MATH 31CH MIDTERM 1 SOLUTIONS

(1) (a) This is just a straightfoward calculation using cofactors. We will proceed downward from the first column.

$$\det\left(\begin{bmatrix} 0 & -2 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}\right) = -1\begin{vmatrix} -2 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix}$$
$$= -1(1) + 2(-4) = -9.$$

(b) We perform a change of basis with the transformation $g(u) := Au = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$. That is, we think of the coordinates x_1, x_2, x_3 as functions of the coordinates of u. Under this transformation we have $x_3 = 2u_1 + u_2$. Therefore

$$\int_{\mathbb{R}^4} \mathbb{1}_P x_3 \, |d^4x| = |-9| \int_0^1 \int_0^1 \int_0^1 \int_0^1 2u_1 + u_2 \; du_1 du_2 du_3 du_4$$
$$= \frac{27}{2}$$

(2) Recall from the definition of integrable that we need to show $\lim_{N\to\infty} U_N(f) - L_N(f) = 0$. As such, we need to show given $\varepsilon > 0$ that there exists $M \in \mathbb{N}$ so that for $N > M$ $U_N(f) - L_N(f) < \varepsilon$. Since A is compact, f is uniformly compact over A. Namely, there exists $\delta > 0$ so that whenever $\|x - y\| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. We would like to choose N so that the diameter of the dyadic cubes is less than δ to ensure $|M_C(f) - m_C(f)| < \varepsilon$. One can check that the diameter of a n-dimensional dyadic cube in the N-th dyadic partition is $\frac{\sqrt{n}}{2^N}$. Therefore, we choose $N > \log_2\left(\frac{\sqrt{n}}{\delta}\right)$. For such N, we have

$$U_N(f) - L_N(f) = \sum_{C\in D_N} (M_C(f) - m_C(f))\text{vol}(C)$$
$$\leq \varepsilon \sum_{C\in\mathcal{D}_N} \text{vol}(C).$$

A is bounded so there exists some $\alpha \in \mathbb{R}$ so that $A \subset [0,\alpha]^n \subset \mathbb{R}^n$. It follows that $\sum_{C\in D_N} \text{vol}(C) \leq \alpha^n$ for all $N \geq 2$. Hence $U_N(f) - L_N(f) < \varepsilon\alpha^n$. Taking ε to zero yields the claim.

(3) We use spherical coordinates. Because we are only integrating over the first octant we have $\theta, \varphi \in [0, \pi/2]$.

$$\int_A (x^2 + y^2 + z^2)^{1/2} |d^3x| = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^3 \cos\varphi \; d\rho d\varphi d\theta = \frac{\pi}{8}.$$

(4) This just amounts to invoking Fubini's theorem. Note that $\frac{\sin(x)}{x}$ is a continuous function on our (compact) region so it is indeed integrable. Therefore, Fubini applies.

$$\int_0^\pi \int_y^\pi \frac{\sin(x)}{x}\, dxdy = \int_0^\pi \int_0^x \frac{\sin(x)}{x}\, dydx = \int_0^\pi \sin(x)\, dx = 2.$$