Math 31CH Spring 2018 Written Homework 3, due 5/4/2018 in HW box in the basement of AP&M by 4 pm

Reading

Read Sections 7.3 and 7.6 of the text. In Section 7.5, read the proof of Proposition 5.13. We will only lightly cover Section 7.4, so you are responsible only for the topics we cover in class.

Exercises to submit on Friday 4/20

Exercises from the text

Write out each answer as a careful proof, in full sentences.

Section 7.2: #7, 20, 23, 24

Hint: for #20, the fundamental theorem of calculus (see problem A below) may be useful.

Additional problems (not from the text)

A. Given a rational number $x \in \mathbb{Q}$, there are unique integers $p, q \in \mathbb{Z}$ such that $q > 0$ and $x = p/q$ is in lowest terms, in other words, there is no integer $n > 1$ such that $p = np'$ and $q = nq'$ for integers p', q'. Note that under this definition the fraction in lowest terms representing 0 is $0/1$.

Define a function $[0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/q & \text{if } x \in \mathbb{Q} \text{ with } x = p/q \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that f is integrable and that in fact $\int_0^1 f dx = 0$.

(Hint: first show that for any natural number n, there are only finitely many values of $x \in [0,1]$ such that $f(x) \geq 1/n$. Given a partition $\mathcal{P}$, on the subintervals not containing any of these values the maximum of the function will be at most $1/n$. By choosing a fine enough partition, the subintervals containing those finitely many values can be chosen to have arbitrarily small total length.)

B. Prove the fundamental theorem of calculus:

Theorem 0.1 *let $f : [a,b] \to \mathbb{R}$ be continuous. Define $F : [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)dt.$$

(By convention $F(a) = \int_a^a f(t)dt = 0$).

Prove that F is differentiable on (a,b) and that $F'(x) = f(x)$ for $x \in (a,b)$.

(Hint: to prove $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$, a variation of Exercise 7.1 #7 may be useful. You'll want to consider the limits from the left and from the right separately.)