Math 31CH Spring 2018 Written Homework 5, due 6/01/2018 in HW box in the basement of AP&M by 4 pm

Reading

Read Sections 8.5-8.6 of the text.

Exercises to submit on Friday 6/1

These problems involve the concept of orientation. Recall that we take the following as our definition: An orientation on a k-manifold $M \subseteq \mathbb{R}^n$ is given by a smooth k-form ω on $\mathbb{R}^n$ which is nowhere-zero on M in the following sense: for each $p \in M$ and basis $\vec{v}_1, \dots, \vec{v}_k$ of the tangent space T_pM, we have $\omega(p)(\vec{v}_1, \dots, \vec{v}_k) \neq 0$. When this holds, the basis $\vec{v}_1, \dots, \vec{v}_k$ of $T_p(M)$ is called positive when $\omega(p)(\vec{v}_1, \dots, \vec{v}_k) > 0$ and negative when $\omega(p)(\vec{v}_1, \dots, \vec{v}_k) < 0$. Note that which bases are called positive and which negative depends on the choice of form. When M has an orientation, we call it *orientable*. The following problems are a series; each continues with the notation of the previous ones.

A. Let $M \in \mathbb{R}^n$ be a smooth $(n-1)$-dimensional manifold which is defined as a level set $F = 0$ of a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ such that $DF(p)$ has rank 1 for all $p \in M$.

For each $p \in M$, show that $\vec{n}(p) = DF(p)^T/||DF(p)^T||$ is a nonzero normal unit vector for M at p; in other words, $\vec{n}(p)$ is in the orthogonal complement to the tangent space T_pM. Show that $\vec{n} : M \to \mathbb{R}^n$ is a smooth function, in other words, the unit normal vectors vary smoothly as we move around the manifold.

B. Find a smooth $(n-1)$-form ω which is nowhere-zero on M and such that the corresponding orientation given by this form can also be described as follows: given a basis $\vec{v}_1, \ldots, \vec{v}_{n-1}$ of the tangent space T_pM of M at p, then the basis is positive if and only if the matrix whose first column is $\vec{n}_p$ and whose remaining columns are $\vec{v}_1, \ldots, \vec{v}_{n-1}$ in that order has positive determinant. Conclude that M is orientable.

C. Prove that the form ω found in (B) also has the following property: for each $p \in M$, $\omega(p) \in \Lambda^{n-1}(\mathbb{R}^n)^*$ is the function that assigns to each basis $\vec{v}_1, \ldots, \vec{v}_{n-1}$ of $T_p(S)$ the signed (n-1)-dimensional area of the parallelepiped they span. By signed area we mean this is equal to the area if $\vec{v}_1, \ldots, \vec{v}_{n-1}$ is a positive basis for the orientation found in (B), and it is minus the area if it is a negative basis for the orientation.

D. Because of part (C), $\int_M \omega$ represents the $(n-1)$-dimensional surface area of M. Now let $n \geq 1$ and let $S^{n-1} \subseteq \mathbb{R}^n$ be the $(n-1)$-dimensional unit sphere consisting of points $\{\vec{x} \in \mathbb{R}^n \big| \, ||\vec{x}|| = 1\}$, which is the level set over 0 of $F : \mathbb{R}^n \to \mathbb{R}$ given by $F(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 - 1$. Find the $(n-1)$-dimensional surface area of S^{n-1} for $n = 2, 3, 4$.

E. Let $B^n = \{\vec{x} \in \mathbb{R}^n \big| \, ||\vec{x}|| \leq 1\}$ be the unit ball in $\mathbb{R}^n$. Show that for any $n \geq 1$, the $(n-1)$-dimensional surface area of S^{n-1} is equal to n times the n-dimensional volume of B^n. For example, when $n = 2$, the unit circle has 1-dimensional surface area (length) equal to 2π, which is 2 times the 2-volume (area) π of the unit disc. (Hint: Stoke's theorem, which is in Section 8.5. We won't cover this in class until Wednesday 5/30.)