf $\mathcal{F}$, write $\operatorname{reg}\mathcal{F} = \operatorname{reg}_{\mathcal{N}}\mathcal{F}$ for the Castelnuovo-Mumford regularity of $\mathcal{F}$. Set $r = \max\{1, \operatorname{reg}\mathcal{O}_X\}$. Also, for $m \geq 1$ we abbreviate $\mathcal{L}^{\otimes m}$ by $\mathcal{L}^m$.

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For any $m \geq m_0$, part (1) provides a short exact sequence

$$0 \to \mathcal{K}_m \to \mathrm{H}^0(X, \mathcal{I} \otimes \mathcal{L}^m) \otimes \mathcal{O}_X \to \mathcal{I} \otimes \mathcal{L}^m \to 0, \tag{1.3}$$

for some sheaf $\mathcal{K}_m$. The proof of [KRS, Proposition 4.12(2)] can be used unchanged to find $M \geq 0$ such that

$$\operatorname{reg}(\mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) \leq 1 \qquad \text{for all } m \geq M. \tag{1.4}$$

In particular, $H^1(X, \mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) = 0$. From here on, however, we need to be a little more careful.

Tensoring (1.3) with $(\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma$ gives the exact sequence

$$0 \to \mathcal{G} \xrightarrow{\theta} \mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma \longrightarrow \mathrm{H}^0(X, \mathcal{I} \otimes \mathcal{L}^m) \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma \to \mathcal{I} \otimes \mathcal{L}^m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma \to 0. \tag{1.5}$$

Since $(\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma$ is locally free outside a finite set, $\mathcal{G}$ has finite support and so $\mathrm{H}^j(X, \mathcal{G}) = 0$ for $j > 0$. Setting $\mathcal{H} = \operatorname{Coker} \theta$, it follows that $\mathrm{H}^1(X, \mathcal{H}) = \mathrm{H}^1(X, \mathcal{K}_m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) = 0$. Thus taking cohomology of (1.5) shows that the induced map

$$\mathrm{H}^0(X, \mathcal{I} \otimes \mathcal{L}^m) \otimes \mathrm{H}^0(X, (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) \longrightarrow \mathrm{H}^0(X, \mathcal{I} \otimes \mathcal{L}^m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma)$$

is a surjection. On the other hand, the multiplication map $\mu : \mathcal{I} \otimes \mathcal{L}^m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma \to \mathcal{I}_{n+1} \otimes \mathcal{L}_{n+1}^m$ is surjective with a kernel of finite length. Hence $\mathrm{H}^1(X, \operatorname{Ker}(\mu)) = 0$ and we obtain a surjection

$$\mathrm{H}^0(X, \mathcal{I} \otimes \mathcal{L}^m \otimes (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) \twoheadrightarrow \mathrm{H}^0(X, \mathcal{I}_{n+1} \otimes \mathcal{L}_{n+1}^m).$$

Combining the last two displayed equations shows that

$$\mathrm{H}^0(X, \mathcal{I} \otimes \mathcal{L}^m) \otimes \mathrm{H}^0(X, (\mathcal{I}_n \otimes \mathcal{L}_n^m)^\sigma) \longrightarrow \mathrm{H}^0(X, \mathcal{I}_{n+1} \otimes \mathcal{L}_{n+1}^m)$$

is also surjective. This is equivalent to the assertion of part (2). □

The next example is mentioned after the proof of [RS2, Lemma 4.7], which we state for the reader's convenience. The natural map $\text{Gr-}R \to \text{Qgr-}R$ is denoted by π.

Lemma 1.6. ([RS2, Lemma 4.7]) *Suppose that* $\mathcal{M} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \text{gr-}\mathcal{R}$ *is coherent, and that* $\mathcal{N} = \bigoplus \mathcal{G}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \text{Gr-}\mathcal{R}$ *is definable by products in the sense of* [RS2, Section 4]*.*

1. *For some* n_1*, there is a natural isomorphism* $\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{M}), \pi(\mathcal{N})) \cong \lim_{n \geq n_1} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}_n, \mathcal{G}_n)$.
2. *Suppose that* $\mathcal{M}$ *and* $\mathcal{N}$ *are coherent Goldie torsion modules and, by* [RS2, Lemma 4.1]*, write* $\mathcal{F}_n = \mathcal{F}$ *and* $\mathcal{G}_n = \mathcal{G}$ *for* $n \gg 0$*. Then there is a natural isomorphism*

$$\operatorname{Hom}_{\text{Qgr-}\mathcal{R}}(\pi(\mathcal{M}), \pi(\mathcal{N})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$ □

An important special case of Lemma 1.6 occurs when $\mathcal{M} = \mathcal{R}$. In this case, [KRS, Lemma 6.4] shows that Lemma 1.6(1) holds for all $\mathcal{R}$-modules $\mathcal{N}$ provided one naïvely blows up a single point. However, when one blows up more than one point at once, then Lemma 1.6(1) can fail, as the next example illustrates.

Example 1.7. Let $\mathcal{R} = \mathcal{R}(\mathbb{P}^2, Z, \mathcal{O}(1), \sigma)$ where Z is the reduced subscheme $\{p, \sigma^{-1}(p)\}$ for some closed point p lying on a critically dense σ-orbit. The key property here is that the natural surjective map $\mathcal{I}_n \otimes \mathcal{I}^{\sigma^n} \to \mathcal{I}_{n+1}$ is never an isomorphism for $n \geq 1$, since locally at the point $q = \sigma^{-n}(p)$ it looks like the map $\mathfrak{m}_q \otimes \mathfrak{m}_q \to \mathfrak{m}_q^2$ in the local ring $\mathcal{O}_{X,q}$. Now let $\mathcal{M} = \mathcal{R} = \bigoplus_{n\geq 0} \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ where $\mathcal{F}_n = \mathcal{I}_n$ and fix some $i \geq 0$. Define $\mathcal{N}^{(i)}$ by putting $\mathcal{H} = \mathcal{I}_i \otimes \mathcal{L}_\sigma^{\otimes i}$ into degree i and letting $\mathcal{N}^{(i)} = \mathcal{H} \otimes \mathcal{R}$; thus $\mathcal{N}^{(i)} = \bigoplus_{n\geq 0} \mathcal{G}_n \otimes \mathcal{L}_\sigma^{\otimes n}$ where $\mathcal{G}_n = 0$ for $n < i$ and $\mathcal{G}_n = \mathcal{I}_i \otimes \mathcal{I}_{n-i}^{\sigma^i}$ for $n \geq i$. Then the structure map $\theta_i : \mathcal{G}_i \otimes \mathcal{I}^{\sigma^i} \to \mathcal{G}_{i+1}$ is simply the isomorphism $\mathcal{I}_i \otimes \mathcal{I}^{\sigma^i} \to \mathcal{I}_i \otimes \mathcal{I}^{\sigma^i}$.

Now suppose we take the identity morphism $Id \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{G}_i) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_i, \mathcal{I}_i)$. Then this determines the identity morphism $Id \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_i \otimes \mathcal{I}^{\sigma^i}, \mathcal{G}_i \otimes \mathcal{I}^{\sigma^i}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_i \otimes \mathcal{I}^{\sigma^i}, \mathcal{I}_i \otimes \mathcal{I}^{\sigma^i})$. However, there is no reasonable way for this to determine a morphism in $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_{i+1}, \mathcal{G}_{i+1}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{i+1}, \mathcal{I}_i \otimes \mathcal{I}^{\sigma^i})$. So for this $\mathcal{N}$, we need to take $n_1 \geq i+1$ in order to define the direct limit in Lemma 1.6(1). Now take $\mathcal{N} = \bigoplus_{i\geq 0} \mathcal{N}^{(i)} \in \text{Gr-}\mathcal{R}$. Then there is no choice of n_1 for which the direct limit in Lemma 1.6(1) is sensible. □

If $k(x)$ is the skyscraper sheaf at a closed point $x \in X$, set $\overline{x} = \bigoplus_{n\geq 0}(k(x) \otimes \mathcal{L}_\sigma^{\otimes n}) \in \text{Gr-}\mathcal{R}$ and write $\widetilde{x} = \pi(\overline{x}) \in \text{Qgr-}\mathcal{R}$. By [RS2, Lemma 4.1(2)], $\overline{x} \in \mathrm{GT}\,\text{gr-}\mathcal{R}$ and so $\widetilde{x} \in \text{qgr-}\mathcal{R}$. Define C_X to be the smallest localizing subcategory of $\mathcal{O}_X\text{-Mod}$ containing all the modules $\{k(c) | c \in \bigcup_{i\in\mathbb{Z}} \sigma^i(S)\}$, where $S = \mathrm{Supp}\,\mathcal{O}_X/\mathcal{I}$. Similarly, write $C_\mathcal{R}$ for the localizing subcategory of Qgr-$\mathcal{R}$ generated by the modules $\widetilde{c}$ for $c \in \bigcup_{i\in\mathbb{Z}} \sigma^i(S)$. The next result gives the analogue for naïve blowups of the classic result that, if $\rho : \widetilde{X} \to X$ is the (classical) blowup of X at a smooth point x, then $X \smallsetminus \{x\} \cong \widetilde{X} \smallsetminus \rho^{-1}(x)$.

Proposition 1.8. [RS2, Proposition 4.12]. *Keep the hypotheses of Assumptions 1.1. Then there is an equivalence of categories* $\mathcal{O}_X\text{-Mod}/C_X \simeq \text{Qgr-}\mathcal{R}/C_\mathcal{R}$.

Proof. Given a sheaf $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$, write $\overline{\mathcal{F}}$ for the corresponding object in $\mathcal{O}_X\text{-Mod}/C_X$. Set $D_X = C_X \cap \mathcal{O}_X\text{-mod}$ and $D_\mathcal{R} = C_\mathcal{R} \cap \text{qgr-}\mathcal{R}$. Define a map $\theta' : \mathcal{O}_X\text{-mod} \to \text{qgr-}\mathcal{R}$ by $\mathcal{F} \mapsto \pi(\mathcal{F} \otimes \mathcal{R})$.

Suppose that $c \in X$ is a closed point, with ideal sheaf $\mathcal{J}_c$, and consider $\bigoplus_{n\geq 0}(\mathcal{I}_n/\mathcal{J}_c\mathcal{I}_n) \otimes \mathcal{L}_n$. Note that $\mathcal{I}_n/\mathcal{J}_c\mathcal{I}_n \cong \bigoplus_{i=1}^{d_n} k(c)$ as $\mathcal{O}_X$-modules, where d_n is the minimum number of generators of $(\mathcal{I}_n)_c$ over the local ring $\mathcal{O}_{X,c}$. Since the points in $\mathrm{Supp}\,Z$ lie on infinite σ-orbits, the stalks $(\mathcal{I}_n)_c$ are all the same for $n \gg 0$, so d_n is a constant value d for $n \gg 0$. It is then easy to check that $\theta'(k(c)) = \bigoplus_{i=1}^{d} \widetilde{c}$ as objects in qgr-$\mathcal{R}$.

It follows that θ' induces a functor $\theta : \mathcal{O}_X\text{-mod}/D_X \to \text{qgr-}\mathcal{R}/D_\mathcal{R}$. Conversely, let $\mathcal{M} = \bigoplus \mathcal{F}_n \otimes \mathcal{L}_\sigma^{\otimes n} \in \text{gr-}\mathcal{R}$. Then [RS2, Lemma 4.4(2)] implies that, for some ω,

$$\mathcal{F}_n \mathcal{I}^{\sigma^n} \cong \mathcal{F}_{n+1} \qquad \text{for } n \geq \omega. \tag{1.9}$$

Thus, $\overline{\mathcal{F}}_n \cong \overline{\mathcal{F}}_{n+1}$ for all $n \geq \omega$ and so the rule $\psi' : \mathcal{M} \mapsto \overline{\mathcal{F}}_\omega$ defines a functor from qgr-$\mathcal{R}$ to $\mathcal{O}_X\text{-mod}/D_X$. For all $c \in \bigcup_{i\in\mathbb{Z}} \sigma^i(S)$, the object $\widetilde{c}$ maps to 0 and so there is an induced functor $\psi : \text{qgr-}\mathcal{R}/D_\mathcal{R} \to \mathcal{O}_X\text{-mod}/D_X$.

Next, we claim that for $\mathcal{F} \in D_X$, then $\mathcal{N} = \bigoplus_{n\geq 0} \mathcal{F} \otimes (\mathcal{L}_\sigma)^{\otimes n}$ satisfies $\pi(\mathcal{N}) \in D_{\mathcal{R}}$. To see this, we first note that if $\operatorname{Supp} \mathcal{F} \cap \{\sigma^{-i}(c) | i \geq 0, c \in \operatorname{Supp} Z\} = \emptyset$, then $\mathcal{N} = \mathcal{F} \otimes \mathcal{R} = \theta'(\mathcal{F})$ and we have already seen that $\pi(\mathcal{N}) \in D_{\mathcal{R}}$ in this case. But since every point of $\operatorname{Supp} Z$ lies on a dense orbit and $\operatorname{Supp} \mathcal{F}$ is finite, we can choose $m \geq 0$ so that $\mathcal{G} = (\mathcal{F} \otimes \mathcal{L}_m)^{\sigma^{-m}}$ has support disjoint from $\{\sigma^{-i}(c) | i \geq 0, c \in \operatorname{Supp} Z\}$. Then $\mathcal{N}' = \bigoplus_{n\geq 0} \mathcal{G} \otimes (\mathcal{L}_\sigma)^{\otimes n}$ satisfies $\pi(\mathcal{N}') \in D_{\mathcal{R}}$, and using [RS2, Lemma 4.3] we have that $\pi(\mathcal{N}') \cong \pi(\mathcal{N}[m])$ in qgr-$\mathcal{R}$. Since $D_{\mathcal{R}}$ is clearly closed under shifts, the claim is proved.

Now we need to check that θ and ψ are inverse functors. First, since $\mathcal{I}_n$ and $\mathcal{O}_X$ are isomorphic modulo D_X, it is clear that $\psi'\theta'(\mathcal{F}) = \overline{\mathcal{F}}$ for $\mathcal{F} \in \mathcal{O}_X$-mod. On the other hand, let $\mathcal{M} \in$ gr-$\mathcal{R}$ and ω satisfy (1.9), so that $\theta\psi'(\mathcal{M})$ is the image of the module $\mathcal{M}' = \mathcal{F}_\omega \otimes \mathcal{R} \in$ gr-$\mathcal{R}$. Then

$$\mathcal{M}'_{\geq \omega} = \bigoplus_{r \geq \omega} \mathcal{F}_\omega \otimes \mathcal{I}_\omega \mathcal{I}_{r-\omega}^{\sigma^\omega} \otimes \mathcal{L}_\sigma^{\otimes r}$$

whereas $\mathcal{M}_{\geq \omega} = \bigoplus_{r\geq \omega} \mathcal{F}_\omega \mathcal{I}_{r-\omega}^{\sigma^\omega} \otimes \mathcal{L}_\sigma^{\otimes r}$, by the choice of ω. Thus, there is a natural multiplication map $\alpha : \mathcal{M}'_{\geq \omega} \to \mathcal{M}_{\geq \omega}$. Then

$$\operatorname{Ker}(\alpha) = \bigoplus_{r\geq \omega} \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}_\omega, \mathcal{O}_X/(\mathcal{I}_\omega \mathcal{I}_{r-\omega}^{\sigma^\omega})) \otimes \mathcal{L}_\sigma^{\otimes r} \quad \text{and} \quad \operatorname{Coker}(\alpha) = \bigoplus_{r\geq \omega} (\mathcal{F}_\omega \mathcal{I}_{r-\omega}^{\sigma^\omega})/(\mathcal{F}_\omega \mathcal{I}_\omega \mathcal{I}_{r-\omega}^{\sigma^\omega}) \otimes \mathcal{L}_\sigma^{\otimes r}.$$

Notice that $\operatorname{Ker}(\alpha)$ must be a coherent $\mathcal{R}$-module, and it is clearly Goldie-torsion. Thus [RS2, Lemma 4.1] implies that the individual sheaves $\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}_\omega, \mathcal{O}_X/(\mathcal{I}_\omega \mathcal{I}_{r-\omega}^{\sigma^\omega}))$ have some stable value for $r \gg 0$, which is obviously in D_X. Similarly, $(\mathcal{F}_\omega \mathcal{I}_{r-\omega}^{\sigma^\omega})/(\mathcal{F}_\omega \mathcal{I}_\omega \mathcal{I}_{r-\omega}^{\sigma^\omega})$ has a stable value in D_X for $r \gg 0$. Hence both $\operatorname{Ker}(\alpha)$ and $\operatorname{Coker}(\alpha)$ have image under π which is in $D_{\mathcal{R}}$, by the claim proved in the previous paragraph.

Therefore, both $\theta\psi$ and $\psi\theta$ are naturally isomorphic to the identity and thus we have an equivalence $\mathcal{O}_X\text{-mod}/D_X \simeq$ qgr-$\mathcal{R}/D_{\mathcal{R}}$. It now follows formally, as in the proof of [KRS, Theorem 6.7], that this induces the desired equivalence $\mathcal{O}_X\text{-Mod}/C_X \simeq$ Qgr-$\mathcal{R}/C_{\mathcal{R}}$. □

The result [RS2, Theorem 6.9] gives upper bounds for the global dimension and cohomological dimension of Qgr-S for a generalized naïve blowup algebra S. As mentioned after the proof of that result, one can also prove the following lower bounds for these dimensions.

Theorem 1.10. *Let $S = S(X, \{\mathbb{I}_n\}, \mathcal{L}, \sigma)$ be a nontrivial generalized naïve blowup algebra in the sense of* [RS2, Definition 5.2]. *Then one has* $\dim X - 1 \leq \operatorname{cd}(\text{Qgr-}S)$. *If X is smooth, then* $\dim X \leq \operatorname{gld}(\text{Qgr-}S)$.

Proof. We start by considering cohomological dimension. As in the proof of [RS2, Theorem 6.9], we may first use [RS2, Corollary 5.4(4)] and [RS2, Lemma 6.8] to replace S by some large Veronese ring $S^{(p)}$ and thus assume by [RS2, Lemma 5.3] that $S = R(X, Z_{\mathcal{I}}, \mathcal{L}, \sigma)$ is a naïve blowup algebra. By [RS2, Proposition 2.10], it suffices to prove the result for $\mathcal{L} = \mathcal{O}_X$.

Fix some $c \in \operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ such that $x = \sigma(c) \notin \operatorname{Supp} \mathcal{O}_X/\mathcal{I}$. Now follow the proof of the lower bound in [KRS, Theorem 8.2], except that replace in order of appearance [KRS, Proposition 3.5] by [RS2, Proposition 2.10]; [KRS, Theorem 6.7] by [RS2, Theorem 4.10]; and [KRS, Theorem 6.8] by [RS2, Corollary 4.11(1)], to conclude that $\operatorname{Ext}^d_{\text{Qgr-}\mathcal{R}}(\widetilde{x}, \widetilde{x}) = \operatorname{Ext}^d_{\mathcal{O}_X}(k(x), k(x)) \neq 0$. Suppose that $\mathcal{F}$ is a coherent sheaf with finite support such that $x \in \operatorname{Supp} \mathcal{F}$. Since $\operatorname{gld} \mathcal{O}_{X,x} = d$, it easily follows that $\widetilde{\mathcal{F}} = \bigoplus_{n \geq 0} \mathcal{F} \otimes (\mathcal{O}_X)_\sigma^{\otimes n} \in \text{Qgr-}\mathcal{R}$ satisfies $\operatorname{Ext}^d_{\text{Qgr-}\mathcal{R}}(\widetilde{\mathcal{F}}, \widetilde{x}) = \operatorname{Ext}^d_{\mathcal{O}_X}(\mathcal{F}, k(x)) \neq 0$ as well.

Since $\mathcal{L} = \mathcal{O}_X$ we have $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{I}_n$ and $\mathcal{R}[+1] = \bigoplus_{n \geq 0} \mathcal{I}_{n+1}^{\sigma^{-1}}$. Clearly there is an inclusion of ideal sheaves $\mathcal{I}_{n+1}^{\sigma^{-1}} \subseteq \mathcal{I}_n$ for all $n \geq 0$ and so one obtains a short exact sequence in qgr-$\mathcal{R}$ of the form $0 \to \mathcal{R}[+1] \to \mathcal{R} \to \widetilde{\mathcal{F}} \to 0$. Here $\widetilde{\mathcal{F}} = \bigoplus_{n \geq 0} \mathcal{F}_n$ with $\mathcal{F}_n = \mathcal{I}_n / \mathcal{I}_{n+1}^{\sigma^{-1}}$ constant, say equal to a sheaf $\mathcal{F}$, for $n \gg 0$. Note in particular that $\mathcal{F}$ has finite support, indeed $\operatorname{Supp} \mathcal{F} \subseteq \operatorname{Supp} \mathcal{O}_X / \mathcal{I}^{\sigma^{-1}}$, and that $x \in \operatorname{Supp} \mathcal{F}$ by choice of c. Then applying $\operatorname{Hom}(-, \widetilde{x})$ to the short exact sequence, we have the following portion of the long exact sequence:

$$\operatorname{Ext}^{d-1}_{\text{Qgr-}\mathcal{R}}(\mathcal{R}[+1], \widetilde{x}) \to \operatorname{Ext}^d_{\text{Qgr-}\mathcal{R}}(\widetilde{\mathcal{F}}, \widetilde{x}) \to \operatorname{Ext}^d_{\text{Qgr-}\mathcal{R}}(\mathcal{R}, \widetilde{x}).$$

Now $\operatorname{Ext}^d_{\text{Qgr-}\mathcal{R}}(\mathcal{R}, \widetilde{x}) = 0$ by the first part of the proof of [KRS, Theorem 8.2], since $\widetilde{x}$ is Goldie torsion. We showed $\operatorname{Ext}^d_{\text{Qgr-}\mathcal{R}}(\widetilde{\mathcal{F}}, \widetilde{x}) \neq 0$ above, and so $0 \neq \operatorname{Ext}^{d-1}_{\text{Qgr-}\mathcal{R}}(\mathcal{R}[+1], \widetilde{x}) = \operatorname{Ext}^{d-1}_{\text{Qgr-}\mathcal{R}}(\mathcal{R}, \widetilde{x}[-1])$, completing the proof of the lower bound for cohomological dimension.

As noted in the proof of [KRS, Theorem 8.3], the calculation that $\operatorname{Ext}^d_{\text{Qgr-}\mathcal{R}}(\widetilde{x}, \widetilde{x}) \neq 0$ above proves the desired lower bound on $\operatorname{gld}(\text{Qgr-}R)$. $\square$

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